Mathematical Statistics–1

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Introduction

We have to know some terms which are very important in *probability theory*

1. A *Random Experiment* is an experiment or process for which the outcome can not be predicted with certainty.

Example 1.1 Three coins are tossed and let r.v. represents the number of heads then x may take values $x = 1, 2, 3, .$
$S.S = \{HHH, HTH, THH, HHT, TTH, THT, HTT, TTT\}$.
Then, $x = 0, 1, 2, 3$.

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2. The *Sample Space* Ω is the collection of all possible outcomes of a *Random Experiment*

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1. A *Random Experiment* is an experiment or process for which the outcome can not be predicted with certainty.
2. The *Sample Space* Ω is the collection of all possible outcomes of a *Random Experiment*
3. An *Event* is a subset of the *Sample Space*.

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Random Variable and Function of Random Variable

Remark If x_1 and x_2 are two r.v.s and c_1, c_2 are constants, then:

1. $c_1 x_1 + c_2 x_2$ is r.v.

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3. $\max\{x_1, x_2\}$ is r.v.
4. $\min\{x_1, x_2\}$ is r.v.

Discrete Random Variable

Definition

If x is discrete r.v. with counting values $x_1, x_2, \ldots$ then the function denoted by $p_x(x)$ and defined as follows:-

$$p_x(x) = \begin{cases} p(x = x_j) & x = x_j \qquad j = 1, 2, 3, 4, , \cdots , \\ 0 & x \neq x_j \end{cases} \tag{1}$$

the above equation is called p.m.f.

Remark

1. $Pr(a \leq x \leq b) = \sum_{x=a}^{b} p(x)$.

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Discrete Random Variable

Properties of p.m.f

1. $p_x(x) \geq 0. \qquad \textit{for all } x = 0, 1, 2, 3, 4, \ldots$

Remark

1. $\sum_{\textit{for all } x} x = \frac{n(n+1)}{2}.$

Discrete Random Variable

Properties of p.m.f

1. $p_x(x) \geq 0. \qquad \textit{for all } x = 0, 1, 2, 3, 4, \ldots$
2. $\sum_{\textit{for all } x} p_x(x) = 1.$

Remark

1. $\sum_{\textit{for all } x} x = \frac{n(n+1)}{2}.$
2. $\sum_{\textit{for all } x} x^2 = \frac{n(n+1)(2n+1)}{6}.$

Discrete Random Variable

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1. $\sum_{\textit{for all } x} x = \frac{n(n+1)}{2}.$
2. $\sum_{\textit{for all } x} x^2 = \frac{n(n+1)(2n+1)}{6}.$
3. $\sum_{\textit{for all } x} x^3 = \left[\frac{n(n+1)}{2}\right]^2.$

Discrete Random Variable

Practical 1.1

1. Let

$$p_x(x) = \begin{cases} \frac{x}{10} & x = 1, 2, 3, 4. \\ 0 & \text{otherwise} \end{cases}$$

1− Prove that $p_x(x)$ is a p.m.f.?
2− Sketch the graph of $p_x(x)$?
3− Find the $p(x = 1)$, $p(x = 5)$ and $p(x = \frac{1}{2})$?
4− Find $p(x \leq 3)$, $p(|x| < 2)$?

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4− Find $p(x \leq 3)$, $p(|x| < 2)$?

2. Determine the constant c so that $p(x)$ is p.m.f.
1− $p(x) = c\left[\frac{1}{3}\right]^x \qquad x = 1, 2, 3, \ldots$
2− $p(x) = cx \qquad x = 1, 2, 3, \ldots, 10.$

Discrete Random Variable

Practical 1.1

3. Let a r.v. x has p.m.f $x = 0, 1, 2, 3, 4, 5, 6, 7, 8.$
   and $p(x) = a, 3a, 5a, 7a, 9a, 11a, 13a, 15a, 17a.$
   $1-$ Determine the value of a.?
   $2-$ Find $p(x < 2), \quad p(x \leq 6), \quad$ and$p(3 < x < 5)$?

Continuous Random Variable

Definition

If x is continuous random variable then $f(x)$ is called probability density function p.d.f.. The properties of p.d.f. :

1. $f(x) \geq 0 \quad \forall x$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

Remark

1. $Pr(a < x < b) = Pr(a \leq x \leq b) = \int_a^b f(x)dx$.
2. $Pr(x = a) = 0$. for continuous random variable.
3. $Pr(x = a) = Pr(a)$. for discrete random variable.

Continuous Random Variable

Example

Let $f(x) = cx \quad 0 < x < 1$ where $f(x)$ is p.d.f. : $-$

1. Find the constant c ?
2. Sketch the graph of $f(x)$?
3. Find $Pr(\frac{1}{2} < x < \frac{3}{4})$ and $Pr(-\frac{1}{2} < x < \frac{1}{2})$?

Continuous Random Variable

Practical 1.2

1. Let the r.v x have:

$$f(x) = \begin{cases} \frac{\sin x}{2} & 0 \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

   Prove that the $f(x)$ is p.d.f of x and compute the $Pr(x \geq \frac{\pi}{3})$?

2. Determine the value of k which would make:

$$f(x) = \begin{cases} kx & |x-2| < 1 \\ 0 & |x-2| > 1 \end{cases}$$

   a p.d.f of x ?

Cumulative distribution function c.d.f

If x is a r.v. having p.m.f and p.d.f such as $p(x)$ and $f(x)$. Then the cumulative distribution function is defined as follows:

1. $F_X(x) = Pr(X \leq x)$.
2. $F_X(x) = Pr(X \leq x) = \sum_{X \leq x} p(X) \quad d.r.v$
3. $F_X(x) = Pr(X \leq x) = \int_{X \leq x} f(X) \quad c.r.v$

Properties of c.d.f

1. $0 \leq F_X(x) \leq 1$ because $0 \leq p(X \leq x) \leq 1$.
2. $F(X)$ is a non-decreasing function of x.
3. $F(\infty) = \lim_{x \to \infty} F(x) = 1$ and $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$. Because the set $[x : x \leq \infty]$ is entire one dimensional space, the set $[x : x \leq -\infty]$ is a null set.
4. $F(x)$ is continuous to the right side.

Cumulative distribution function c.d.f

Practical 1.2

1. Prove that the above properties are TRUE ?
2. Let N be a positive integer and let

$$p(x) = \begin{cases} \frac{2x}{N(N+1)} & x = 1, 2, 3, \dots, N \\ 0 & \text{Otherwise} \end{cases}$$

1− Show that $p(x)$ is p.m.f?
2− Find c.d.f of $p(x)$?

3. Let the r.v. x have

$$f(x) = \begin{cases} \frac{\sin x}{2} & 0 \leq x \leq \pi \\ 0 & \text{Otherwise} \end{cases}$$

1− Prove that the $f(x)$ is p.d.f ?
2− Determine the c.d.f of x and sketch the graph of c.d.f ?
3− Find $Pr(x \geq \frac{\pi}{3})$ and $Pr(x \geq m) = \frac{1}{2}$?

Cumulative distribution function c.d.f

Homework 1.1

1. A r.v. has c.d.f

$$F(x) = \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}(x)\right]$$

- Find the p.d.f of x ?
- Determine $Pr(|x| < 1)$?

Mixed Distribution

Since the function F is right-continuous, it is dis-continuous at the point x_0, iff $F_{(x_0')} < F_{(x_0)}$. We can say that the difference will be called the jump $p_{(x_0)}$ at the point x_0. Then , we can write the function as follows:

$$F_{(x)} = \alpha F_c + (1-\alpha)F_d, \qquad 0 \leq \alpha \leq 1.$$

where F_c is a continuous c.d.f., and F_d is a discrete c.d.f..

1. If $\alpha = 0$, then $F_{(x)}$ is a discrete function.
2. If $\alpha = 1$, then $F_{(x)}$ is a continuous function.
3. Otherwise, the distribution $F_{(x)}$ will be called *mixed distribution*. It means that the *mixed distribution* is combination of discrete and continuous.

Mixed Distribution

Practical 1.3

1− Let x be a random variable. If the mixed distribution have

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \leq x < 1 \\ \frac{x+1}{4} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$a-$ Sketch the graph of $F(x)$?
$b-$ Find the p.d.f of x ?
$c-$ Find $Pr(\frac{1}{4} < x < 1), Pr(x = 1),$ and $Pr(x = \frac{1}{2})$?

Mixed Distribution

HomeWork 1.2

1− Let x be a random variable. If the mixed distribution have

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x+1}{2} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

$a-$ Sketch the graph of $F(x)$?
$b-$ Find the p.d.f of x ?
$c-$ Find $Pr(x = 1), Pr(x = \frac{1}{2}), Pr(1 < x \leq 2), Pr(x > \frac{1}{2})$and $Pr(|x| \leq 1)$?

Mixed Distribution

HomeWork 1.2

2− Let x be a random variable. If the mixed distribution have

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{3} & 0 \leq x < 1 \\ \frac{x}{2} & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$a-$ Sketch the graph of $F(x)$?
$b-$ Find the p.d.f of x ?
$c-$ Find $Pr(\frac{1}{2} \leq x \leq \frac{3}{2}), Pr(\frac{1}{2} \leq x \leq 1)$ and $Pr(1 \leq x \leq \frac{3}{2})$?

Mixed Distribution

HomeWork 1.2

3− Leting c.d.f of discrete random variable

$$F(x) = \begin{cases} \frac{32}{31}\left[1 - (\frac{1}{2})^x\right] & x = 1, 2, 3, 4, 5 \\ 0 & x < 1 \\ 1 & x > 5 \end{cases}$$

$a-$ Find the p.m.f of x ?
$b-$ Find $Pr(x < 2), Pr(1 \leq x \leq 5)$,
$Pr(|x| \leq 3)$ and $Pr(x \leq \frac{5}{2})$?

Mathematical Expectation

Definition

If x is a r.v. and $u(x)$ is a function of r.v. x, then the *Mathematical Expectation* or *Expected value* for $u(x)$ is defined as follows:

$$E[u(x)] = \sum_{\forall j} u(x_j)p(u_j) \quad \text{d.r.v}$$

$$E[u(x)] = \int_{\forall x} u(x)f(x)dx \quad \text{c.r.v}$$

Properties of Mathematical Expectation

1. $E(c) = c$ where c is constant.
2. $E[cu_1(x)] = cE[u_1(x)]$.
3. $E[c_1 u_1(x) + c_2 u_2(x)] = c_1 E[u_1(x)] + c_2 E[u_2(x)]$.
4. $E[u_1(x)] \leq E[u_2(x)]$ if $u_1(x) \leq u_2(x)$.
5. $$\begin{aligned} \mu = E(x) &= \sum_{\forall x} x p(x) \quad \text{d.r.v} \\ &= \int_{-\infty}^{\infty} x f(x) dx \quad \text{c.r.v} \end{aligned}$$
6. $$\begin{aligned} var(x) &= \sum_{\forall x} (x-\mu)^2 p(x) \quad \text{d.r.v} \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx \quad \text{c.r.v} \end{aligned}$$

Mathematical Expectation

Example

The p.d.f. of x is:

$$f(x) = \begin{cases} 2\exp(-x) & 0 \leq x \leq \ln 2 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the c.d.f of x?
2. Find $E(x)$ and $E[\exp(2x)]$?
3. Letting $g(x)$ a function of x where $g(x) = 2x + 1$.
   Find $E(2x + 1)$?

The Moment

1. Non-Central Moment
   If x is a r.v., the r^{th} non-central moment of x usually denoted by m_r as $m_r = E(x)^r$ where r is a positive integer number. For example, $m_1 = E(x), m_2 = E(x^2), \cdots$, etc.
2. Central Moment
   If x is a r.v., the r^{th} central moment of x around a is defined as $E(x-a)$. If $a=\mu$, then the r^{th} central moment of x,i.e., μ_x denoted by μ'_r as: $\mu'_r = E(x-\mu_r)^r$.

Remark

$$\begin{aligned}
\mu'_1 &= E(x-\mu_1) = E(x) - \mu_1 = \mu_1 - \mu_1 = 0. \\
\mu'_2 &= E(x-\mu)^2 = var(x) = E(x^2) - (EX)^2 \\
\mu'_3 &= E(x-\mu)^3 = E(x^3) - 3\mu E(x^2) + 3\mu^2 Ex - \mu^3, \text{generally,} \\
\mu'_r &= E\left[\sum_{i=0}^{r} \binom{r}{i} (-1)^i (\mu_1)^i x^{r-i}\right]
\end{aligned}$$

The Moment

HomeWork

1. Find the relationship between *central* and *non-central* moments?
2. Let

$$p(x) = \begin{cases} \frac{1}{3} & x = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

   $1-$ Prove that $p(x)$ is p.m.f? $2-$ Find the c.d.f of x? $3-$ Find the variance of x? $4-$ Find $Pr(x = -1)$ and $Pr(-\frac{1}{2} < x < \frac{1}{2})$?
3. Let x has p.m.f $p(x)$ is positive where $x = -1, 0, 1$. If $f(0) = \frac{1}{2}, E(x) = \frac{1}{6}$. Find $E(x^2)$ and determine $f(1)$ and $f(-1)$?

Factorial Moment

Definition

If x is a r.v., the r^{th} factorial moment is defined as:

$$\mu_{[r]} = E\left[x(x-1)(x-2)\cdots(x-r+1)\right],$$

where r is a positive integer number.

$$\begin{aligned}
\mu_{[1]} &= E(x) \\
\mu_{[2]} &= E[x(x-1)] = E(x^2) - E(x) \\
\mu_{[3]} &= E[x(x-1)(x-2)] = E(x^3) - 3E(x^2) + 2E(x)
\end{aligned}$$

Factorial Moment

Example

Let

$$f(x) = \begin{cases} \frac{2x}{a^2} & 0 \leq x \leq a \\ 0 & \text{Otherwise} \end{cases}$$

1. Find the expectation of x?
2. Find the second non-central moment of x?
3. Find the second central moment of x?
4. Find the third factorial moment of x?

Moment Generating Function M.G.F

Definition

The Moment Generating Function of a random variable x denoted by $M_x(t)$. It can be defined as follows:

$$M_x(t) = E[\exp(tx)] = \int_{-\infty}^{\infty} \exp(tx) f(x) dx \quad c.r.v.$$

$$M_x(t) = E[\exp(tx)] = \sum_{-\infty}^{\infty} \exp(tx) p(x) \quad d.r.v.$$

where h is a positive number, $-h < t < h$.

If we differinate $M.G.F$ r times with respect to t, then

$$\frac{\partial^r M_x(t)}{\partial t^r} = \int_{-\infty}^{\infty} x^r \exp(tx) f(x) dx$$

$$\frac{\partial^r M_x(t)}{\partial t^r}\Big|_{t=0} = \int_{-\infty}^{\infty} x^r f(x) dx$$

Properties of M.G.F

1. If $y = ax + b$ and $m_x(t)$ is a moment generating function of x then: $M_y(t) = M_x(at) \times \exp(bt)$.
2. If $z = y + x$ and $M_x(t), M_y(t)$ are M.G.F of two independent r.v. of (y, x) then: $M_z(t) = M_y(t) \times M_x(t)$.
3. Let $x_1, x_2, \cdots, x_n$ be a random sample from distribution with M.G.F, then: $M_{\overline{x}}(t) = \left[M_x(\frac{t}{n})\right]^n$.

Example

Suppose that r.v. y has M.G.F $M_y(t) = [1 - t]^{-r} \quad r < 1$.
FInd $E(y)^r, r = 1, 2, 3, \cdots$, then find the mean and the variance?

Homework

If the M.G.F of $\mu_x(t) = \frac{2}{5}\exp(t) + \frac{1}{5}\exp(2t) + \frac{2}{5}\exp(3t)$. Find the mean and variance of x and defined the p.d.f of x ?

Factorial Moment Generating Function

Let x be a r.v. the factorial M.G.F. is defined as :

$$\begin{aligned} \Psi_x(t) = E(t^x) &= \int_{\forall x} t^x f(x) dx \qquad \text{c.r.v} \\ \Psi_x(t) = E(t^x) &= \sum_{\forall x} t^x p(x) \qquad \text{d.r.v} \end{aligned}$$

Example

Prove that

$$\Psi_x^r(t) = E[x(x-1)(x-2)\dots(x-r+1)]?$$

Characteristic Function

In some cases, the distribution does not have M.G.F then there are another techinque in which called *Characteristic Function* denoted by $\phi_x(t)$. It can be defined as follows:

$$\phi_x(t) = E\exp(itx) = \int_{\forall x} \exp(itx) f(x) dx \qquad \text{c.r.v.}$$

$$\phi_x(t) = E\exp(itx) = \sum_{\forall x} \exp(itx) p(x) \qquad \text{d.r.v.}$$

Properties of Characteristic Function

$$\begin{aligned} 1 - \phi_x(0) &= 1 \\ 2 - \phi_x(t) &= E[\cos(tx) + i\sin(tx)] \\ 3 - |\phi_x(t)| &\leq 1 \\ 4 - \phi_x(-t) &= \phi_x(t) \end{aligned}$$

Characteristic Function

Some Theories

1. $\phi_{cx}(t) = \phi_x(ct)$.
2. If x_1 and x_2 are two independent r.v. then
$$\phi_{x_1+x_2}(t) = \phi_{x_1}(t) + \phi_{x_2}(t)$$
3. If x is a r.v. with characteristic function $\phi_x(t)$ and $\mu_r = Ex^r$ exists then
$$\mu_r = \left[\frac{1}{i}\right]^r \left[\frac{\partial^r \phi_x(t)}{\partial t^r}\right]_{t=0}$$

Example Let x be c.r.v. having p.d.f:

$$f(x) = \begin{cases} \frac{1}{2}\exp(-|x|) & -\infty < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

show that $\phi_x(t) = \frac{1}{(1+t^2)}$?

The Median of distribution

A *median* of any distribution for one r.v. can be computed as follows:

$$\begin{aligned}
p(x \leq m) &= \sum_{-\infty}^{m} p(x) \geq \frac{1}{2} \quad \text{or} \\
p(x < m) &= \sum_{-\infty}^{m-1} p(x) \leq \frac{1}{2} \qquad \text{d.r.v.} \\
f(x \leq m) &= \int_{-\infty}^{m} f(x)dx = \frac{1}{2} \quad \text{or} \\
f(x \geq m) &= \int_{m}^{\infty} f(x)dx = \frac{1}{2} \qquad \text{c.r.v.}
\end{aligned}$$

The Median of distribution

Examples

1. Find the median of the following p.d.f:

$$f(x) = \begin{cases} 3x^2 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Let

$$p(x) = \begin{cases} \binom{4}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{4-x} & x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

find the median of $p(x)$?

The Mode of distribution

A *mode* of any distribution of discrete or continuous r.v. is the value of x when maxizing $f(x)$.

Examples

1. find the mode of the following p.m.f

$$p(x) = \begin{cases} (\frac{1}{2})^x & x = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

2. Let

$$f(x) = \begin{cases} \frac{1}{2}x^2 \exp(-x) & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

find the mode of x ?

Joint, Marginal and Conditional distribution

Definition

Let x and y be two r.vs *discrete* or *continuous* the $f(x,y)$ is called *Joint function* or *bivariate distribution* of x and y.

$$\int_{\forall x}\int_{\forall y} f(x,y)dxdy = 1 \qquad f(x,y) \geq 0 \qquad \text{c.r.v}$$

$$\sum_{\forall x}\sum_{\forall y} p(x_i,y_j) = 1 \quad p(x_i,y_j) \geq 0 \quad i,j = 1,2,\ldots \text{d.r.v}$$

Marginal Function

Let $f(x,y)$ be the joint p.d.f or p.m.f of x and y, then:

$$f(x) = \int_{\forall y} f(x,y)dy \qquad \text{c.r.v.}$$

$$f(y) = \int_{\forall x} f(x,y)dx \qquad \text{c.r.v.}$$

Joint, Marginal and Conditional distribution

$$f(x) = \sum_{\forall y} p(x, y) \qquad \text{d.r.v.}$$

$$f(y) = \sum_{\forall x} p(x, y) \qquad \text{d.r.v.}$$

Conditional distribution

The conditional distribution is defined as follows:

$$f(x|y) = \frac{f(x, y)}{f(y)} \qquad f(y) \neq 0$$

$$f(y|x) = \frac{f(x, y)}{f(x)} \qquad f(x) \neq 0$$

is the conditional distribution a p.d.f. Prove that?

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Remark

1. If $f(x|y)$ is p.d.f then we can compute;

$$Pr(a < x < b|y) = \int_a^b f(x|y)dx,$$

and

$$Pr(c < y < d|x) = \int_c^d f(y|x)dy.$$

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Conditional Expectation

Let $u(x)$ be a function of x, then the *Conditional Expectation* is defined as:

$$\begin{aligned} E[u(x)|y] &= \int u(x)f(x|y)dx \qquad \text{c.r.v} \\ &= \sum u(x)f(x|y) \quad \text{d.r.v} \end{aligned}$$

If $u(x) = x$ then

$$\begin{aligned} E(x|y) &= \int xf(x|y)dx \\ &= \sum xf(x|y) \\ var(x|y) &= E(x^2|y) - [E(x|y)]^2 \end{aligned}$$

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Example

Let

$$p(x_1, x_2) = \frac{x_1 + x_2}{21} \quad x_1 = 1, 2, 3 \text{ and } x_2 = 1, 2$$

1. Show that $p(x_1, x_2)$ is p.m.f?
2. Find $p(x_1)$ and $p(x_2)$?
3. Find $p(x_1|x_2)$ and $p(x_2|x_1)$?
4. Find $E(x_1|x_2)$ and $E(x_2|x_1)$?
5. Find $Pr(x_1 = 3), Pr(x_2 = 2), Pr(x_1 \leq 3, x_2 \leq 2), Pr(1 < x_1 \leq 3, x_2 \leq 2), Pr(0 < x_1 < 3|x_2 = 1)$ and $Pr(0 < x_2 < 2|x_1 = 2)$?

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Some Theories

1. Let (x, y) be two r.vs then $E[E(g(y)|x)] = E[g(y)]$ in particular $E[E(y|x)] = E(y)$ and $E[E(g(x)|y)] = E[g(x)]$ in particular $E[E(x|y)] = E(x)$.
2. $var(y) = E[var(y|x)] + var[E(y|x)]$.

Correlation Coefficient

Distribution of Random Variable

Discrete Distribution

Continuous Distribution

Ditributions of functions of random variable