1.1 Euclidean space and vectors

Def: 1) The set of all ordered n-tuples of real numbers is called n-dimensional. Euclidean space and is denoted by R^n .we will denote such n-tuples either by writing out the component or by single boldface letters

$X = (x_1, x_2, ..., x_n)$

2) The n-tuple whose components are all zero is denoted 0

$0 = (0,0,0,0,0,0,.........,0)$

When n = 2 or 3, we shall often write (x, y) or (x, y, z) unstead of (x_1, x_2) or (x_1, x_2, x_3) but we use X as a single symbol to denote the ordered pair or triple .

3) Addition $X + Y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$

Scalar multiplication $cX = (cx_1, cx_2, ..., cx_n)$

Dot product $X.Y = (x_1 y_1 + x_2 y_2 + ... + x_n y_n)$

4) If $x \in R^n$, then the norm of X is defined to be $|X| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = \sqrt{X.X}$

1.1 (**Cauchy's Inequality)**. For any $a, b \in R^n$, $|a.b| \le |a||b|$

Proof: If b = 0 , then both sides are 0 otherwise Let $t \in R$ and consider the function

$f(t) = |a - tb|^2 = (a - tb).(a - tb) = |a|^2 - 2t\, a.b + t^2 |b|^2$

$f'(t) = 0 - 2a.b + 2t|b|^2$

f has its minimum value when $f'(t) = 0$

$f'(t) = 0 = -2a.b + 2t|b|^2 \Rightarrow$ so f has the min value at $t = \dfrac{a.b}{|b|^2}$

And the min. value is $f(\dfrac{a.b}{|b|^2}) = |a|^2 - \dfrac{(a.b)^2}{|b|^2}$

On the other hand , $f(t) \ge 0$, for all t ,so $|a|^2 - \dfrac{(a.b)^2}{|b|^2} \ge 0$

$\Rightarrow (a.b)^2 \le |a|^2 |b|^2 \Rightarrow |a.b| \le |a||b|$

1.2 The Triangle Inequality

For any $a, b \in R^n , |a + b| \le |a| + |b|$

Proof : we have $|a + b|^2 = (a + b).(a + b) = |a|^2 + 2a.b + |b|^2$

By Cauchy inequality , this last sum is at most

$|a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2$, so the result follows by taking square roots

$|a + b|^2 \le (|a| + |b|)^2 \Rightarrow |a + b| \le |a| + |b|$

Def : The distance between two points X and Y in 3-space is given by

$$\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}=|X-Y|$$

We shall take this as definition of distance in n –space for any n.
Distance from X and Y $=|X-Y|$
By taking $a=X-Y \ and \ b=Y-Z$ in the triangle inequality we see that
$|X-Z|\le|X-Y|+|Y-Z|$ For all $x,y,z\in R^n$

Def: The angle θ between two vectors X and Y is

$$\theta=\cos^{-1}\left(\frac{X.Y}{|X|\ |Y|}\right) \quad where \ \theta\in[0,\pi]$$

Def: If $X.Y=0$, then X and Y are said to be orthogonal to each other.

Remark : Let $X=(x_1,x_2,..............,x_n)$
Let M be the largest of the numbers $|x_1|,|x_2|,..............,|x_n|$
Then $M^2\le x_1^2+x_2^2+..............+x_n^2$ because M^2 is one of the numbers in the righte and $x_1^2+x_2^2+..............+x_n^2\le nM^2$ because each number on the left is at most M^2

In other words ,

$$\max(|x_1|,|x_2|,..............,|x_n|)\le|X|\le\sqrt{n}\max(|x_1|,|x_2|,..............,|x_n|)$$

Cross product
Let $a=(a_1,a_2,a_3)=a_1i+a_2j+a_3k\in R^3$
$b=(b_1,b_2,b_3)=b_1i+b_2j+b_3k\in R^3$
The cross product is defined by

$$a\times b=\begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Remark : 1) $(c_1a_1+c_2a_2)\times b=c_1(a_1\times b)+c_2(a_2\times b)$
$a\times(c_1b_1+c_2b_2)=c_1(a\times b_1)+c_2(a\times b_2)$

2) $a\times b=-b\times a \qquad not \ commutitve$
$a\times(b\times c)\ne(a\times b)\times c$ in General
3) $a\times(b\times c)+b\times(c\times a)+c\times(a\times b)=0 \qquad Jacobi \ identity$
4) $|a\times b|^2=|a|^2|b|^2-(a.b)^2$

If θ is the angle between a and b where $(0 \le \theta \le \pi)$, then $a.b = |a|\,|b|\,\cos\theta$

So, $|a \times b|^2 = |a|^2\ |b|^2\ (1-\cos^2\theta)$ or $|a \times b| = |a|\,|b| \sin\theta$

5) $|a \times b| =$ the area of the parallelogram generated by a and b .

6) $a.(a \times b) = b.(a \times b) = 0$

7) $a \times b$ is orthogonal to both a and b .

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1) Let $x = (3,-1,-1,1)$, $y = (-2,2,1,0)$ compute the norm of x and y and the angle between them

$$\theta = \cos^{-1}\left(\frac{x.y}{|x|\,|y|}\right) = \cos^{-1}\left(\frac{-6-2-1+0}{\sqrt{12}\ \ \sqrt{9}}\right) = \cos^{-1}\left(\frac{-9}{3.2\sqrt{3}}\right) = \cos^{-1}\left(\frac{-3}{2\sqrt{3}}\right) = \frac{5\pi}{6}$$

6) Show that $\|a| - |b\| \le |a-b|$ for every $a,b \in R^n$

Solution : $|a| = |a-b+b| \le |a-b| + |b|$ .....(1)

$|b| = |b-a+a| \le |a-b| + |a|$ .....(2)

$from\ (1)$ $|a| - |b| \le |a-b|$

$from\ (2)$ $|b| - |a| \le |a-b|$

$\Rightarrow \|a| - |b\| \le |a-b|$ for every $a,b \in R^n$

7) Suppose that $a, b \in R^3$

a) Show that if $a.b = 0$ and $a \times b = 0$, then either a = 0 or b = 0

Solution : If $a.b = 0 \Rightarrow either\ \theta = \dfrac{\pi}{2} between\ a\ and\ b\ or\ either\ a\ or\ b\ is\ zero$

If $\theta = \dfrac{\pi}{2} \Rightarrow |a \times b| = |a|\,|b| \sin\theta \ne 0$ contradiction So, a = 0 or b = 0

b) $a.c = b.c \Rightarrow (a-b).c = 0$

$c \ne 0 \Rightarrow a-b = 0 \Rightarrow a = b$

$a \times c = b \times c \Rightarrow (a-b) \times c = 0$

$c \ne 0 \Rightarrow a-b = 0 \Rightarrow a = b$

c) $(a \times a) \times b = a \times (a \times b)$ iff a and b are proportional

Let $b = ra$

$(a \times a) \times b = (a \times a) \times (ra) = r(a \times a) \times a = a \times (a \times ra) = ra \times (a \times a) = 0$

1.2 Subsets of Euclidean space R^n

Def: The set of all points whose distance from a fixed point a is equal to some number r is called the sphere of radius r about a and the set of points whose distance from a is less than r is called the (open) ball of radius r about a .
We use the notation $B(r,a)$ for the ball of radius r about a.

$$B(r,a)=\{x\in R^n :|x-a|<r\}$$

In a space R^1 of one dimensional a ball is an open interval ,and in dimension 2 , the words “ disc” and circle used in place of ball and sphere .

A set $S\subset R^n$ is called bounded if it is contained in some ball about the origin, that is, if there is a constant C such that $|x|<C$ for every $x\in S$
Where $|X|=\sqrt{x_1^2+x_2^2+\dots\dots\dots+x_n^2}=\sqrt{X.X}$

Def : Let S be a subset of R^n

1) The complement of S is the set of all points in R^n that are not in S , we denote it by R^n/S or by S^c , $S^c=R^n/S=\{x\in R^n : x\notin S\}$
2) A point $x\in R^n$ is called an interior point of S if all points sufficiently close to x (including x itself) are also in S , that is if S contains some ball centered at X.
3) The set of all interior points of S is called the interior of S and is denoted by S^{int} , $S^{\text{int}}=\{x\in S: B(r,x)\subset S \text{ for some } r>0\}$
4) A point $x\in R^n$ is called a boundary point of S if every ball centered at x contains both points in S and points in S^c (Note that if x is a boundary point of S , x may belong to either S or S^c). The set of all boundary points of S is called the boundary of S and is denoted by $\partial S=\{x\in R^n : B(r,x)\cap S\neq\varphi \text{ and } B(r,x)\cap S^c\neq\varphi \text{ for every } r>0\}$
5) S is called open if it contains none of its boundary points.
6) S is called closed if it contains all of its boundary points.
7) The closure of S is the union of S and all its boundary points . It is denoted by $\bar{S}$: $\bar{S}=S\cup\partial S$
8) Finally, a neighborhood of a point $x\in R^n$ is a set of which x is an interior point. That is, S is neighborhood of x iff x is an interior point of S.

Remark :

1) The boundary points of S are the same as the boundary points of S^c
2) If x is neither an interior point of S nor an interior point of S^c , then x must be a boundary point of S .

3) Given $S \subset R^n$ and $x \in R^n$, there are exactly three possibilities: x is an interior point of S or x is an interior point of S^c , or x is a boundary point .

1.3 **Proposition** : Suppose $S \subset R^n$

a) $S \ is \ open \Leftrightarrow$ every point of S is an interior point .

b) $S \ is \ closed \Leftrightarrow S^c$ is open

Proof : Every point of S is either an interior point or boundary point , thus S is open $\Leftrightarrow$ every point of S is an interior point and S is closed $\Leftrightarrow$ it contains all of ∂S, which is the same as $\partial(S^c)$; this happens when S^c contains none of its boundary points , that is S^c is open .

Example 1) : Let S be $B(\rho,0)$, the ball of radius ρ about the origin . First given $x \in S$ .Let $r = \rho - |x|$, If $|y-x| < r$, then by the triangle inequality we have $|y| \le |y-x| + |x| < \rho$, So that $B(r,x) \subset S$

Therefore , every $x \in S$ is an interior point of S , so S is open .

Second a similar calculation shows that if $|x| > \rho$ then $B(r,x) \subset S^c$ where $r = |x| - \rho$

So every point with $|x| > \rho$ is an interior point of S^c .

On the other hand , if $|x| = \rho$,then $cx \in S$ for $0 < c < 1$ and $cx \in S^c$ for $c \ge 1$, and $|cx - x| = |c-1|\rho$ can be as small as we please , so x is a boundary point .In the other words , the boundary of S is the sphere of radius ρ about the origin ,and the closure of S is the closed ball $\{x : |x| \le \rho\}$

Example 2) : Let S be the ball of radius ρ about the origin together with the upper hemisphere ,of its boundary : $S = B(\rho,0) \cup \{x \in R^n : |x| = \rho \ and \ x_n > 0\}$

$S^{\text{int}} = B(\rho,0)$, $\partial S = \{x : |x| = \rho\}$

And $\bar{S} = \{x : |x| \le \rho\}$

The set S is neither open nor closed .

Example3) : In the real line (n = 1) , let S be the of all rational numbers ,since every ball in R -that is every interval contains both rational and irrational numbers , every point of R is a boundary point of S .The set S is neither open nor closed , its interior is empty ,and its closure is R .

1.3 Limits and continuity

Def: A function $f(x)$ of one variable is said to approach a limit $L \in R$ as x approach a if and only if for any positive real no.

$\varepsilon > 0\ , \exists \delta > 0 , \ni whenever\ \ 0 < |x - a| < \delta \Rightarrow | f(x) - L| < \varepsilon$

In symbols we write $\lim_{x \to a} f(x) = L$

Example:

$\lim_{x \to 1} \sqrt{x} = 1$

Let $\varepsilon > 0$, we will find $\delta > 0$ such that whenever

$0 < |x - 1| < \delta \Rightarrow |\sqrt{x} - 1| < \varepsilon$

$-\varepsilon < \sqrt{x} - 1 < \varepsilon$

$1 - \varepsilon < \sqrt{x} < 1 + \varepsilon$

$(1-\varepsilon)^2 < x < (1+\varepsilon)^2$

$(1-\varepsilon)^2 - 1 < x - 1 < (1+\varepsilon)^2 - 1$

$\delta = \min.\{1-(1-\varepsilon)^2, (1+\varepsilon)^2 - 1\} = \{2\varepsilon - \varepsilon^2, \varepsilon^2 + 2\varepsilon\}$

Another solution :

Let $\varepsilon > 0$, let $\delta = \varepsilon$

We have $|\sqrt{x} - 1| = \frac{|x-1|}{|\sqrt{x} + 1|}$

If $0 < |x - 1| < \delta = \varepsilon$ we obtain $|\sqrt{x} - 1| < \frac{\varepsilon}{|\sqrt{x} + 1|} < \varepsilon$

Example: Show that $f(x) = \sin\frac{1}{x}$ has no limit as $x \to 0$

Solution : Suppose that $f(x) = \sin\frac{1}{x}$ has a limit as $x \to 0$, then choose $\varepsilon = \frac{1}{2}$, we can find a $\delta > 0$ such that $|\sin\frac{1}{x} - L| < \frac{1}{2}\ whenever\ 0 < |x| < \delta$

Let n be any integer whose absolute value is so large that bothe points

$$x_1 = \frac{1}{(2n + \frac{1}{2})\pi},\ \text{and } x_2 = \frac{1}{(2n - \frac{1}{2})\pi}$$

belong to the neighborhood $0 < |x| < \delta$, then

$\sin\frac{1}{x_1} = \sin(2n + \frac{1}{2})\pi = \sin\frac{1}{2}\pi = 1$

while $\sin\frac{1}{x_2} = \sin(2n - \frac{1}{2})\pi = \sin -\frac{1}{2}\pi = -1$

It follows that $|\sin\frac{1}{x_1} - L| = |1 - L| < \frac{1}{2}$ and $|\sin\frac{1}{x_2} - L| = |-1 - L| < \frac{1}{2} \Rightarrow |1 + L| < \frac{1}{2}$

Then $|1 - L| < \frac{1}{2} \Rightarrow -\frac{1}{2} < 1 - L < \frac{1}{2} \Rightarrow -\frac{3}{2} < -L < -\frac{1}{2} \Rightarrow \frac{1}{2} < L < \frac{3}{2} \Rightarrow L > \frac{1}{2}$

and $|1 + L| < \frac{1}{2} \Rightarrow -\frac{1}{2} < 1 + L < \frac{1}{2} \Rightarrow -\frac{3}{2} < L < -\frac{1}{2} \Rightarrow L < -\frac{1}{2}$

Contradiction , Thus the assumption that $\sin\frac{1}{x}$ has a limit at x = 0 leads to a contradiction , therefore $\sin\frac{1}{x}$ does not have a limit at x = 0

Def: $f(x)$ is said to a approach $L \in R$ as $x \to a$ from the right and denoted by $\lim_{x\to a^+} f(x) = L$ provided that for each $\varepsilon > 0$, $\exists \delta > 0 \ni 0 < x - a < \delta \Rightarrow | f(x) - L| < \varepsilon$ and is said to approach $L \in R$ as $x \to a$ from left , denoted by $\lim_{x\to a^-} f(x) = L$ provided that for each $\varepsilon > 0$, $\exists \delta > 0, \ni 0 < a - x < \delta \Rightarrow | f(x) - L| < \varepsilon$

Example: Let $f(x) = \begin{cases} x + 1 & for \ |x| \le 1 \\ 0 & for \ |x| > 1 \end{cases}$

Then $\lim_{x\to 1^-} f(x) = 2$

$\lim_{x\to 1^+} f(x) = 0$

$\Rightarrow \lim_{x\to 1} f(x)$ does not exist .

Remark : For a function $f(x)$, we can define the one sided limits as $x \to a$ from right and left as $\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a \\ x>a}} f(x)$ and $\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a \\ x<a}} f(x)$.

Remark : The ordinary limit as $x \to a$ for $f(x)$ is called the two sided limit and $\lim_{x\to a} f(x)$ exists whenever $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

Def of Continuity : A function $f(x)$ is called cont. at x = a provided that $\lim_{x\to a} f(x) = f(a)$, and we write $f(x) \in C$.

The function $f(x)$ belongs to the class of cont. function or we can write for
$\varepsilon > 0$, $\exists \delta > 0, \ni |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Def: $f(x)$ is called cont. from right at x = a provided that $\lim_{x \to a^+} f(x) = f(a)$
And $f(x)$ is called cont. from left at x = a provided that $\lim_{x \to a^-} f(x) = f(a)$
If $f(x)$ is cont. at x = a , we say that $f(x) \in C$ at x = a .
If $f(x)$ is cont. at each x of the interval (a,b) we say that $f(x) \in C$ for
a < x < b or $f(x) \in C(a,b)$.
If $f(x) \in C$, a < x < b and $f(x)$ is cont. at a from the right and cont. at b from the left we say that $f(x) \in C$, $a \le x \le b$ or $f(x) \in C[a,b]$

Theorem : $\lim_{x \to a} f(x)$ exists iff $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$
Proof: Let $\lim_{x \to a} f(x) = L \in R$
$\forall \varepsilon > 0$, $\exists \delta > 0, \ni 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
So ,if $0 < x - a < \delta \Rightarrow 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
So , $\lim_{x \to a^+} f(x) = L$
Also , if $0 < a - x < \delta \Rightarrow 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$
So , $\lim_{x \to a^-} f(x) = L$
$\Leftarrow$ suppose that $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$
$\forall \varepsilon > 0$, $\exists \delta_1 > 0, \ni 0 < x - a < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$
Also , $\exists \delta_2 > 0, \ni 0 < a - x < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$
Let $\delta = \min.(\delta_1, \delta_2)$
If $0 < |x-a| < \delta \Rightarrow 0 < x - a < \delta$ or $0 < a - x < \delta$
If $0 < x - a < \delta \Rightarrow 0 < x - a < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$
$0 < a - x < \delta \Rightarrow 0 < a - x < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$
So , $\lim_{x \to a} f(x) = L$

Functions of several variables :
Def: A function $f(x,y)$ approaches a limit $L \in R$ as x approaches a and y approaches b , denoted by $\lim_{\substack{x \to a \\ y \to b}} f(x,y) = L$ provided that

$\forall \varepsilon > 0$, $\exists \delta > 0, \ni |x-a| < \delta$, $|y-b| < \delta$ and $(x-a)^2 + (y-b)^2 > 0 \Rightarrow |f(x,y) - L| < \varepsilon$

Example: $f(x,y) = x^2 + y^2$

Show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0$

Proof: Let $\varepsilon>0$, *choose* $\delta=\sqrt{\dfrac{\varepsilon}{2}}$

Let $|x-0|<\delta=\sqrt{\dfrac{\varepsilon}{2}}$, $|y-0|<\delta=\sqrt{\dfrac{\varepsilon}{2}}$

$\Rightarrow x^2+y^2<\varepsilon$

$\Rightarrow |f(x,y)-0|=|x^2+y^2-0|\le x^2+y^2<\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}=\varepsilon$

$\Rightarrow \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=0$

Remark : It is not true in general that $\lim\limits_{y\to b}(\lim\limits_{x\to a} f(x,y))=\lim\limits_{x\to a}(\lim\limits_{y\to b} f(x,y))$

Example : Let $f(x,y)=\begin{cases}\dfrac{x-y}{x+y} & x\ne -y\\ 1 & x=-y\end{cases}$

$\lim\limits_{x\to 0}(\lim\limits_{y\to 0}\dfrac{x-y}{x+y})=\lim\limits_{x\to 0}\dfrac{x}{x}=1$ but $\lim\limits_{y\to 0}(\lim\limits_{x\to 0}\dfrac{x-y}{x+y})=\lim\limits_{y\to 0}-\dfrac{y}{y}=-1$

The limit $(\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{x-y}{x+y})$ does not exists .

Let $y=mx$

$\lim\limits_{x\to 0}\dfrac{x-y}{x+y}=\lim\limits_{x\to 0}\dfrac{x-mx}{x+mx}=\lim\limits_{x\to 0}\dfrac{x(1-m)}{x(1+m)}=\dfrac{(1-m)}{(1+m)}$ *along the line* $y=mx$

So, the limit as (x,y) approaches (0,0) along the line y = mx is $\dfrac{(1-m)}{(1+m)}$ which changes as m change .So , the limit does not exist .

Note : $(\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$ exists when the limit along any path passes through the point (a,b) is a unique .

Example: Let $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & if\ (x,y)\ne(0,0)\\ 0 & if\ (x,y)=(0,0)\end{cases}$

Show that $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist .

Solution : Let y = cx

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\,.cx}{x^2+(cx)^2} = \lim_{x\to 0} \frac{x^2}{x^2}\frac{c}{1+(c)^2} = \frac{c}{1+(c)^2}$$

The limit changes as c changes, so the limit does not exist .

Example : Let $g(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & if \ (x,y) \neq (0,0) \\ 0 & if \ (x,y) = (0,0) \end{cases}$

Solution : Let $y = cx^2$

$$\lim_{x\to 0} g(x,cx^2) = \lim_{x\to 0} \frac{cx^4}{x^4+c^2x^4} = \frac{c}{1+c^2}$$

The limit changes as c changes, so the limit does not exist .

Def: we say that $f(x,y) \in C$ at (a,b) iff $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$

Example: $\phi(x,y) = \dfrac{\sin(3x+2y)}{x^2-y}$ is cont. every where except along the parabola $y = x^2$

TH : The sum , product , or difference of two cont. function is cont. , the quotient of two cont. function is cont. on the set where the denominator is nonzero .

Example: $h(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & if \ (x,y) \neq (0,0) \\ 0 & if \ (x,y) = (0,0) \end{cases}$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy\,(x^2-y^2)}{x^2+y^2} = 0 \ \text{since}\ |h(x,y)| \leq \frac{|xy|\,|x^2-y^2|}{|x^2+y^2|} \leq |xy|$$

So, $h(x,y)$ is cont. at (x,y)=(0,0) as the limit approaches 0.
The limit of $h(x,y)$ exists on any path and equal to zero , so it is cont. at (0,0) and at any point .

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1) For the following functions f ,show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

a) $f(x,y)=\dfrac{x^2+y}{\sqrt{x^2+y^2}}$

Let $y=mx$

$$\lim_{x\to 0}\frac{x^2+mx}{\sqrt{x^2+m^2x^2}}=\lim_{x\to 0}\frac{x(x+m)}{x\sqrt{1+m^2}}=\frac{m}{\sqrt{1+m^2}}$$

The limit changes as m change so , the limit does not exist .

2) For the following function f , show that $\lim_{(x,y)\to(0,0)} f(x,y)=0$

a) $f(x,y)=\dfrac{x^2y^2}{x^2+y^2}$

Let $y=mx$

$$\lim_{x\to 0}\frac{x^2m^2x^2}{x^2+m^2x^2}=\lim_{x\to 0}\frac{m^2x^2}{1+m^2}=0$$

Let $y=mx^2$

$$\lim_{x\to 0}\frac{x^2m^2x^4}{x^2+m^2x^4}=\lim_{x\to 0}\frac{m^2x^4}{1+m^2x^2}=0$$

For $(x,y)\neq(0,0)$, we have $0\le f(x,y)=x^2\dfrac{y^2}{x^2+y^2}\le x^2\dfrac{(y^2+x^2)}{(x^2+y^2)}=x^2$

since $\lim_{x\to 0}x^2=0$, so $\lim_{(x,y)\to(0,0)} f(x,y)=0.$

b) $\lim_{x\to 0}\dfrac{3x^5-xy^4}{x^4+y^4}=\lim_{x\to 0}\dfrac{x(3x^4-y^4)}{x^4+y^4}$

Let $y=mx$

$$\lim_{x\to 0}\frac{x(3x^4-m^4x^4)}{x^4+m^4x^4}=\lim_{x\to 0}\frac{x(3-m^4)}{1+m^4}=0$$

A long $y=mx^2$

$$\lim_{x\to 0}\frac{3x^5-x.m^4x^8)}{x^4+m^4x^8}=\lim_{x\to 0}\frac{3x-m^4x^5}{1+m^4x^4}=0$$

$\forall\varepsilon>0,\exists\delta>0,\ni|x|<\delta,|y|<\delta\quad and\quad x^2+y^2>0$

$$\Rightarrow\left|\frac{3x^5-xy^4}{x^4+y^4}\right|<3\frac{|x|x^4}{x^4+y^4}+|x|\frac{y^4}{x^4+y^4}\le 3|x|+|x|=4|x|.$$

As $\lim_{x\to 0} 4|x| = 0,$ so, the limit exist by sandwich theorem for any path and equals to zero.

3) Let $f(x,y) = x^{-1}\sin(xy)$ for $x \neq 0$ .How should you define $f(0,y)$ *for* $y \in R$ So as to make f a cont. function on all of R^2?

Solution : $f(0,y)$ is a cont. if $\lim_{x\to 0} f(x,y) = f(0,y)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\frac{\sin xy}{x} = \lim_{x\to 0} y\frac{\sin xy}{xy} = y$$

So, $f(0,y) = y$

4) Let $f(x,y) = \dfrac{xy}{x^2+y^2}$ as in Example 1. Show that , although f is discont. At (0,0) $f(x,a)$ *and* $f(a,y)$ are cont. functions of x and y , resp. for any $a \in R$ (including a = 0). we say that f is separately continuous in x and y.

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$$f(x,a) = \frac{xa}{x^2+a^2}$$

$$\lim_{x\to 0} f(x,a) = \lim_{x\to 0}\frac{xa}{x^2+a^2} = \frac{0}{a^2} = 0 \quad if \ a \neq 0$$

If $a = 0 \Rightarrow f(x,a) = \dfrac{0}{x^2+0} = 0 \Rightarrow f(x,a) = f(x,0)$

So, $f(x,a)$ is cont. $\forall a \in R$ Similarly $f(a,y)$

5) Let $f(x,y) = \dfrac{y(y-x^2)}{x^4}$ *if* $0 < y < x^2$, $f(x,y) = 0$

Otherwise. At which points is f discont. ?

Solution : along y= mx

$$\lim_{x\to 0}\frac{mx(mx-x^2)}{x^4} = \lim_{x\to 0}\frac{mx^2(m-x)}{x^4} = \lim_{x\to 0}\frac{m(m-x)}{x^2}$$ does not exist .

Along $y = mx^2$

$$\lim_{x\to 0}\frac{mx^2(mx^2-x^2)}{x^4} = \lim_{x\to 0}\frac{mx^4(m-1)}{x^4} = m(m-1)$$

The limit changes as m change so the limit does not exist .

So, $f(x,y)$ not cont. at (0,0)
For any other point $f(x,y)$ is cont.

6) Let $f(x)=x$ if x is rational , $f(x)=0$ if x is irrational
Show that f is cont. at x = 0 and nowhere else.
Solution:

Note that $f(0)=0$. Let $\varepsilon>0$ be arbitrary. Take $\delta=\varepsilon$.
Let $x\in R$ such that $|x|<\delta$.
If x is rational then $|f(x)-f(0)|=|x-0|=|x|<\delta=\varepsilon$.
If x is irratinal then $|f(x)-f(0)|=0<\varepsilon$.
In both cases, we have $|f(x)-f(0)|<\varepsilon$ whenever $|x|<\delta$.
Therfore, f is continuous at 0.
To show that f is discontinuous at any point $a\neq 0$.
let $a\neq 0$.
Case 1: If a is rational, then f(a)=a. Take $\varepsilon_0=|a|/2>0$. Let $\delta>0$ be arbitrary.
Choose x_δ to be an irrational number in the interval $(a-\delta,a+\delta)$, then we have
$|x_\delta-a|<\delta$ and $|f(x_\delta)-f(a)|=|0-a|=|a|\geq\varepsilon_0$.
therfore f is not continuous at a.
Case 2: If a is irrational, then f(a)=0. Take $\varepsilon_0=|a|/2>0$.
Let $\delta>0$ be arbitrary. choose x_δ to be a rational number in the intervbal
$(a-\delta,a+\delta)\cap(a-\varepsilon_0,a+\varepsilon_0)$, then we have $|x_\delta-a|<\delta$. Also $|x_\delta-a|<\varepsilon_0$,
we obtain
$|f(x_\delta)-f(a)|=|x_\delta|\geq|a|-|x_\delta-a|\geq|a|-\varepsilon_0=|a|/2=\varepsilon_0$.
therfore f is not continuous at a.

CH 2 Differential Calculus

2.1 Differentiability in one variable

Def: 1) $f'(a)=\lim_{h\to0}\dfrac{f(a+h)-f(a)}{h}$ is the derivative of $f(x)$ at a .

2) $f'_+(a)=\lim_{h\to0^+}\dfrac{f(a+h)-f(a)}{h}$ is the derivative of $f(x)$ at x = a from right .

3) $f'_-(a)=\lim_{h\to0^-}\dfrac{f(a+h)-f(a)}{h}$ is the derivative of $f(x)$ at x = a from left .

Def: $f'(a^+)=\lim_{x\to a^+}f'(x)$.

Example: Let $f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & if \quad x\neq0 \\ 0 & if \quad x=0 \end{cases}$

Find $f_+^{'}(0) \quad and \quad f^{'}(0^+)$

Solution : $f_+^{'}(0) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2 \sin\frac{1}{h} - 0}{h} == \lim_{h\to 0^+} h \sin\frac{1}{h} = 0 \ (by \ sandwich \ theorem)$

$f^{'}(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

$f^{'}(0^+) = \lim_{x\to 0^+} f^{'}(x) = \lim_{x\to 0^+} 2x\sin\frac{1}{x} - \cos\frac{1}{x} = 0 - \lim_{x\to 0^+} \cos\frac{1}{x}$ does not exist .

Def: $f^{''}(x) = (f^{'}(x))^{'}$

$f^{n}(x) = (f^{n-1}(x))^{'} \quad for \ n \geq 0$

Def: we say that $f(x) \in C^n$ provided that $f^n(x) \in C^n \quad n = 1,2,3,......$

The mean value theorem :

Proposition: Suppose f is defined on an open interval I and $a \in I$. If f has a local maximum or minimum at the point $a \in I$ and f is differentiable at a ,then $f^{'}(a) = 0$

Proof: Let f has a local min. value at x = a. In the difference quotient $\frac{f(a+h) - f(a)}{h}$,

$f(a+h) - f(a) \geq 0$ for all h near zero

Since $f(a+h) \geq f(a) \Rightarrow$

For $h > 0 \Rightarrow f_+^{'}(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \geq 0$

For $h < 0 \Rightarrow f_-^{'}(a) = \lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h} \leq 0$

$\Rightarrow f_+^{'}(a) \geq 0$, $f_-^{'}(a) \leq 0$, Since $f^{'}(a) = f_+^{'}(a) = f_-^{'}(a) \Rightarrow f^{'}(a) = 0$

The same result obtained if f has local max. at x = a .

Lemma : (Roll's theorem) Suppose f is cont. on [a,b] and differentiable on (a,b). If $f(a) = f(b)$, then there is at least one point $c \in (a,b)$ such that $f^{'}(c) = 0$

Proof: Since f is cont. at [a,b] , then f assumes a maximum value and a minimum value on [a,b]

Case 1) If the max. and min. values occurs at an end point , then f is constant on [a,b] , because $f(a) = f(b)$ so $f^{'}(x) = 0 \quad \forall x \in (a,b)$

Case 2) Otherwise at least one of them occurs at some interior point $c \in (a,b)$ and $f^{'}(c) = 0$, by previous proposition .

Theorem : (Mean value theorem I) Suppose f is cont. on [a,b] and differentiable on (a,b) . There is at least one point $c \in (a,b)$ such that

$$f^{'}(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Let $L(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$

Let $g(x) = f(x) - L(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$

1) g(a) = 0 , g(b) = 0
2) g(x) is cont. on [a,b] and diff. on (a,b) so , g(x) satisfies the conditions of Roll's theorem so , $\exists c \in (a,b) \ni g^{'}(c) = 0$

$$g^{'}(c) = 0 = f^{'}(c) - 0 - \frac{f(b) - f(a)}{b - a}(1 - 0)$$

$$f^{'}(c) = \frac{f(b) - f(a)}{b - a}$$

Def: We say that a function f is increasing (res. Strictly increasing) on an interval I. If $f(a) \le f(b)$ $(resp.\ f(a) < f(b))$ whenever $a,b \in I$ and a < b. Similarly for decreasing and strictly decreasing .

Theorem : Suppose f is differentiable on the open interval I

a) If $|f^{'}(x)| \le C$ for all $x \in I,$ then $|f(b) - f(a)| \le C|b - a|$ *for all* $a,b \in I$
b) If $f^{'}(x) = 0$ for all $x \in I,$ then f is constant on I .
c) If $f^{'}(x) \ge 0$,(resp. $f^{'}(x) > 0, f^{'}(x) \le 0,$ or $f^{'}(x) < 0$, for all $x \in I,$ then f is increasing (resp. strictly increasing , decreasing ,or strictly decresing) on I .

Proof:

a) Let $a,b \in I \Rightarrow \exists C \in (a,b) \ni f^{'}(c) = \dfrac{f(b)-f(a)}{b-a}$

$f(b)-f(a) = f^{'}(c)(b-a)$

$\Rightarrow |f(b)-f(a)| = |f^{'}(c)|\ |(b-a)| < C|b-a|$

Since $|f'(x)| < C$ for all $x \in I$.

If $f^{'}(c) = 0$

b) If $f^{'}(c) = 0 \Rightarrow |f(b)-f(a)| = 0 \Rightarrow f(b) = f(a)\ \ \forall a,b \in Z$, then f is cont.

c) If $f^{'}(c) \geq 0 \Rightarrow f(b)-f(a) \geq 0\ \ for\ \ b > a$

$\Rightarrow f$ is increasing and similarly for the other cases .

TH: (Mean value theorem II): Suppose that f and g are continuous on [a,b] and diff. on (a,b) , and $g^{'}(x) \neq 0$ for all $x \in (a,b)$ .Then there exist $c \in (a,b)$ such that

$$\frac{f^{'}(c)}{g^{'}(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: Let $h(x) = [f(b)-f(a)][g(x)-g(a)] - [g(b)-g(a)][f(x)-f(a)]$

Then h is cont. on [a,b] and diff. on (a,b) ,and h(a) = h(b) = 0 , So h satisfies Roll's theorem. There is a point $c \in (a,b)$ such that

$0 = h^{'}(c) = [f(b)-f(a)]g^{'}(c) - [g(b)-g(a)]f^{'}(c)$

Since $g^{'} \neq 0$ on (a,b) ,we have $g^{'}(c) \neq 0$ and $g(b)-g(a) \neq 0$ (by mean value theorem)

Since $g(b)-g(a) = g^{'}(\bar{c})(b-a)\ \ for\ some\ \bar{c} \in (a,b)$. Hence we can divide by both these quantities to obtain the desired result .

$$\frac{f^{'}(c)}{g^{'}(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Theorem : (L'Hopital Rule I) .Suppose f and g are diff. functions on (a,b) and

$$\lim_{x \to a^{+}} f(x) = \lim_{x \to a^{+}} g(x) = 0$$

If $g^{'}(x) \neq 0$ on (a,b) and the limit $\lim\limits_{x \to a^{+}} \dfrac{f^{'}(x)}{g^{'}(x)} = L\ \ exists$, then g never vanishes on (a,b)

and $\lim\limits_{x \to a^{+}} \dfrac{f(x)}{g(x)} = L\ \ exists$

The same result holds for the left hand limit $\lim\limits_{x \to a^{-}}$, if f and g are diff on (d,a)

The two sided limit $\lim_{x \to a}$, if f and g are diff on (d,a) and (a,b)
The limit $\lim_{x \to \infty}$ or $\lim_{x \to -\infty}$, if f and g are diff on an interval (b,∞) *or* $(-\infty,b)$

Proof: If $f(a)$ *and* $g(a)=0$, then f and g are cont. on the interval [a,x] for x<b, by preivious th. $\forall x \in (a,b)$,$\exists c \in (a,x)$ (depending on x) ∍

$$\frac{f(x)-0}{g(x)-0}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^{'}(c)}{g^{'}(c)}$$

Since $c \in (a,x)$, c approaches a^{+}, as x does , so $\lim_{x \to a^{+}}\frac{f(x)}{g(x)}=\lim_{c \to a^{+}}\frac{f^{'}(c)}{g^{'}(c)}=L$

The proof for left –hand limit is similar , and the case of two –sided –limits is obtained, by combining right – hand and left –hand limits .

Finally, for the case $a=\pm\infty$, we set $y=\frac{1}{x}$ and consider the function

$F(y)=f(\frac{1}{y})$ *and* $G(y)=g(\frac{1}{y})$. Since $F^{'}(y)=-\frac{f^{'}(\frac{1}{y})}{y^{2}}$ *and* $G^{'}(y)=-\frac{g^{'}(\frac{1}{y})}{y^{2}}$

We have $\frac{F'(y)}{G'(y)}=\frac{f^{'}(\frac{1}{y})}{g^{'}(\frac{1}{y})}$

So $\lim_{x \to \pm\infty}\frac{f(x)}{g(x)}=\lim_{y \to \pm 0}\frac{F(y)}{G(y)}=\lim_{y \to \pm 0}\frac{F^{'}(y)}{G^{'}(y)}=\lim_{y \to \pm 0}\frac{f^{'}(\frac{1}{y})}{g^{'}(\frac{1}{y})}=\lim_{x \to \pm\infty}\frac{f'(x)}{g'(x)}=L.$

Remark : It may well happen that $f^{'}(x)$ and $g^{'}(x)$ tend to zero also , so that the limit of $\frac{f^{'}(x)}{g^{'}(x)}$ can not be evaluated immediately
In this case we apply the previous theorem again to evaluate the limit by examining $\frac{f^{''}(x)}{g^{''}(x)}$.
More generally , if the functions $f, f^{'},......., f^{k-1}, g, g^{'},..............., g^{k-1}$ all tend to zero as x tend to a^{+} or a^{-} or $\pm\infty$, but $\frac{f^{k}(x)}{g^{k}(x)} \to L$ *,then* $\lim_{x \to}\frac{f(x)}{g(x)}=L$

Example: Let $f(x) = 2x - \sin 2x$, $g(x) = x^2 \sin x$, a=0, then f, g and their first two derivatives vanishes at x = a , but the third derivative do not , so

$$\lim_{x\to 0} \frac{2x - \sin 2x}{x^2 \sin x} = \lim_{x\to 0} \frac{2 - 2\cos 2x}{2x\sin x + x^2 \cos x} = \lim_{x\to 0} \frac{4\sin 2x}{(2 - x^2)\sin x + 4x\cos x}$$

$$\lim_{x\to 0} \frac{8\cos 2x}{(6 - x^2)\cos x - 6x\sin x} = \frac{4}{3}$$

TH: (L'Hopital's Rule II) Previous theorem remains valid when the hypothesis that, lim f(x)=lim g(x) = 0 (as $x \to a^+$, $x \to a^-, etc.$) is replaced by the hypothesis $\lim |f(x)| = \lim |g(x)| = \infty.$

Proof : we consider the case of left –hand limits as $x \to a^-$

Given $\varepsilon > 0$,we must show that $|\frac{f(x)}{g(x)} - L| < \varepsilon$ provided that x sufficiently close to a on the left. Since $\frac{f'(x)}{g'(x)} \to L \ and \ |g(x)| \to \infty$, we can choose $x_0 < a$

So that $|\frac{f'(x)}{g'(x)} - L| < \frac{\varepsilon}{2} \ \ and \ \ g(x) \neq 0 \quad for \ \ x_0 < x < a$

If $x_0 < x < a$ we have, by previous theorem,

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)} \quad for \ some \ c \in (x_0, x) \text{ and}$$

hence , since $x_0 < c < a$,

$$|\frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L| < \frac{\varepsilon}{2} \quad for \ \ x_0 < x < a. \ \ ....(1)$$

Next , division of top and bottom by g(x) yields

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x)}}{1 - \frac{g(x_0)}{g(x)}}$$

Since $|g(x)| \to \infty \ as \ x \to a$, then the quotients $\frac{f(x_0)}{g(x)}$ and $\frac{g(x_0)}{g(x)}$ can be made as close to zero a we please by taking x sufficiently close to a . It follows that for x sufficiently close to a , we have

$$|\frac{f(x) - f(x_0)}{g(x) - g(x_0)} - \frac{f(x)}{g(x)}| < \frac{\varepsilon}{2} ....(2)$$

$$\left|\frac{f(x)}{g(x)}-L\right|=\left|\frac{f(x)}{g(x)}-\frac{f(x)-f(x_0)}{g(x)-g(x_0)}+\frac{f(x)-f(x_0)}{g(x)-g(x_0)}-L\right|$$

$$\leq\left|\frac{f(x)-f(x_0)}{g(x)-g(x_0)}-\frac{f(x)}{g(x)}\right|+\left|\frac{f(x)-f(x_0)}{g(x)-g(x_0)}-L\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

And hence by the proceeding estimates (1,2), $\Rightarrow\left|\frac{f(x)}{g(x)}-L\right|<\varepsilon$ which is what we needed to show

Corollary : For Any a > 0 ,we have

$$\lim_{x\to\infty}\frac{x^a}{e^x}=\lim_{x\to\infty}\frac{\log x}{x^a}=\lim_{x\to 0^+}\frac{\log x}{x^{-a}}=0.$$

That is , the exponential function e^x grows more rapidly than any power of x as $x\to\infty$, where as $|\log x|$ grows more slowly than any positive power of x as $x\to\infty$ and more slowly than any negative power of x as $x\to 0^+$

Proof: Let k be the smallest integer that is $\geq a$. If we apply the previous theorem for k –times, we have,

$$\lim_{x\to\infty}\frac{x^a}{e^x}=\lim_{x\to\infty}\frac{a(a-1).....(a-k+1)x^{a-k}}{e^x}$$ since $a-k\leq 0\Rightarrow$ the later limit is zero .

$$\lim_{x\to\infty}\frac{\log x}{x^a}=\lim_{x\to\infty}\frac{1}{ax^a}=0 \quad , \quad \lim_{x\to 0^+}\frac{\log x}{x^{-a}}=-\lim_{x\to 0^+}\frac{x^a}{a}=0.$$

Remark : By raising the quantities previous corollary to a positive power b and replacing a by $\frac{a}{b}$ we obtain the more general formula,

$$\lim_{x\to\infty}\frac{x^a}{e^{bx}} \quad , \quad \lim_{x\to\infty}\frac{(\log x)^b}{x^a}=\lim_{x\to 0^+}\frac{|\log x|^b}{x^{-a}}=0 \ (a,b>0)$$

Vector – valued functions :

If $f=(f_1,f_2,..........,f_n)\in R^n$ is a vector valued function then its derivative at the point a is defined to be $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

The jth component of the diff. quotient on the right is

$$f_j'(a)=\lim_{h\to 0}\frac{f_j(a+h)-f_j(a)}{h}$$

* f is diff. iff each of its component functions f_j is diff. and that diff. is simply performed componentwise : $f'(a) = (f_1'(a), f_2'(a), \ldots\ldots\ldots, f_n'(a))$.

If ϕ is a scalar function and f is a vector valued function f, then
$(\phi f)' = \phi' f + \phi f'$
If f *and* g are two vector valued functions , then
$(f \cdot g)' = f' \cdot g + f \cdot g'$.
$(f \times g)' = f' \times g + f \times g'$.

Remark 1): The mean value theorem is not valid for a vector valued functions .

Example :

1) $f(t) = (\cos t, \sin t)$ satisfies $f(0) = f(2\pi)$ but $f'(t) = (-\sin t, \cos t)$, so there is no point t where $f'(t) = 0$

2) If $|f'(t)| \le M \quad for\ all\ \ t \in [a,b]$, then $|f(b) - f(a)| \le M|b-a|$

Remark 2) : If $f'(a) = 0$, then the curve may not have tangent line at $f(a)$

Example:
$f(t) = (t^3, |t^3|)$, $f'(0) = (0,0)$ *but the curve is* $y = |x|$, does not have a tangent line at x=0.

Ex. Sec.2.1 1,2,3 ,4,5,6,7,9

2.2 Differentiability in several variables :
Def: The partial derivative of a function $f(x_1, x_2, \ldots\ldots\ldots, x_n)$ with respect to the variable x_j is

$$\frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(x_1, \ldots\ldots\ldots, x_j + h, \ldots\ldots\ldots, x_n) - f(x_1, x_2, \ldots\ldots\ldots, x_n)}{h}$$

Provided that the limit exists .

It may be denoted by f_{x_j} or f_j or $\partial_{x_j} f$ or $\partial_j f$

Example : Let $f(x,y,z) = \dfrac{e^{3x} \sin xy}{1+5y-7z}$

$$\frac{\partial f}{\partial x} = f_x = \partial_1 f = \frac{3e^{3x} \sin xy + e^{3x} y \cos yx}{1+5y-7z}$$

$$\frac{\partial f}{\partial y} = f_y = \partial_2 f = \frac{(1+5y-7z)e^{3x} x \cos xy - 5e^{3x} \sin yx}{(1+5y-7z)^2}$$

$$\frac{\partial f}{\partial z} = f_z = \partial_3 f = \frac{7e^{3x} \sin yx}{(1+5y-7z)^2} .$$

Example : Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$

f is discont. at (0,0) , it approaches different limits as (x,y) approaches the origin a long different straight lines.

$$f_x = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$\Rightarrow f_x = f_y = 0$ exists but $f(x,y)$ is discont. at (0,0).

Def : A function $f(X)$, $(X \in R^n)$ is differentiable at a point $X = a = (a_1, a_2, \ldots\ldots, a_n) \in R^n$ if there is a linear function $L(X)$ such that $L(a) = f(a)$ and the difference $f(X) - L(X)$ approaches to zero faster than $(x-a)$ as x approaches a .

Or if $L(X) = b + c_1 x_1 + c_2 x_2 + \ldots\ldots c_n x_n = b + C.X$
$= b + (c_1, c_2, \ldots\ldots, c_n).(x_1, x_2, \ldots\ldots, x_n)$,

is a general linear function of n variables such that,
$L(a) = f(a) \Rightarrow b = f(a) - C.a$
Then $L(X) = f(a) - C.a + C.x = f(a) + C.(x-a),$
And $f(X) - L(X) = f(X) - f(a) - C.(x-a)$ tends to zero faster that $(X-a)$ as $X \to a.$

Def: A function f defined on an open set $S \subset R^n$ is called differentiable at a point $a \in S \subset R^n$, if there is a vector $C \in R^n$ such that

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-C.h}{|h|}=0 ,$$

where $C=\nabla f\,(a)=\nabla f_{|x=a}=(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},................,\frac{\partial f}{\partial x_j})_{|(x_1,x_2,....,x_n)=(a_1,a_2,.....,a_n)}$.

If $E(h)=f(a+h)-f(a)-\nabla f(a).h$,then $f(a+h)=f(a)+\nabla f(a).h+E(h)$

where $\lim_{h\to 0}\frac{E(h)}{|h|}=0$.

$f(a+h)$ is the linearization of f at x = a which equal $f(a+h)=f(a)+\nabla f(a).h$ near h =0 ,(h = x-a).

If n = 2 , then $Z=f(X)\ \ with\ \ X=(x,y)$ represents a surface in 3-space , and the graph of the eq. $z=f(a)+\nabla f(a).(x-a)$ (x is variable, a is fixed), represents a plane . These two objects both pass through the point $(a,f(a))$
and nearby the points x= a+h , we have

$$z_{surface}-z_{plane}=f(X)-f(a)-\nabla f(a).h=E(h)\ \ and\ \ \frac{E(h)}{h}\to 0\ as\ h\to 0 ,$$

and the surface $z=f(a)+\nabla f(a)(x-a)$ is the tangent plane to the surface $z=f\,(X\,)$ at x =a.

Theorem : If f is diff. at a, then the partial derivatives $\partial_j f(a)$ all exists , and they are components of the vector $C=\nabla f(a)$

Proof: Suppose f is diff. at $a=(a_1,a_2,..............,a_n)$, if $h=(h,0,0,...........,0)$, $h\in R$, we have,

$$C.h=\nabla f\,(a).h=c_1h=\frac{\partial f(a)}{\partial x}h=\partial_1 f\,(a)h \qquad and\ \ |h|=\pm h.$$

Thus $\lim_{h\to 0}\frac{f(a_1+h,a_2,..............,a_n)-f(a_1,..........,a_n)-c_1h}{|h|}=0$

$$\Rightarrow \lim_{h\to 0}\frac{f\,(a_1+h,a_2,..............,a_n)-f\,(a_1,..........,a_n)}{h}-c_1=0$$

$c_1=\partial_1 f(a)=\frac{\partial f}{\partial x}|_{x=a}=\lim_{h\to 0}\frac{f(a_1+h,a_2,..............,a_n)-f(a_1,..........,a_n)}{h}$ exists .

Similarly , $c_j=\partial_j f\,(a)=\frac{\partial f\,(a)}{\partial x_j}\ \ for\ \ j=2,3,..........,n$

TH: If f is diff. at a $\Rightarrow f$ is cont. at a .

Proof: f is diff. at a $\Rightarrow \exists C \in R^n \ni \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)-C.h}{|h|}=0,$

Multiply by |h| , we have $\lim\limits_{h\to 0} f(a+h)-f(a)-C.h=0$

But $\lim\limits_{h\to 0} C.h=0$

$\therefore \lim\limits_{h\to 0} f(a+h)-f(a)=0 \Rightarrow \lim\limits_{h\to 0} f(a+h)=f(a)$

f is cont. at x = a

The converse need not true .

Example : $f(x)=|x|$ is cont. at x = 0 but not diff at $x \neq 0$

Example : Let $f(x,y)=x^2+y^2$ show that f is diff. at every point (a,b) in plane

Solution : $\dfrac{\partial f}{\partial x}|_{(a,b)}=2a \quad , \dfrac{\partial f}{\partial y}|_{(a,b)}=2b$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-\nabla f(a).h}{|h|}=\lim_{h\to 0}\frac{f((a,b)+(h_1,h_2))-(a^2+b^2)-(\frac{\partial f}{\partial x}|_{(a,b)}h_1+\frac{\partial f}{\partial y}|_{(a,b)}h_2)}{|h|}$$

$$\lim_{h\to 0}\frac{(a+h_1)^2+(b+h_2)^2-(a^2+b^2)-2ah_1-2bh_2}{|h|}$$

$$\lim_{h\to 0}\frac{2ah_1+h_1^2+2bh_2+h_2^2-2ah_1-2bh_2}{|h|}=\lim_{h\to 0}\frac{h_1^2+h_2^2}{\sqrt{h_1^2+h_2^2}}=\lim_{h\to 0}\sqrt{h_1^2+h_2^2}=0$$

So , $f(x,y)=x^2+y^2$ is diff. at every point (a,b).

Remark :

1) diff $\Rightarrow$ cont.
2) The existence of partial derivative of f does not imply the differentiability of f .

Example : $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$,

has partial derivatives at (0,0) .

$f_1(0,0) = f_2(0,0) = 0$,

but it is not cont. at origin so it can not be diff. at the origin .

TH : Let f be a function defined on an open set in R^n that contains the point $a \in R^n$. Suppose that the partial derivatives $\partial_j f$ all exists on some neighborhood of a and that they are cont. at a .Then f is diff. at a .

Proof: Let n = 2

We will show that $\Rightarrow \exists C = \nabla f(a) \in R^n \ni \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)-\nabla f(a).h}{|h|}=0,$

$f(a+h)-f(a)=f(a_1+h_1,a_2+h_2)-f(a_1,a_2)$

$=[f(a_1+h_1,a_2+h_2)-f(a_1,a_2+h_2)]+[f(a_1,a_2+h_2)-f(a_1,a_2)]$ .....(1)

Since the partial derivatives $\partial_j f$ exist whenever $|x-a|\leq|h|$, so by the mean value theorem of one variable and if we set $g(t)=f(t,a_2+h_2)$ we have

$f(a_1+h_1,a_2+h_2)-f(a_1,a_2+h_2)=g(a_1+h_1)-g(a_1)$

$=g'(a_1+c_1)h_1=\partial_1 f(a_1+c_1,a_2+h_2)h_1 \quad for\ some\ c_1\in(0,h_1).$

and $f(a_1,a_2+h_2)-f(a_1,a_2)=\partial_2 f(a_1,a_2+c_2)h_2 \quad for\ some\ c_2\in(0,h_2)$

Substituting these results into eq.(1) we get

$=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)-\nabla f(a).h}{|h|}$

$=\lim\limits_{h\to 0}\dfrac{\partial_1 f(a_1+c_1,a_2+h_2)h_1+\partial_2 f(a_1,a_2+c_2)h_2-\partial_1 f(a_1,a_2)h_1-\partial_2 f(a_1,a_2)h_2}{|h|}$

$=\lim\limits_{h\to 0}[\partial_1 f(a_1+c_1,a_2+h_2)-\partial_1 f(a_1,a_2)]\dfrac{h_1}{|h|}+\lim\limits_{h\to 0}[\partial_2 f(a_1,a_2+c_2)-\partial_2 f(a_1,a_2)]\dfrac{h_2}{|h|}=0$ because $\partial_j f$ are cont. at a and $\dfrac{h_1}{|h|}$, $\dfrac{h_2}{|h|}$ are bounded by 1

$\Rightarrow f$ is diff at a.

Def: If f has partial derivatives $\partial_j f$ all exists and are cont. on an open set S there is said to be of class C^1 on S .

i.e. $f\in C^1\ on\ S\ or\ f\in C^1(S).$

If $f\in C^1 \Rightarrow f\ \ is\ diff. \Rightarrow partial\ \ derivatives\ \ exist.$

The converse need not true .

Example : $f(x)=\begin{cases} x^2 \sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$

Is diff. at x = a ,but f $f \notin C^1$ because $\lim_{x\to 0} f'(x)$ does not exist .

Differential

Suppose f is diff. at a, so $f(a+h)-f(a)=\nabla f(a).h+error$. Where the $error \to 0$ as $h\to 0$.

$\nabla f(a).h = f(a+h)-f(a)$ is called the differentail of f at a and is denoted by $df(a,h)$ or $df_a(h)$.

And $df_a(h)=\nabla f(a).h=\partial_1 f(a)h_1+\partial_2 f(a)h_2+\dots\dots\dots\dots+\partial_n f(a)h_n$.

If $u=f(X)$, $h=dX=(dx_1,dx_2,\dots\dots\dots\dots\dots,dx_n)$.

Then $du=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+\dots\dots\dots\dots+\frac{\partial f}{\partial x_n}dx_n$,

$d(f+g)=df+dg$

$d(f.g)=fdg+gdf$

and $d(f/g)=\frac{gdf-fdg}{g^2}$

Example : A right Circular cone has height 5 and base radius 3.

a) About how much does the volume increase if the height is increased to (5.02) and the radius is increased to (3.01)?
b) If the height is increased to (5.02) , by about how much should the radius be decreased to keep the volume constant ?

Solution : a) $V=\frac{1}{3}\pi r^2 h \Rightarrow dV=\frac{2}{3}\pi r h\, dr+\frac{1}{3}\pi r^2\, dh$

If $r=3$, $h=5 \Rightarrow dr=0.01$, $dh=0.02$

$\Rightarrow dV=\frac{2}{3}\pi(3)(5)(0.01)+\frac{1}{3}\pi(3)^2(0.02)=0.16\pi\approx 0.5$

b) If $r=3$, $h=5\Rightarrow dr=?$, $dh=0.02$

$\Rightarrow dV=\frac{2}{3}\pi r h\, dr+\frac{1}{3}\pi r^2\, dh$

If $dV=0=\frac{2}{3}\pi(3)(5)\ dr+\frac{1}{3}\pi(3)^2(0.02)\Rightarrow dr=-0.006$

Directional derivatives:

If a is a point in R^n ,and u is a unit vector in the direction of a line passing through the point a , then the parametric eqs. Of the line are given by $g(t)=a+t(u)$.

Then the directional derivative of f at a in the direction u is defined to be

$\partial_u f\ (a) = \frac{d}{dt} f\ (a+tu)_{|t=0} = \lim_{t\to 0} \frac{f\ (a+tu) - f\ (a)}{t}$ provided that the limit exists .

If $u = (0,0,0,1,......,0)$ is a unit vector in the positive jth coordinate , then $\partial_u f(a) = \partial_j f(a)$.

TH: If f is differentiable at a , then the directional derivative of f at a all exists , and they are given by $\partial_u f(a) = \nabla f(a).u$.

Proof: Since f is diff. at a,

$\Rightarrow \lim_{h\to 0} \frac{f(a+h) - f(a) - \nabla f(a).h}{|h|} = 0$.

Let $h = tu\ ,\ if\ \ t > 0 \Rightarrow |h| = t\ \ and\ \Rightarrow \lim_{h\to 0} \frac{f(a+tu) - f(a)}{t} - \nabla f(a).u = 0$

$\Rightarrow \lim_{h\to 0} \frac{f\ (a+tu) - f\ (a)}{t} = \nabla f\ (a).u$

If $h = tu\ ,\ if\ \ t < 0 \Rightarrow |h| = -t\ \ and\ \Rightarrow -\lim_{h\to 0} \frac{f(a+tu) - f(a)}{t} + \nabla f(a).u = 0$,

then $\lim_{h\to 0} \frac{f(a+tu) - f(a)}{t} = \nabla f(a).u = \partial_u f(a)$

So , $\partial_u f(a)$ exists and equal $\nabla f(a).u$.

If $\nabla f(a) \neq 0$, then $|\partial_u f(a)| = |\nabla f(a).u|\ \ and\ \ \nabla f(a).u = |\nabla f(a)||u|\cos\theta$ where θ is the angle between the vectors $\nabla f(a)$ and u

$\Rightarrow |\partial_u f(a)| \leq |\nabla f(a)|\ |\cos\theta| \leq |\nabla f(a)|$ for every unit vector u .

1) $\partial_u f(a) = |\nabla f(a)|$ when u in the direction of $\nabla f(a)$ and $\partial_u f(a)$ has the largest directional derivative of f at a .

2) $\partial_u f(a) = -|\nabla f(a)|$ when u in the opposite direction of $\nabla f(a)$ and $\partial_u f(a)$ has the smallest directional derivative of f at a.

3) $\partial_u f(a) = 0\ \ where\ \ u \perp \nabla f$

Example: Let $f(x,y) = x^2 + 5xy^2\ \ , a = (-2,1)$

a) Find the directional derivative of f at a direction of $v = (12,5)$
b) What is the largest of the directional derivative of f at a and in what direction does it occur ?

Solution :a) $\nabla f = (2x + 5y^2, 10xy) \quad so, \nabla f(-2,1) = (1,-20)$

The unit vector in the direction of v is $u = (\frac{12}{13}, \frac{5}{13})$, so the directional derivative in this direction is $\nabla f(a).u = (1,-20).(\frac{12}{13}, \frac{5}{13}) = \frac{-88}{13}$.

b) $|\nabla f(a)| = \sqrt{401}$ is the largest directional derivatives at a and occurs in the direction of $u = \frac{1}{\sqrt{401}}(1,-20)$.

Ex. 1,2,3(a),5,7

2.3 The chain Rule

Let $f(x_1, x_2, \ldots\ldots, x_n)$ be a function of variables $x_1, x_2, \ldots\ldots, x_n$ and $x_j = g_j(t)$ for j = 1,2,3,……….,n and let $X = g(t) = (g_1(t), g_2(t), \ldots\ldots, g_n(t)) = (x_1, x_2, \ldots\ldots, x_n)$,then

$$\phi(t) = f(g(t)) = f(g_1(t), g_2(t), \ldots\ldots, g_n(t))$$

$$g'(t) = (g_1'(t), g_2'(t), \ldots\ldots, g_n'(t))$$

$$\nabla f = (\frac{\partial f}{\partial g_1}, \frac{\partial f}{\partial g_2}, \ldots\ldots, \frac{\partial f}{\partial g_n})$$

$$\phi'(t) = \nabla f(t).g'(t) = \frac{\partial f}{\partial g_1}.\frac{dg_1}{dt} + \frac{\partial f}{\partial g_2}.\frac{dg_2}{dt} + \ldots\ldots + \frac{\partial f}{\partial g_n}.\frac{dg_n}{dt}.$$

TH: Chain Rule I . Suppose that g(t) is diff . at t = a,

$f(X) = f(x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$ is diff. at $X = b = g(a) = (g_1(a), g_2(a), \ldots\ldots\ldots\ldots\ldots\ldots, g_n(a))$ then the composite function $\phi(t) = f(g(t))$ is diff. at t = a , and its derivative is given by $\phi'(a) = \nabla f(b).g'(a)$ or on Leibniz notation , with $w = f(X)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}.\frac{dx_2}{dt} + \ldots\ldots\ldots\ldots + \frac{\partial w}{\partial x_n}.\frac{dx_n}{dt}.$$

Proof: Since f *and* g are diff. at the points b and a resp. then

$$f(b+h) = f(b) + \nabla f(b).h + E_1(h) \quad \ldots\ldots\ldots.(1) \quad where \ \frac{E_1(h)}{|h|} \to 0 \ as \ h \to 0$$

$$g(a+u) = g(a) + u\,g'(a) + E_2(u) \quad \ldots\ldots\ldots\ldots(2) \quad where \ \frac{E_2(u)}{|u|} \to 0 \ as \ u \to 0$$

Let $h = g(a+u) - g(a)$ int *the first eq.*(1)

Then from (2) $h = u\,g'(a) + E_2(u)$, since $b = g(a)$ so,

$$\phi(a+u) = f(g(a+u)) = f(b+h) = f(b) + \nabla f(b).h + E_1(h)$$
$$= f(g(a)) + \nabla f(b).[u\,g'(a) + E_2(u)] + E_1(h)$$
$$= \phi(a) + u\,\nabla f(b).g'(a) + E_3(u) \quad \ldots\ldots\ldots.(3)$$

Where $E_3 = \nabla f(b).E_2(u) + E_1(h)$

We claim that $E_3(u)$ satisfies $\frac{E_3(u)}{u} \to 0$ as $u \to 0$, Now by using the triangle inequality, we have

$$\left|\frac{E_3(u)}{u}\right| = \left|\frac{\nabla f(b).E_2(u) + E_1(h)}{u}\right| \le |\nabla f(b)|\,\frac{|E_2(u)|}{|u|} + \frac{|E_1(h)|}{|u|} \to 0 \ as \ u \to 0$$, since

$\frac{E_2(u)}{u} \to 0$ as $u \to 0$, so $E_2(u) \le u$, then

$$|h| = |u\,g'(a) + E_2(u)| \le |(g'(a)+1)|\,|u| \Rightarrow \frac{1}{|u|} \le \frac{|g'(a)+1|}{|h|} \Rightarrow$$

$$\frac{E_1(h)}{|u|} = \frac{E_1(h)}{|h|}(|(g'(a)+1)|) \to 0 \ as \ h \to 0$$

From (3) $\frac{\phi(a+u) - \phi(a)}{u} = \nabla f(b).g'(a) + \frac{E_3(u)}{u} \to \nabla f(b).g'(a)$ as $u \to 0$

So , $\phi'(a) = \nabla f(b).g'(a)$

Example : $w = f(x, y, z)$ is diff. function of (x, y, z) and $x = t^4 - t$, $y = \sin 3t$ *and* $z = e^{-2t}$

$$\frac{dw}{dt} = \frac{d}{dt} f(t^4 - t, \sin 3t, e^{-2t}) = (\partial_1 f).(4t^3 - 1) + (\partial_2 f).(3\cos 3t) + (\partial_3 f).(-2e^{-2t})$$

* If $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ are function of a family of variables $t = (t_1, t_2, \ldots\ldots\ldots\ldots, t_m)$ say $x_j = g_j(t_1, t_2, \ldots\ldots\ldots\ldots, t_m)$ or $X = g(t)$

If f is diff. function of x ,we have $\phi(t) = f(g(t))$, and to find partial derivative of ϕ with respect to t_k , we fix all but one of those variables and apply the Chain Rule

$$\frac{\partial \phi(a)}{\partial t_k}\Big|_{a=(t_1, t_2, \ldots\ldots\ldots, t_m)} = \nabla f(b) . \frac{\partial g(a)}{\partial t_k} \quad (b = g(a))$$

Or $w = f(X)$

$$\frac{\partial w}{\partial t_k} = \frac{\partial w}{\partial x_1} . \frac{\partial x_1}{\partial t_k} + \frac{\partial w}{\partial x_2} . \frac{\partial x_2}{\partial t_k} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\partial w}{\partial x_n} . \frac{\partial x_n}{\partial t_k}$$

Th: chain Rule II .Suppose that $g_1, g_2, \ldots\ldots\ldots\ldots, g_n$ are function of $t = (t_1, t_2, \ldots\ldots\ldots\ldots\ldots, t_m)$ and f is a function of $X = (x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n)$

Let $b = g(a)$ *and* $\phi = f \circ g$. If $g_1, g_2, \ldots\ldots\ldots\ldots, g_n$ are diff. at a (resp. of class C^1 near a) and f is differentiable at b (resp. of class C^1 near b), then φ is diff. at a (res. of class C^1 near a) , and its partial derivatives are given by

$$\frac{\partial \phi}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \ldots\ldots\ldots\ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k}$$

where the derivatives $\frac{\partial f}{\partial x_j}$ are evaluated at b and the derivatives $\frac{\partial \phi}{\partial t_k}$ and $\frac{\partial x_j}{\partial t_k} = \frac{\partial g_j}{\partial t_k}$ are evaluated at a .

Example : Suppose that f is a diff. function of x and y and that $x = s\log(1+t^2)$ and $y = \cos(s^3 + 5t)$, then the partial derivatives of the composite function $z = f(s\log(1+t^2), \cos(s^3 + 5t))$ are given by

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = f_x \log(1+t^2) + f_y(-3s^2)\sin(s^3 + 5t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = f_x \frac{2st}{1+t^2} + f_y(-5)\sin(s^3 + 5t)$$

Let $w = f(X)$, $X = (x_x, x_2, \ldots\ldots\ldots\ldots\ldots, x_n)$, then the differential of w is

$$dw = \frac{\partial w}{\partial x_1}dx_1 + \frac{\partial w}{\partial x_2}dx_2 + \ldots\ldots\ldots\ldots + \frac{\partial w}{\partial x_n}dx_n \qquad \ldots\ldots\ldots\ldots(1)$$

If each of $x_x, x_2, \ldots\ldots\ldots\ldots, x_n$ are functions of $t_x, t_2, \ldots\ldots\ldots\ldots, t_m$ and $w = f(x)$, $t = (t_x, t_2, \ldots\ldots\ldots\ldots, t_m)$, then

$$dx_j = \frac{\partial x_j}{\partial t_1} dt_1 + \frac{\partial x_j}{\partial t_2} dt_2 + ... + \frac{\partial x_j}{\partial t_n} dt_n \qquad ....(2)$$

And $dw = \frac{\partial w}{\partial t_1} dt_1 + \frac{\partial w}{\partial t_2} dt_2 + ... + \frac{\partial w}{\partial t_m} dt_m \qquad ....(3)$

If we substitute the expression (2) for dx_j into (1) , and regroup the terms , we obtain

$$dw = \frac{\partial w}{\partial x_1}[\frac{\partial x_1}{\partial t_1} dt_1 + ... + \frac{\partial x_1}{\partial t_m} dt_m] + ... + \frac{\partial w}{\partial x_n}[\frac{\partial x_n}{\partial t_1} dt_1 + ... + \frac{\partial x_n}{\partial t_m} dt_m]$$

$$dw = [\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + ... + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}]dt_1 + ... + [\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + ... + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}]dt_m$$

If $w = f(x,y,z,t),\ where\ (x,y,z)$ are functions of t ,then

$w = f(x(t), y(t), z(t), t)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt} + \frac{\partial w}{\partial t}$$

If $w = f(x,y,t,s)$ where x ,y are themselves are function of t, s

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial w}{\partial t}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}.\frac{\partial y}{\partial s} + \frac{\partial w}{\partial s}$$

If $w = f(x,y,t,s),\ x = \phi(t,s),\ y = \psi(t,s)$ then

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}.\frac{\partial \phi}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial \psi}{\partial t} + \frac{\partial f}{\partial t} = f_1\phi_1 + f_2\psi_1 + f_3.$$

Def : A function f on R^n is called (positively) homogeneus of degree a $(a \in R)$ if $f(tX) = t^a f(X)$ for all t > 0 and $x \neq 0$.

Example : Let $f(x,y) = x^2 + y^2$, show that f is homog.

Solution : $f(tx,ty) = t^2x^2 + t^2y^2 = t^2(x^2 + y^2) = t^2 f(x)$

Then f is homog. of degree 2.

TH : Euler's theorem . If f is homog. of degree a, then at any point X where f is diff. we have

$$x_1\partial_1 f(X) + x_2\partial_2 f(X) + ... + x_n\partial_n f(X) = af(X).$$

Proof : Let $\phi(t) = f(tX) = t^a f(X)$. Now differentiate with respect to, we get

$\phi'(t) = a t^{a-1} f(X) = a t^{a} t^{-1} f(X) = a t^{-1} f(tX)$.

We have from definition, $\phi'(t) = \nabla f(tX).\frac{d}{dt}(tX) = X.\nabla f(tX)$.

Let t=1, then

$\phi'(1) = X.\nabla f(X)$.

$x_1\partial_1 f(X) + x_2\partial_2 f(X) + ... + x_n\partial_n f(X) = af(X)$

Def: The differentiable function $F(x,y,z)=0$ is called smooth surface, in the set $S \subseteq R^3$, if $\frac{\partial F}{\partial x},\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ exist, and all are not zero.

Let $g(t)=(x,y,z)$ is aparametric representation of a smooth curve on S $\Rightarrow F(g(t))=0$, and

$\frac{dF(g(t))}{dt} = \nabla f(g(t)).g'(t) = 0$.

Then ∇f is orthogonal to the tangent vector of any curve on S at each point on the curve .

i.e. At any point a, the ∇f is orthogonal to the tangent vector $g'(t)$ of any curve g(t) on S .

Th: Suppose that F is a diff. function on some open set $u \subset R^3$,and suppose that the set $S = \{(x,y,z) \in u \ , F(x,y,z)=0\}$ is a smooth surface . If $a \in S$ and $\nabla F(a) \neq 0$, then the vector $\nabla f(a)$ is perpendicular , or normal to the surface S at a .

Corollary : Under the conditions of the theorem the eq. of the tangent plane to S at a is $\nabla F(a).(x-a)=0$.

Ex.(6) Find the tangent plane to the surface in R^3 described by the given eq. at the given point $a \in R^3$.

a) $z = x^2 - y^3$, $a=(2,-1,5)$

$F(x,y,z) = x^2 - y^3 - z = 0$

$\nabla F|_a = (2x, -3y^2, -1)|_a = (4,-3,-1)$.

The tangent plane is

$\nabla F(a).(x-a)=0$

$(4,-3,-1)((x-2),(y+1),(z-5))=0$

$4(x-2)-3(y+1)-1(z-5)=0$

$z=4x-3y-6.$

Ex.sec.2.3: 1,2,3,5,6.

Ex(3) c) Show that $u=f(xz,yz)$ satisfies $x\partial_x u+y\partial_y u=z\partial_z u$

Solution : $\partial_x u=\dfrac{\partial u}{\partial x}=f_1.z+f_2.0=f_1.z$

$$\partial_y u=\frac{\partial u}{\partial y}=f_2.z+f_1.0=f_2.z$$

$$\partial_z u=\frac{\partial u}{\partial z}=x\,f_1+y\,f_2$$

$$\Rightarrow x\,\partial_x u+y\,\partial_y u=x\,f_1.z+y\,f_2.z=z\,(x\,f_1+y\,f_2)=z\,\partial_z u\,.$$

2.5 Functional Relations and implicit functions . A first look

Let $F(x_1, x_2, \ldots\ldots\ldots\ldots, x_n, y) = 0$ where $y = g(x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$

If we differentiate with respect to x_j we have

$$\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x_j} = 0 \Rightarrow \frac{\partial F}{\partial x_j} = -\frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x_j}$$

$$\frac{\partial y}{\partial x_j} = -\frac{\partial F}{\partial x_j} \bigg/ \frac{\partial F}{\partial y}$$

$$\frac{\partial g}{\partial x_j} = -\frac{\partial F}{\partial x_j} \bigg/ \frac{\partial F}{\partial g}$$

Example : Let $F(x, y) = x - y - y^5 = 0$ where y is a function of x

Find $\frac{dy}{dx}$

Solution : $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{\partial F}{\partial x} \bigg/ \frac{\partial F}{\partial y} \quad , \quad F_x = 1 \quad , \quad F_y = -1 - 5y^4$$

$$\frac{dy}{dx} = \frac{-1}{-(1 + 5y^4)} = \frac{1}{(1 + 5y^4)}$$

2) Let $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ where z is a function in x and y

Find $\frac{\partial z}{\partial x}$

$$F_x + F_z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z} = -\frac{x}{z}$$

Let $w = \phi(x_1, x_2, \ldots\ldots\ldots\ldots, x_n, y)$ *where* $x_1, x_2, \ldots\ldots\ldots\ldots, x_n, y$

satisfy the relation $F(x_1, x_2, \ldots\ldots\ldots\ldots, x_n, y) = 0$

If the eq. $F(x_1, x_2, ..., x_n, y) = 0$ can be solved for y , say $y = g(x_1, x_2, ..., x_n)$,then $w = \phi(x_1, x_2, ..., x_n, g(x_1, x_2, ..., x_n))$

$$\frac{\partial w}{\partial x_j} = \partial_j \phi + \frac{\partial \phi}{\partial g} \cdot \frac{\partial g}{\partial x_j}$$

$\frac{\partial g}{\partial x_j}$ evaluated by the eq. $\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial g} \cdot \frac{\partial g}{\partial x_j} = 0$

$$\frac{\partial g}{\partial x_j} = -\frac{\partial F}{\partial x_j} \Big/ \frac{\partial F}{\partial g}$$

Now , Suppose $w = \phi(x, y, z)$ where x, y, z are constrained to satisfy $F(x, y, z) = 0$, and suppose we can solve the letter eq. for any one of the three variables in terms of the other two .

If we take x as independent variable , the meaning of $\frac{\partial w}{\partial x}$ depends on whether we take y or z as the other independent variable .

Example 2) : Let $w = x^2 + y^2 + z$ and $F(x, y, z) = x + y + z = 0$

If we take x ,y as independent variables then $z = -(x + y)$ is dependent variable

So, $w = x^2 + y^2 - x - y$ and $\frac{\partial w}{\partial x} = 2x - 1$

If we take x ,z as independent variables then $y = -(x + z)$ is dependent variable

So, $w = x^2 + (x + z)^2 + z = 2x^2 + 2xz + z^2 + z$ and $\frac{\partial w}{\partial x} = 4x + 2z$

$\frac{\partial w}{\partial x}|_y =$ derivative of w with respect to x when y is fixed , then from the pervious example we have $\frac{\partial w}{\partial x}|_y = 2x - 1$, $\frac{\partial w}{\partial x}|_z = 4x + 2z$

Let $F(x, y, u, v) = 0$

$G(x, y, u, v) = 0$

If u , v is a dep. Variables with respect to the independent variables x ,y then to find $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$, we diff. both of F, G with respect to x by helding y fixed ,we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad ....(1)$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad ....(2)$$

From (1)

$$\frac{\partial u}{\partial x}\frac{\partial F}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial F}{\partial v}=-\frac{\partial F}{\partial x}$$

$$\frac{\partial u}{\partial x}\frac{\partial G}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial G}{\partial v}=-\frac{\partial G}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix}-\frac{\partial F}{\partial x} & \frac{\partial F}{\partial v}\\ -\frac{\partial G}{\partial x} & \frac{\partial G}{\partial v}\end{vmatrix}}{\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}\\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v}\end{vmatrix}} \quad , \quad \frac{\partial v}{\partial x}=\frac{\begin{vmatrix}\frac{\partial F}{\partial u} & -\frac{\partial F}{\partial x}\\ \frac{\partial G}{\partial u} & -\frac{\partial G}{\partial x}\end{vmatrix}}{\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}\\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v}\end{vmatrix}}$$

Example3) : Suppose the quantities x, y and z are initially equal to 1,0,and 2 resp. , and are constrained to satisfy the eq. $x^5 + x(y^3+1)z - 2yz^5 = 3$ and $yz = \sin(2x+y-z)$. By about how much do y and z change if x is changed to 1.02 ?

Solution : Let $F(x,y,z) = x^5 + x(y^3+1)z - 2yz^5 - 3 = 0$

$G(x,y,z) = yz - \sin(2x+y-z) = 0.$

We will find $\frac{dy}{dx}$ and $\frac{dz}{dx}$

In the two eq. y ,z are dependent variables and x is the only indep. variable .
By differentiating the two eqs. With respect to , treating y , z are dept. variable
We have

$$5x^4 + (y^3+1)z + 3xy^2 z y' + x(y^3+1)z' - 2z^5 y' - 10yz^4 z' = 0$$

$$zy' + yz' - \cos(2x+y-z).(2+y'-z') = 0$$

At $(x,y,z) = (1,0,2)$ we have

$$5 + (1)\,2 + 1.z' - 2.(32)y' - 0 = 0$$

$$64.y' - z' = 7 \qquad ...(1)$$

$$2y' - 1.(2+y'-z') = 0$$

$$y' + z' = 2 \qquad ....(2)$$

Summation (1) with (2)

$$65y' = 9 \Rightarrow y' = \frac{9}{65}$$

$$z' = \frac{121}{65}$$

$$dy = \frac{9}{65}dx \quad , \quad dz = \frac{121}{65}dx$$

If $dx = 0.02$

$$dy = \frac{9}{65}(0.02) = \frac{9}{3250} \quad , \quad dz = \frac{121}{65}(0.02) = \frac{121}{3250}$$

Ex. 1,2,3,4,5,6

2.6 Higher –order Partial Derivatives :

Let f be a diff. function on an open set $S \subset R^n$ .The first partial derivative of f with respect to x_j is denoted by $\frac{\partial f}{\partial x_j} = \partial_j f$

The partial derivative of $\frac{\partial f}{\partial x_j}$ with respect to x_i is the second order derivative

$\frac{\partial}{\partial x_i}[\frac{\partial f}{\partial x_j}]$

Or can be written as

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\ ,\ f_{x_j x_i}\ ,\ f_{ji}\ , \partial x_i \partial x_j f\ , \partial_i \partial_j$$

And $\frac{\partial^2 f}{\partial x_j^2}\ ,\ f_{x_j x_j}\ ,\ f_{jj}\ , \partial^2 x_j f\ , \partial^2{}_j f$

Def: If the function f and all its partial derivatives of order $\le k$ exist and cont. on an open set u , then $f \in C^k$ (The class C^k and it is of class C^∞ if and all its partial derivatives of all order cont. on u) .

Def: for $i \ne j$ $\partial_i \partial_j f$ is called mixed second order, partial derivative of f

Remark: It is not true in general $\partial_i \partial_j f = \partial_j \partial_i f$

Example : if $g(x, y) = x\sin(x^3 + e^{2y})$, we have

$\partial_x g = \sin(x^3 + e^{2y}) + 3x^2 \cos(x^3 + e^{2y})$

$\partial_y g = 2xe^{2y} \cos(x^3 + e^{2y})$

Differentiating $\partial_x g$ with respect to y and $\partial_y g$ with respect to x yields

$\partial_y \partial_x g(x, y) = 2e^{2y} \cos(x^3 + e^{2y}) - 6x^3 e^{2y} \sin(x^3 + e^{2y}) = \partial_x \partial_y g(x, y)$

$\partial_y \partial_x g(x, y) = \partial_x \partial_y g(x, y)$

Example : Let $f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$ $\quad if\ (x, y) \ne (0,0) \qquad f(0,0) = 0$

$f(x,0) = f(0, y) = 0 \quad \forall x, y$

$\partial_x f(0,0) = \partial_y f(0,0) = 0$

$$\partial_x f(x,y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

$$\partial_y f(x,y) = \frac{x^5 - 4x^3 y^2 - xy}{(x^2 + y^2)^2}$$

$\partial_x f(0,y) = -y$,and $\partial_y f(x,0) = x$ for all x , y.

So, $\partial_y \partial_x f(0,0) = \lim_{h\to 0} \frac{\partial_x f(0,h) - \partial_x f(0,0)}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$

$$\partial_x \partial_y f(0,0) = \lim_{h\to 0} \frac{\partial_y f(h,0) - \partial_y f(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$\partial_y \partial_x f(0,0) \neq \partial_x \partial_y f(0,0)$ but $\partial_y \partial_x f(x,y) = \partial_x \partial_y f(x,y) \;\; \forall \;\; (x,y) \neq (0,0)$.

Th : Let f be a function defined in an open set $S \subset R^n$ and suppose $a \in S$ and $i, j \in \{1,2,........,n\}$ .If the derivative $\partial_i f \;, \partial_j f \;, \partial_i \partial_j f \;\; and \;\; \partial_j \partial_i f$ exist in S , and if $\partial_i \partial_j f \;\; and \;\; \partial_j \partial_i f$ are cont. at a , then $\partial_i \partial_j f(a) = \partial_j \partial_i f(a)$

Proof : see the book .

Corollary : If f is of class C^2 on an open set , then $\partial_i \partial_j f(a) = \partial_j \partial_i f(a)$ on S , for all $i \;\; and \;\; j$

Th: If f is of class C^k on an open set then $\partial_{i1} \partial_{i2} .... \partial_{ik} f(a) = \partial_{j1} \partial_{j2} .... \partial_{jk} f(a)$ on S , whenever the seq. $\{j_1, j_2, ..., j_k\}$ is a reordering of the seq. $\{i_1, i_2, ..., i_k\}$

If $w = f(x,y)$, x, y are function of s,t

Assume that all the functions belongs to C^2 by chain Rule

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial^2 w}{\partial s^2} = \frac{\partial}{\partial s}[\frac{\partial w}{\partial x}]\frac{\partial x}{\partial s} + \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial}{\partial s}[\frac{\partial w}{\partial y}]\frac{\partial y}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial s^2} ....(1)$$

$$\frac{\partial}{\partial s}[\frac{\partial w}{\partial x}] = \frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial y}{\partial s} ....(2)$$

$$\frac{\partial}{\partial s}[\frac{\partial w}{\partial y}] = \frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial s} ......(3)$$

Now , we substitute eq.2) and eq.3) in eq.1) to obtain $\frac{\partial^2 w}{\partial s^2}$

Example : Let $u = f(x,y),\ x = s^2 - t^2\ \ , y = 2st$

Assume f is of class C^2, find $\frac{\partial^2 u}{\partial s \partial t}$ in terms of derivative of f

Solution : $\frac{\partial u}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t} = -2tf_x + 2sf_y$

So, $\frac{\partial^2 u}{\partial s \partial t} = -2t[2sf_{xx} + 2tf_{xy}] + 2s[2sf_{xy} + 2tf_{yy}] + 2f_y$

$$= -4stf_{xx} + 4(s^2 - t^2) f_{xy} + 4stf_{yy} + 2 f_y$$

Example: Let $u = f(x,y),\quad f \in C^2$

Let $x = r\cos\theta\ , y = r\sin\theta$

Then , $\frac{\partial u}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} = (\cos\theta) f_x + (\sin\theta) f_y$

$$\frac{\partial u}{\partial \theta} = f_x \frac{\partial x}{\partial \theta} + f_y \frac{\partial y}{\partial \theta} = -(r\sin\theta) f_x + (r\cos\theta) f_y$$

Proceeding to the second derivatives :

$$\frac{\partial^2 u}{\partial r^2} = (\cos\theta)\frac{\partial f_x}{\partial r} + (\sin\theta)\frac{\partial f_y}{\partial r} = (\cos^2\theta)\ f_{xx} + (2\cos\theta\sin\theta) f_{xy} + (\sin^2\theta) f_{yy}$$

$$\frac{\partial^2 u}{\partial \theta^2} = -(r\cos\theta) f_x - (r\sin\theta)\frac{\partial f_x}{\partial \theta} - (r\sin\theta) f_y + r(\cos\theta)\frac{\partial f_y}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f_{xx} + f_{yy}$$

The expression is called Laplacian of f

Propostion: suppose u is a C^2 function of $f(x,y)$ in some open set in R^2 If (x,y) is related to (r,θ) by $x = r\cos\theta$, $y = r\sin\theta$,we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Multi index Notation :

Def : A multi index is an n-tuple of nonnegative integers multi – indices are generally denoted by the Greek letters $\alpha\ or\ \beta$

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)\quad , \beta = (\beta_1, \beta_2, ..., \beta_n)\ \ \alpha_j, \beta_j \in \{0,1,..................\}$$

If α is a multi index ,we define

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n\quad , \ \alpha! = \alpha_1!\alpha_2!...........\alpha_n!$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ....x_n^{\alpha_n}\ where\ \ X = (x_1, x_2, ..., x_n) \in R^n$$

$$\partial^{\alpha} f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots\ldots\ldots\ldots\ldots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots\ldots\ldots\ldots \partial x_n^{\alpha_n}}$$

Def : $|\alpha| = \alpha_1 + \alpha_2 + \ldots\ldots\ldots\ldots + \alpha_n$ is called the order or degree of α .
If $f \in C^k \Rightarrow$ then the k-th order partial derivative $\partial^{\alpha} f$ with $|\alpha| = k$ exists.

Example : If n =3 , and $X = (x, y, z)$, we have
$\partial^{(0,3,0)f} = \dfrac{\partial^3 f}{\partial y^3}$, $X^{(2,1,5)} = x^2 y z^5$

Th: The (multinomial theorem) For any $X = (x_1, x_2, \ldots\ldots, x_n)$ and any positive integer k,

$(x_1 + x_2 + \ldots\ldots + x_n)^k = \sum_{|\alpha|=k} \dfrac{k!}{\alpha!} X^{\alpha}$ where

$\alpha! = \alpha_1! \alpha_2! \ldots\ldots\ldots \alpha_n!$, $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots\ldots\ldots\ldots\ldots x_n^{\alpha_n}$

Proof:
For n = 2

$$(x_1 + x_2)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x_1^j x_2^{k-j} = \sum_{\alpha_1+\alpha_2=k} \frac{k}{\alpha_1! \alpha_2!} x_1^{\alpha_1} x_2^{\alpha_2} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} X^{\alpha}$$

Where $\alpha_1 = j$, $\alpha_2 = k - j$, $\alpha = (\alpha_1, \alpha_2)$
By induction suppose the result is true for n<N and $X = (x_1, x_2, \ldots\ldots\ldots, x_N)$
By using the result for n = 2 and the result fro n = N-1 , we obtain

$$(x_x + x_2 + \ldots\ldots + x_N)^k = [(x_x + x_2 + \ldots\ldots + x_{N-1}) + x_N]^k$$

$$= \sum_{i+j=k} \frac{k!}{i!\; j!} (x_1 + x_2 + \ldots\ldots\ldots\ldots + x_{N-1})^i x_N^j$$

$$= \sum_{i+j=k} \frac{k!}{i!\; j!} \sum_{|\beta|=i} \frac{i!}{\beta!} \breve{x}^{\beta}\, x_N^j$$

Where $\beta = (\beta_1, \beta_2, \ldots\ldots\ldots\ldots\ldots, \beta_{N-1})$ *and* $\breve{X} = (x_1, x_2, \ldots\ldots\ldots\ldots, x_{N-1})$
If $\alpha = (\beta_1, \beta_2, \ldots\ldots\ldots\ldots\ldots, \beta_{N-1}, j)$ *so,* $\beta!\, j! = \alpha!$ *and* $\breve{X}^{\beta} x_N^j = X^{\alpha}$

Observing that α runs over all multi-index of order k when β runs over all multi-indices of order i = k-j and j runs from 0 to k , we obtain $\sum_{|\alpha|=k} \frac{k\,!}{\alpha\,!} X^{\alpha}$

EX. 1,2,3,4,5,6,7,9,11

2.7 Taylor's Theorems:

Def : The Taylor polynomial of order k for f at a is defined by

$$P = P_{a,k}(h) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j .$$

Def : The Taylor remainder of order k is defined as

$$R_{a,k}(h) = f(a+h) - P_{a,k}(h) = f(a+h) - \sum_{j=0}^{k} \frac{f^{j}(a)h^j}{j!} \quad .....(1)$$

TH: (Taylor's theorem with integral Remainder I)

Suppose that f is of class C^{k+1} $(k \geq 0)$ on an interval $I \subset R$ *, and* $a \in I$. Then the remainder $R_{a,k}(h)$ is defined above in eq.(1) given by

$$R_{a,k}(h) = f(a+h) - \sum_{j=0}^{k} \frac{f^{(j)}(a)h^j}{j!} = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k f^{(k+1)}(a+th)dt$$

Proof: For k = 0

$$R_{a,0} = f(a+h) - f(a) = h\int_0^1 f'(a+th)dt$$

If $u = a+th \Rightarrow du = hdt \Rightarrow \frac{du}{h} = dt$

$$h\int_0^1 f'(a+th)dt = \int_a^{a+h} f'(u)du = f(a+h) - f(a)$$

The result holds

Let $I = h\int_0^1 f'(a+th)dt$, If we integrate by parts choosing

$u = f'(a+th) \qquad dv = dt$

$du = f''(a+th)hdt \quad , v = t-1 = -(1-t).$

Then $I = h\int_0^1 f'(a+th)dt = h(t-1)f'(a+th)\,|_0^1 - h\int_0^1 (t-1)f''(a+th)hdt$

$$= f'(a)h + h^2\int_0^1 (1-t)f''(a+th)dt$$

$$\Rightarrow f(a+h) - f(a) = f'(a)h + h^2\int_0^1 (1-t)f''(a+th)dt$$

For k = 1

$$f(a+h) - f(a) - f'(a)h = h^2\int_0^1 (1-t)f''(a+th)dt$$

$$R_{a,1}(h) = f(a+h) - P_{a,1}(h) = h^2 \int_0^1 (1-t) f''(a+th)dt$$

So we obtain the result for k = 1.
If we integrate again by parts

$$h^2 \int_0^1 (1-t) f''(a+th)dt = h^2 \left(\frac{-(1-t)^2}{2}\right) f''(a+th) |_0^1 + h^2 \int_0^1 \frac{(1-t)^2}{2} f'''(a+th)hdt$$

$$= \frac{f''(a)h^2}{2} + \frac{h^3}{2} \int_0^1 (1-t)^2 f'''(a+th)dt$$

We obtain the threorem for k = 2 . The result holds if we integrate bu parts k- time .

TH : (Taylor's theorem with integral Remainder II).
Suppose that f is of class C^k $(k \geq 1)$ on an interval $I \subset R$, *and* $a \in I$, Then the remainder $R_{a,k}$ defined above in eq.(1) is given by

$$R_{a,k}(h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} [f^{(k)}(a+th) - f^{(k)}(a)]dt$$

Proof: By previous theorem .If we replace k with (k-1) we get

$$R_{a,k-1}(h) = f(a+h) - \sum_{j=0}^{k-1} \frac{f^{(j)}(a)}{j!} h^j = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt$$

Sustracting $\frac{f^k(a)}{k!} h^k$ from both sides gives

$$f(a+h) - \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt - \frac{f^{(k)}(a)}{k!} h^k \;.....(2)$$

Since $\int_0^1 (1-t)^{k-1} dt = -\frac{(1-t)^k}{k} |_0^1 = 0 + \frac{1}{k} = \frac{1}{k}$

$$\frac{h^k}{k!} = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} dt = \frac{h^k}{(k-1)!} \frac{1}{k}$$ substiuting in (2)

We have $R_{a,k}(h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} [f^{(k)}(a+th) - f^{(k)}(a)]dt$.

Corollary : If f is of class C^k on I , then $\frac{R_{a,k}(h)}{h^k} \to 0 \quad as \quad h \to 0$

Proof: $f^{(k)}$ is cont. at a , so for $\varepsilon > 0, \exists \delta > 0, \ni |f^{(k)}(y) - f^{(k)}(a)| < \varepsilon when\ |y - a| < \delta$
$\Rightarrow | f^k(a+th) - f^k(a)| < \varepsilon \ \ for\ \ 0 \le t \le 1\ when\ \ |h| < \delta$.

$$\text{Since,} |R_{a,k}(h)| = | \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} [f^{(k)}(a+th) - f^{(k)}(a)] dt |$$

$$\le \frac{|h|^k}{(k-1)!} \int_0^1 (1-t)^{k-1} \varepsilon \ dt = \frac{\varepsilon}{k!} |h|^k \quad for\ \ |h| < \delta,$$

and $\frac{h^k}{k!} = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} dt \quad whenever\ \ |h| < \delta.$

$$\Rightarrow | \frac{R_{a,k}(h)}{h^k} | < \frac{\varepsilon}{k!} \to 0 \quad as \quad h \to 0$$

Corollary : If f is of class C^{k+1} on I and $|f^{(k+1)}(x)| \le M \ \ for\ \ x \in I$,then
$|R_{a,k}(h)| \le \frac{M}{(k+1)!} |h|^{k+1} \ ,(a+h \in I)$

Proof: since $|R_{a,k}(h)| = | \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k f^{(k+1)}(a+th) dt |$

$$|R_{a,k}(h)| \le \frac{|h|^{k+1}}{k!} \int_0^1 (1-t)^k M \ \ dt |$$

Since , $\frac{h^{k+1}}{(k+1)!} = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^k dt = \frac{h^{k+1}}{(k)!} . \frac{(1-t)^{k+1}}{k+1} |_0^1 = \frac{h^{k+1}}{(k+1)!}$

Then , $|R_{a,k}(h)| \le \frac{|h|^{k+1}}{k!} | \int_0^1 (1-t)^k M \ \ dt | = \frac{M}{(k+1)!} |h|^{k+1}$.

Lemma: Suppose g is k+1 differentiable on [a,b] . If g(a)= g(b) and $g^j(a) = 0$ for $1 \le j \le k$, then there is a point $c \in (a,b)$ such that $g^{k+1}(c) = 0$.

Proof: g satisfies Roll's theorem , there is a point $c_1 \in (a,b)$ such that $g'(c_1) = 0$. Since g' is cont. on $[a, c_1]$ and diff. on (a, c_1) and $g'(a) = g'(c_1) = 0$, there is a point $c_2 \in (a, c_1)$ such that $g''(c_2) = 0$. Proceeding induction , we find that for $1 \le j \le k+1$ that is a point $c_j \in (a, c_{j-1})$ such that $g^j(c_j) = 0$, and the final case j = k+1 is the desired result.

TH: (Taylor's theorem with Lagrange Remainder)

Suppose f is k+1 times diff. on an interval $I \subset R$, *and* $a \in I$, For each $h \in R$ such that $a+h \in I$, there is a point c between 0 and h such that $R_{a,k}(h) = f^{(k+1)}(a+c)\dfrac{h^{k+1}}{(k+1)!}$

Proof: Fix a particular h , and suppose for now that h > 0 .

Let $g(t) = R_{a,k}(t) - \dfrac{R_{a,k}(h)}{h^{k+1}} t^{k+1}$

$$= f(a+t) - f(a) - f'(a)t - \cdots\cdots - \frac{f^{k}(a)t^{k}}{k!} - \frac{R_{a,k}(h)\, t^{k+1}}{h^{k+1}}$$

$g(h)=0$, *and* $g(0)=0$, *Somilarly for* $j \le k$ *we have*

$$g^{j}(t) = f^{j}(a+t) - f^{j}(a) - \cdots\cdots - \frac{f^{(k)}(a)\, t^{k-j}}{(k-j)!} - \frac{R_{a,k}(h)(k+1)t^{k}}{h^{k+1}} \cdots\cdots (k+2-j)\, t^{k+1-j},$$

So , $g^{j}(0)=0$, Therefore by preivious lemma , there is a point $c \in (0,h)$ such that

$$0 = g^{k+1}(c) = f^{(k+1)}(a+c) - \frac{R_{a,k}(h)(k+1)!}{h^{k+1}}$$

$$R_{a,k}(h) = \frac{f^{(k+1)}(a+c)\, h^{k+1}}{(k+1)!}$$

The case h<0 is handeled similarly by considering the function $\breve{g}(t) = g(-t)$ on the interval $[0,|h|]$

Propostion : The Taylor polynomials of degree k a bout a = 0 of the functions .
e^{x} , $\cos x$, $\sin x$, $(1-x)^{-1}$ *are resp.*

$$e^{x} = \sum_{j=0}^{k} \frac{x^{j}}{j!} \quad , \ \cos x = \sum_{j=0}^{k/2} \frac{(-1)^{j} x^{2j}}{(2j)!} \quad , \ \sin x = \sum_{j=0}^{(k-1)/2} \frac{(-1)^{j} x^{2j+1}}{(2j+1)!} \quad , \quad (1-x)^{-1} = \sum_{j=0}^{k} x^{j} \ .$$

Example : Use Taylor expansion to evaluate $\displaystyle\lim_{x\to 0} \frac{x^{2} - \sin x^{2}}{x^{4}(1-\cos x)}$

$$x^{2} - \sin x^{2} = x^{2} - (x^{2} - \frac{1}{6}x^{6} + \ldots\ldots\ldots) = \frac{1}{6}x^{6} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$x^{4}(1-\cos x) = x^{4}(1-(1-\frac{1}{2}x^{2} + \ldots\ldots\ldots\ldots)) = \frac{1}{2}x^{6} + \ldots\ldots\ldots\ldots$$

Where the dots denote error terms that vanish faster than x^{6} *as* $x \to 0$, therefore

$$\lim_{x\to 0}\frac{x^2-\sin x^2}{x^4(1-\cos x)}=\lim_{x\to 0}\frac{\frac{1}{6}x^6+..........}{\frac{1}{2}x^6+..........}=\frac{1}{3}(by\ L'Hopital\ \ rule)$$

Agenerlaization of function on R^n :

Def: A set $s\in R^n$ is called convex if wherever $a,b\in S$, the line segment from a to b also lies in S .

Suppose $f:R^n\to R$ is of class C^k on a convex open set S . we will derive a Taylor expansion for $f(X)$ about a point $a\in S$ by looking at the restriction of f to the line joining a and x . That is we set h = x-a and $g(t)=f(a+t(x-a))=f(a+th)$
By he chain Rule

$$g^{'}(t)=h.\nabla f(a+th)=h_1\frac{\partial f(a+th)}{\partial x_1}+h_2\frac{\partial f(a+th)}{\partial x_2}+.............+h_n\frac{\partial f(a+th)}{\partial x_n}$$

$g^j(t)=(h.\nabla)^j f(a+th)\ where\ (h.\nabla)^j$ denote the result of applying the operation

$h.\nabla=h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+............+h_n\frac{\partial}{\partial x_n}$ n times on f

The Taylor formula for g with a = 0 and h = 1

$$g(1)=\sum_{j=0}^{k}\frac{g^j(0)}{j!}1^j+remainder$$

$$f(a+h)=\sum_{j=0}^{k}\frac{(h.\nabla)^j f(a)}{j!}+R_{a,k}(h)...............(1)$$

Where formula for $R_{a,k}(h)$ can be obtained from previous formulas and theorems.
By appling the multinomial theorem of $(h.\nabla)^j$ we get

$$(h.\nabla)^j=\sum_{|\alpha|=j}\frac{j!}{\alpha!}h^\alpha\partial^\alpha...............(2)$$

Substituting this in (1) we obtain the following theorem.

TH: (Taylor's theorem in several variables)
Suppose $f:R^n\to R$ is of class C^k on an open convex set S . If $a\in S\ \ and\ \ a+h\in S$, then

$$f(a+h)=\sum_{|\alpha|\le k}\frac{\partial^\alpha f(a)}{\alpha!}h^\alpha+R_{a,k}(h)$$

Where $R_{a,k}(h)=k\sum_{|\alpha|=k}\frac{h^\alpha}{\alpha!}\int_0^1(1-t)^{k-1}[\partial^\alpha f(a+th)-\partial^\alpha f(a)]dt$

If f is of class C^{k+1} on S ,we also have

$$R_{a,k}(h) = (k+1)\sum_{|\alpha|=k+1} \frac{h^\alpha}{\alpha!}\int_0^1 (1-t)^k \partial^\alpha f(a+th)\,dt$$

And

$$R_{a,k}(h) = \sum_{|\alpha|=k+1} \partial^\alpha f(a+ch)\frac{h^\alpha}{\alpha!} \quad for\ \ some\ \ c \in (0,1)$$

The Taylor polynomial of second order of f at x = a is given by

$$P_{a,2}(h) = f(a) + \sum_{j=1}^{n} \partial_j f(a)\, h_j + \frac{1}{2}\sum_{j,k=1}^{n} \partial_j \partial_k f(a)\, h_j\, h_k$$

$$P_{a,2}(h) = f(a) + \sum_{j=1}^{n} \partial_j f(a)\, h_j + \frac{1}{2}\sum_{j=1}^{n} \partial_j^2 f(a)\, h_j^2 + \sum_{1 \le j \le n} \partial_j \partial_k f(a)\, h_j\, h_k$$

Example : If $f(x,y) \in C^{n+1}$, $x - a = h_1$, $y - b = h_2$, $h = (h_1, h_2)$ $\mathrm{a} = (a,b)$

$$f(\mathrm{a}+h) = f(a+h_1, b+h_2) = \sum \frac{1}{j!}(h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y})^j f(a,b) + R_{a,k}(h)$$

The Taylor poly. Of second order

$$P_{a,2}(h) = f(a,b) + \frac{1}{1!}(h_1 \frac{\partial}{\partial x} f(a,b) + h_2 \frac{\partial}{\partial y} f(a,b)) + \frac{1}{2!}(h_1 \frac{\partial}{\partial x} f(a,b) + h_2 \frac{\partial}{\partial y} f(a,b))^2$$

$$P_{a,2}(h) = f(a,b) + (x-a)\frac{\partial}{\partial x} f(a,b) + (y-b)\frac{\partial}{\partial y} f(a,b) + \frac{1}{2}(x-a)^2 \frac{\partial^2}{\partial x^2} f(a,b)$$

$$+ \frac{1}{2}(y-b)^2 \frac{\partial^2}{\partial y^2} f(a,b) + (x-a)(y-b)\frac{\partial^2}{\partial x \partial y} f(a,b)$$

Example : Find the 3rd order Taylor polynomial of $f(x,y) = e^{x^2+y}$ about $(x,y) = (0,0) = a$, $(a = 0, b = 0)$

Solution :

$$P_{a,3}(h_1,h_2) = f(0,0) + \frac{1}{1!}(h_1 \frac{\partial}{\partial x} f(0,0) + h_2 \frac{\partial}{\partial y} f(0,0)) + \frac{1}{2!}(h_1 \frac{\partial}{\partial x} f(0,0) + h_2 \frac{\partial}{\partial y} f(0,0))^2$$

$$+ \frac{1}{3!}(h_1 \frac{\partial}{\partial x} f(0,0) + h_2 \frac{\partial}{\partial y} f(0,0))^3$$

$$P_{a,2}(h) = f(0,0) + (x-0)\frac{\partial}{\partial x} f(0,0) + (y-0)\frac{\partial}{\partial y} f(0,0) + \frac{1}{2}(x-0)^2 \frac{\partial^2}{\partial x^2} f(0,0)$$

$$+\frac{1}{2}(y-0)^2 \frac{\partial^2}{\partial y^2} f(0,0) + (x-0)(y-0)\frac{\partial^2}{\partial x \partial y} f(0,0) + \frac{1}{3!}(x^3 \frac{\partial^3}{\partial x^3} f(0,0)$$

$$+3x^2 y \frac{\partial^3}{\partial x^2 \partial y} f(0,0) + 3y^2 x \frac{\partial^3}{\partial x \partial y^2} f(0,0) + y^3 \frac{\partial^3}{\partial y^3} f(0,0))$$

$$e^{x^2+y} = 1 + (x^2 + y) + \frac{1}{2}(x^2+y)^2 + \frac{1}{6}(x^2+y)^3 + (order > 3)$$

$$= 1 + x^2 + y + \frac{1}{2}(x^4 + 2x^2 y + y^2) + \frac{1}{6}(x^6 + 3x^4 y + 3x^2 y^2 + y^3) + (order > 3)$$

$$= 1 + y + x^2 + \frac{1}{2}y^2 + x^2 y + \frac{1}{6}y^3 + (order > 3)$$

If we have thrown the terms $x^4, x^6, x^4 y$ *and* $x^2 y^2$ since are themselves of order >3

Thus the answer $P_{a,3}(x,y) = 1 + y + x^2 + \frac{1}{2}y^2 + x^2 y + \frac{1}{6}y^3$.

Ex . 1,2,4,5,6,7

2.8 Critical points:

Def: Suppose f is a diff. function on some open set $S \subset R^n$. The point $a \in S$ is called a critical point for f if $\nabla f(a) = 0$.

To find the critical points of f we solve the n.eqs. $\partial_1 f(x) = 0, \partial_2 f(x) = 0, ..., \partial_n f(x) = 0$ Simultaneously for the n quantities $x_1, x_2, ..., x_n$.

Def: we say that f has a local max (or local min) at a if $f(x) \le f(a)$ $(or \;\; f(x) \ge f(a))$ for all x in some neighborhood of a

Proposition: If f has a local max. Or min at a and f is diff. at a, then $\nabla f(a) = 0$

Proof: If f has a local max. Or min at a, then for any unit vector u, the function $g(t) = f(a + tu)$ has a local max. Or min at t = 0 So, $g'(0) = \partial_u f(a) = 0$ In particular $\partial_j f(a) = 0$,for all j, so $\nabla f(a) = 0$.

Def: we say that f on an open set in R^n has a saddle point at if f has neither a max. nor min., and its graph goes up in one direction and down in some other direction.

Th: Suppose f is of class C^2 on an open set in R^2 containing the point a, and suppose $\nabla f(a) = 0$, Let $\alpha = \partial_1^2 f(a)$, $\beta = \partial_1 \partial_2 f(a)$, $\gamma = \partial_2^2 f(a)$. Then

a) If $\alpha\gamma - \beta^2 < 0$, f has a saddle point at a .

b) If $\alpha\gamma - \beta^2 > 0$, and $\alpha > 0$, f has a local min. at a .

c) If $\alpha\gamma - \beta^2 > 0$, and $\alpha < 0$, f has a local max. at a .

d) If $\alpha\gamma - \beta^2 = 0$, no conclusion can be drawn .

Example: Find and classify the critical point of the function $f(x, y) = xy(12 - 3x - 4y)$

Solution: we have

$\partial_x f = 12y - 6xy - 4y^2 = y(12 - 6x - 4y),$

$\partial_y f = 12x - 3x^2 - 8xy = x(12 - 3x - 8y).$

Thus if $\partial_x f = 0$, then $y = 0$ or $12 - 6x - 4y = 0$ and $\partial_y f = 0$, then $x = 0$ or $12 - 3x - 8y = 0$, so there are four possibilities

$x = y = 0$, $y = 12 - 3x - 8y = 0 \quad (y = 0 \Rightarrow x = 4)$

$x = y = 0$, $x = 12 - 6x - 4y = 0 \quad (x = 0 \Rightarrow y = 3)$, and

$12 - 6x - 4y = 0,\ 12 - 3x - 8y = 0 \Rightarrow x = \frac{4}{3},\ y = 1.$

Solving these given the critical points
(0, 0), (4, 0) , (0, 3) , (4/3, 1)
Since $\alpha = \partial_1^2 f(a) = -6y$, $\gamma = \partial_2^2 f(a) = -8x$, $\beta = \partial_1\partial_2 f(a) = 12 - 6x$
By pervious Theorem

At (0, 0) we have $\alpha\gamma - \beta^2 = 0 - (12)^2 < 0$
(0, 0) is a saddle point

At (4, 0) we have $\alpha\gamma - \beta^2 = 32 - (12 - 6(4))^2 < 0$
(4, 0) is a saddle point

At (0, 3) is a saddle point
But (4/3, 1) is a local max because $\alpha\gamma - \beta^2 = 48 > 0,\ \alpha < 0.$

Example: Find and classify the critical points of the function $f(x, y) = y^3 - 3x^2 y$

Solution $\partial_x f = -6xy$, $\partial_y f = 3y^2 - 3x^2$
If $\partial_x f = 0 \Rightarrow x = 0 \ or \ y = 0$
And $\partial_y f = 0 \Rightarrow x^2 = y^2 \Rightarrow x = y = 0$

So (0, 0) is the only critical point

$\alpha = \partial_1^2 f(a) = -6y$, $\beta = \partial_1\partial_2 f(a) = -6x$ $\gamma = \partial_2^2 f(a) = 6y$ all are vanishes at (0, 0), so by the previous test is failure .
Since $f(x, y) = y(y - \sqrt{3}x)(y + \sqrt{3}x)$

and the lines $y=0$, $y=\sqrt{3}x, y=-\sqrt{3}x$ separate the plane into six region on which f is alternatively positive and negative , and these region all meet at the origin . Thus f has neither a max. or a min. at the origin , So f has a saddle point called " monkey saddle" .

Ex 1) a,b,c,d,e

2.9 Extreme value problems :

Th: Let f be continuous function on an unbounded closed set $S \subset R^n$.

a) If $f(x) \to \infty \quad as \ |x| \to \infty \ (x \in S)$, then f has as absolute minimum but no absolute maximum on S .

b) If $f(x) \to 0 \quad as \ |x| \to \infty \ (x \in S)$ and there is a point $x_0 \in S$ where $f(x_0) > 0$ (resp. $f(x_0) < 0$) , then f has absolute maximum (resp. minimum) on S .

Example : Find the absolute max. and min. values of the function

$f(x,y) = \dfrac{x}{x^2 + (y-1)^2 + 4}$ on the first quadrant $S = \{(x,y) : x, y \geq 0\}$

Solution : for $x, y \geq 0$, $f(x,y) \geq 0 \ and \ f(0,y) = 0$,

so the minimum is zero . achievedat all points on the y –axis.

$$f(x,y) = \frac{x}{x^2 + (y-1)^2 + 4} \leq \frac{x}{x^2} = \frac{1}{x} \leq \frac{1}{x} + \frac{1}{(y-1)^2}$$

$$So, \ f(x,y) \leq \frac{1}{x} \ and \ f(x,y) \leq \frac{1}{(y-1)^2} \quad f(x,y) \to 0 \ \ as \ |(x,y)| \to \infty$$

So by previous theorem f has a maximum value on S which must occur either in the interior of S or on the positive x – axis

$$\frac{\partial f}{\partial x} = \frac{(x^2 + (y-1)^2 + 4) - (2x)(x)}{(x^2 + (y-1)^2 + 4)^2} = 0$$

$$(x^2 + (y-1)^2 + 4) - (2x)(x) = 0$$

$$(y-1)^2 - x^2 + 4 = 0$$

$$\frac{\partial f}{\partial y}=\frac{-x(2(y-1))}{(x^2+(y-1)^2+4)^2}=0$$

$x=0$, $y-1=0\Rightarrow y=1$

If $y=1\Rightarrow x^2=4\Rightarrow x=2$

So at (2, 1) there is a critical point and $f(2,1)=\frac{1}{4}$

Also, $f(x,0)=\frac{x}{x^2+5}$

$$\frac{\partial f}{\partial x}=\frac{(x^2+5)-2x^2}{(x^2+5)^2}=0$$

$x^2+5-2x^2=0\Rightarrow x^2=5\Rightarrow x=\sqrt{5}$ is a critical point and $f(\sqrt{5},0)=\frac{\sqrt{5}}{10}<\frac{1}{4}$,

So the max value of f on S is $\frac{1}{4}$.

Lagrange multiplier method:

Let f and g have a continuous first partial derivatives on an open set containing the surface or the curve S which is the graph of the eq. $g(X)=0$, Let $\nabla g(X)\neq 0$ on S and suppose that $f(X)$ has a constrained local extrema at the point a of S, then there is a number λ such that

$\nabla f(a)=\lambda\nabla g(a)$, that is the gradients of f and g are parallel at a

Example : what is the maximum area of a rectangle with perimeter P

Solution : Let $f(x,y)=xy$

and $g(x,y)=2x+2y-p=0$

$\nabla f=(y,x)$

$\nabla g=(2,2)$

$\nabla f=\lambda\nabla g\Rightarrow y=2\lambda$, $x=2\lambda$, $2x+2y=p$

Solving the first two equations give $y=x$, substituting into the third eq. given that

$2x+2x=p\Rightarrow x=\frac{1}{4}p=y$

So the max of f is $f(x,y)=\frac{1}{16}p^2$

The min. on this set namely 0 , is achieved when $x=0, y=\frac{1}{2}p$ *or vise versa*

Example: Find the absolute max. and min. of $f(x,y)=x^2+y^2+y$ on the disc $x^2+y^2\le 1$

Solution : $f_x = 2x = 0$, $f_y = 2y+1=0$, thus the only critical points is at $(0,-\frac{1}{2})$ lies on the disc , at which $f(0,-\frac{1}{2})=-\frac{1}{4}$

On the boundary we use Lagrange multiplier method with $g(x,y)=x^2+y^2-1.$

$\nabla g=(2x,2y)$

$\nabla f=\lambda\nabla g$ we solve the eqs.

$2x=2x\lambda$ and $2y+1=2y\lambda$, $x^2+y^2=1$

The first eq. implies

$x(1-\lambda)=0 \Rightarrow x=0 \ or \ \lambda=1$

If $\lambda=1 \Rightarrow$ *the second eq.* $2y+1=2y$ has no solution , So $\lambda=1$ is impossible ,So $x=0$, Then from the third eq. $y=\pm 1$

$\Rightarrow f(0,1)=2$, $f(0,-1)=0$, so the abs. max is 2 at (0,1) and the abs. min. is $-\frac{1}{4}$ *at* $(0,-\frac{1}{2})$.

We can analyze f on the boundary by parametring the latter as $x=\cos\theta$, $y=\sin\theta$ (*because* $x^2+y^2=1$). *Then* $f(\cos\theta,\sin\theta)=1+\cos\theta$ which has a max. value of 2 at $\theta=0$ an min. value of 0 at $\theta=\pi$

Ex. 1,2,3,6,7,11,12,14

2.10 Vector –valued functions and their derivatives

The vector valued function from R^n to R^m where n and m are any positive integers is defined by bold face $\mathbf{f} : R^n \to R^m \quad X \in R^n$

$\mathbf{f}(X) = (f_1(X), f_2(X), f_3(X), ..., f_m(X))$

$\mathbf{f}$ is called linear mapping from R^n to R^m , if it satisfy

$\mathbf{f}(aX + bY) = a\,\mathbf{f}(X) + b\,\mathbf{f}(Y) \quad (a, b \in R, X, Y \in R^n)$ .And these maps can be represented by an m x n matrix.

$A = (A_{jk})$ with m rows and n columns

If the elements of R^n to R^m are rep. as column vectors , $\mathbf{f}(X)$ is just the matrix product AX, and $f_j(X) = \sum_{k=1}^{n} A_{jk} x_k$

Differentiability of vector valued function :

A mapping f from an open set $S \subset R^n$ into R^m is said to be differentiable at $a \in S$.
If there is an m x n matrix L such that

$$\lim_{h \to 0} \frac{| f(a+h) - f(a) - Lh |}{| h |} = 0$$

The matrix L is a unique matrix defined as $D\,\mathbf{f}(a)$,$\mathbf{f}'(a)$ and $d\,\mathbf{f}_a$ its called the Frechet derivative of $\mathbf{f}$ at a .

We define this matrix in the following proposition

Proposition: An R^m- valued function f is differentiable at a precisely when each of its components $f_1,\ldots\ldots\ldots,f_n$ is differentiable at a . In this cases $Df(a)$ is the matrix whose jth row is the row vector $\nabla f_j(a)$. In other words

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Example : Let $f(x,y,z) = (u,v) = (xyz^2 - 4y^2, 3xy^2 - yz)$ Compute $Df(x,y,z)$

Solution : $$Df(x,y,z) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} yz^2 & xz^2 - 8y & 2xyz \\ 3y^2 & 6xy - z & -y \end{pmatrix}$$

TH: (Chain Rule) Suppose $g: R^k \to R^n$ is diff. at $a \in R^k$ and $f: R^n \to R^m$ is diff. at $g(a) \in R^n$. Then $H = f \circ g: R^k \to R^m$ is diff. at a and $DH(a) = Df(g(a))\, Dg(a)$ where the expression on the right is the product of the matrices $Df(g(a))$ and $Dg(a)$

Example : Define $f: R^2 \to R^3$ by $f(u,v) = (u^2 - 5v, ve^{2u}, 2u - \log(1+v^2))$

a) Compute $Df(u,v)$, what is $Df(0,0)$

Solution : Let $s = u^2 - 5v$, r=ve^{2u}, t=2u-log(1+r^2),

$$Df(u,v) = \begin{pmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & -5 \\ 2ve^{2u} & e^{2u} \\ 2 & \frac{-2v}{1+v^2} \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 0 & -5 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$$

b) Suppose $g : R^2 \to R^2$ is of class C^1 , $g(1,2) = (0,0)$

$and \quad Dg(1,2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$

Compute $D(f \circ g)(1,2)$

Solution : $D(f \circ g)(1,2) = Df\,(g(1,2))\,.Dg(1,2)$

$$= Df\,(0,0).Dg(1,2)$$

$$= \begin{pmatrix} 0 & -5 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -15 & -20 \\ 3 & 4 \\ 2 & 4 \end{pmatrix}$$

Jacobians : If m = n ,then the Frechet determinant Df of a function $f : R^n \to R^n$ is an nxn matrix of functions , defined on the set S where f is diff. , so we can form its determinant on S is called The Jacobian of the mapping f , it is denoting by J_f or if $Y = f(X)$ by $\frac{\partial(y_1, y_2, \ldots\ldots, y_n)}{\partial(x_1, x_2, \ldots\ldots, x_n)} = J_f = Det\ Df$

$$Det\ Df = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & . & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \ldots & & \vdots \\ . & & & \vdots \\ \frac{\partial y_2}{\partial x_1} & \ldots & & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

And if $Y = f(X) \ and \ X = g(t) \ ,(t, X, Y \in R^n)$, then $J_{f \circ g}(t) = J_{f(g(t))}.J_g(t) \ or$

$$\frac{\partial(y_1, y_2, \ldots\ldots, y_n)}{\partial(t_1, t_2, \ldots\ldots, t_n)} = \frac{\partial(y_1, y_2, \ldots\ldots, y_n)}{\partial(x_1, x_2, \ldots\ldots, x_n)} \cdot \frac{\partial(x_1, x_2, \ldots\ldots, x_n)}{\partial(t_1, t_2, \ldots\ldots, t_n)}$$

Example1) : Let $(u,v) = f(x,y,z) = (xyz^2 - 4y^2, 3xy^2 - yz)$

Compute $\dfrac{\partial(u,v)}{\partial(x,y)}$, $\dfrac{\partial(u,v)}{\partial(y,z)}$, $\dfrac{\partial(u,v)}{\partial(x,z)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} yz^2 & xz^2 - 8y \\ 3y^2 & 6xy - z \end{vmatrix} = yz^2(6xy - z) - 3y^2(xz^2 - 8y)$$

Ex : 1,2,3,4,8

Ex. 8) $w = f(x,y,t,s)$, $x(t,s)$, $y(t,s)$, $g(t,s) = (x(t,s), y(t,s), t, s)$

$w = f(g(t,s)) = (f \circ g)(t,s)$

$D(f \circ g(t,s)) = Df\ (g(t,s))\ .Dg(t,s)$

$$= \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial t} & \dfrac{\partial f}{\partial s} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial s} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial t} & \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial f}{\partial s} \end{pmatrix}$$

CH3 : The implicit Function theorem and its application :

3.1 The implicit function theorem: in this section we consider the problem of solving an eq. $F(x_1, x_2, x_3, ..., x_n) = 0$ for one of the variables x_j as a function of the remaining (n-1) variables or more generally of solving a system of k such eqs for k of the variables as a functions of the remaining (n-k) variables .

For the case n =2 , we are given an eq. $F(x, y) = 0$ relating the variables x and y , and we ask when we can solve for y as a function of x or vice versa .

If $S = \{(x, y) : F(x, y) = 0 \}$ then our equations is when can S be represented as the graph of a function y=f(x) or x= g(y) ?

For the case n= 3, the set where $F(x, y, z) = 0$ will usually be a surface , and we ask when this surface can be represented as the graph of a function

$z = f(x, y), y = g(x, z) \text{ or } x = h(y, z)$

Example: Consider $F(x, y) = x^2 + y^2 + 1$ (1)

$F(x, y) = x^2 + y^2$ (2)

$F(x, y) = x^2 + y^2 - 1$ (3)

In the first eq.(1) , F(x,y)=0 did not satisfied for any point .

In the second eq.(2) , F(x,y)=0 satisfied for x = y =0 , so $\exists y = f(x) \text{ at } x = 0$.

In the third eq.(3) , F(x,y)=0 satisfied for $-1 < x < 1$, and eq(3) does define two functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ but these functions are not defined in a two sided neighborhood of x =1 or of x = -1 because in this case $F_2(1,0) = 0, F_2(-1,0) = 0$

Now If the number of variables are n+1 and if we denote the last variable by y, we have the problem .

Given a function F(x,y) of class C^1 and a point (a,b) satisfying F(**a**,b)= 0 , when is there

1) A function f(X), defined in some open set in R^n containing $a \in R^n$, and
2) An open set $u \subset R^{n+1}$ containing (a,b) such that for $(X, y) \in u$, $F(X, y) = 0 \Leftrightarrow y = f(X)$?

TH 3.1 The implicit Function theorem for a single equations. Let $F(X, y)$ be a function of class C^1 on some neighborhood of a point $(\mathbf{a}, b) \in R^{n+1}$

Suppose that F(**a**,b) = 0 and $\partial_y F(\mathbf{a}, b) \neq 0$ then there exist very small positive numbers r_0, r_1 such that the following conclusion are valid .

a) For each X in the ball $|X - a| < r_0$ ther is a unique y such that $|y - b| < r_1$ and F(X,y)=0, we denote this y by f(X) , in particular f(**a**) = b.

b) The function f thus defined for $|X - a| < r_0$ is of class C^1, and its partial derivatives are given by $\partial_j f(X) = \dfrac{-\partial_j F(X, f(x))}{\partial_y F(X, f(x))}$

Summary of the theorem

If 1) $F(X, y) \in C^1$

2) $F(a, b) = 0$

3) $F_y(a, b) \neq 0$

$\Rightarrow \exists!\, (a\ unique\ function)\ y = f(X)\ and\ \ \exists r_0, r_1 > 0 \ni$ for all X, $|X - a| < r_0$, $|y - b| < r_1$

A) y(**a**) = f(**a**) =b

B) F(X,f(X)) = 0, where |X-**a**|< r_0.

C) f(x) $\in C^1$, on |X-**a**|< r_0 and $\partial_j f(X) = \dfrac{-\partial_j F(X, f(X))}{\partial_y F(X, f(X))}$.

proof: see the book.

Example (1) : let $F(X, y) = x - y^2 - 1$ for any point $(a, b) \in R^2$ for which $F(a, b) = 0$,

1) $F(x, y) \in C^1$

2) $F(a, b) = a - b^2 - 1$

3) $F_x(a, b) = 1\ \ , F_y(a, b) = -2b$

First $F_x(a, b) = 1 \neq 0$, so the implicit function theorem guarantees that the eq. F(x,y)=0 can be solved for X locally near any point (a,b) for which F(a,b) = 0, So $F(x, y) = x - y^2 - 1 = 0$, can be solved explicitly as x a function of y namely $x = y^2 + 1$ and this solution is valid for any point (a,b).

Next , $F_y(a, b) = 0$ when b =0, so the implicit function theorem quarantees that F(x,y) = 0 can be solved uniqualy for y near any point (a,b) such that F(a,b)=0 and b$\neq$0

In fact the possible solutions are $y = \sqrt{x-1}$ and $y = -\sqrt{x-1}$.

For x very close to a only one of these solutions will be very close to b namely $\sqrt{x-1}\ \ if\ \ b > 0$ and $-\sqrt{x-1}\ \ if\ \ b < 0$ and these solutions are defined only for $x \geq 1$, so $r_0 = a - 1$ (in the therorem)

Finally, we have F(1,0) = 0, but the eq. F(x,y) = 0 can not be solved uniqualy for y as a function of x in any neighborhood of (1,0), if x>1 there are two solution s ,both equally close to 0, and if x<1 there are none.

Example 2 : let $G(x, y) = x - e^{1-x} - y^3$

$\partial_x G(a, b) = 1 + e^{1-a} > 1$ for all (a,b)

So the implicit function theorem guarantees that the eq. G(x,y)= 0 can be solved for x locally near any point (a,b) such that G(a,b) = 0.

Next, $\partial_y G(a,b) = -3b^2$, so the implicit function theorem guarantees that the eq. G(x,y) =0 can be solved for y as a C' function of x locally near any point (a,b) such that G(a,b) = 0 and b$\neq$0. In fact the solution is $y = \sqrt[3]{x - e^{1-x}}$ which is globally uniquely defined but fails to be diff at the point when y =0, x = 1.

Ex3.1

Ex(1) : Investigate the possibility of solving the eq. $x^2 - 4x + 2y^2 - yz = 1$ for each of its variables in terms of the other two near the point (2,-1,3). Do this both by checking the hypotheses of the implicit function theorem and by explicitly computing the solution :

Sol: let $F(x,y,z) = x^2 - 4x + 2y^2 - yz - 1 = 0.$

1) $F(x,y,z) \in C^1$

2) $F(2,-1,3) = 4 - 8 + 2 + 3 - 1 = 0$

3) $F_x(x,y,z) = 2x - 4 |_{(2,-1,3)} = 0$

$F_y(x,y,z) = 4y - z |_{(2,-1,3)} = -4 - 3 = -7 \neq 0$

$F_z(x,y,z) = -y |_{(2,-1,3)} = 1 \neq 0$

So, $F(x,y,z)$ can be solved for y and z at the point (2,-1,3) but not for x.

Ex. 3) Can the eq. $(x^2 + y^2 + 2z^2)^{0.5} = \cos z$ be solved uniquely for y in terms of x and z near (0,1,0) ? For z in terms of x and y ?

Solution : let $F(x,y,z) = (x^2 + y^2 + 2z^2)^{0.5} - \cos z = 0$.

Then

1) $F(x,y,z) \in C'$

2) $F(0,1,0) = 0$

3) $F_y = 0.5(x^2 + y^2 + 2z^2)^{-0.5}(2y) |_{(0,1,0)} = 0.5(0+1)^{-0.5}(2) = 1 \neq 0$

4) $F_z = 0.5(x^2 + y^2 + 2z^2)^{-0.5}(4z) + \sin z \;|_{(0,1,0)} = 0.$

F can be solved for y in terms of x and z near (0,1,0) but not for z in terms of x and y.

The implicit theorem for a system of equations.

If we have k functions $F_1, F_2, ..., F_k$ of n+k variables $x_1, x_2, ..., x_n, y_1, ..., y_k$ and ask when we can solve the equations

$$F_1(x_1, ..., x_n, y_1, ..., y_k) = 0$$
$$F_2(x_1, ..., x_n, y_1, ..., y_k) = 0$$
$$\vdots$$
$$\vdots$$
$$F_k(x_1, ..., x_n, y_1, ..., y_k) = 0$$

For the y's in terms of the x's

We shall use the vector notation to abbreviate as $F(X, Y) = 0$ we assume $F \in C^1$ near the point (a,b) and $F(a,b) = 0$, and we ask when $F(X, Y) = 0$ determines Y as a C^1 function of X in some neighborhood of (**a**,**b**).

Let the matrix $B = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & .... & \frac{\partial F_1}{\partial y_k} \\ \vdots \\ \vdots \\ \frac{\partial F_k}{\partial y_1} & \frac{\partial F_k}{\partial y_2} & .... & \frac{\partial F_k}{\partial y_k} \end{pmatrix}$,

be the partial Frechet derivative of F with respect to the variables Y evaluated at (**a**,**b**).

We have the following theorem.

Th : The implicit function theorem for a system of equations:

Let F(X,Y) be an R^k - valued function of class C^1 on some neighborhood of a point $(a,b) \in R^{n+k}$ and let the matrix B is the parial Frenchet deriavative of F with respect to the variables Y, evaluated at (a,b). Suppose F(a,b)= 0 and let $\det B \neq 0$. Then there exist positive numbers r_0, r_1 such that the following conclusions are valid

a) For each X in the ball $|X - a| < r_0$ there is a unique y such that $|Y - b| < r_1$ and F(X,Y) = 0, we denote this Y by f(**X**) in particular, f(**a**) = **b**.
b) The function f thus defined for $|X - a| < r_0$ is of class C' ,and its partial derivative $\partial_{x_j} f$ can be computed by differentiating the equations F(X,f(X)) =

0 with respect to x_j and solving the resulting linear system of equations for $\partial_{x_j} f_1, \ldots\ldots\ldots, \partial_{x_j} f_k$.

Example 3: Consider the problem of solving the eqs $x - yu^2 = 0 \ \ , xy + uv = 0$ ….(1) for u and v as functions of x and y setting $F = x - yu^2 \ \ and \ \ G = xy + uv$

we see that $\dfrac{\partial(F,G)}{\partial(u,v)} = \det\begin{pmatrix} -2yu & 0 \\ v & u \end{pmatrix} = -2yu^2$.

So the implicit function theorem quarantees a local solution near any point (x_0, y_0, u_0, v_0) at which eqs.(1) holds provided that $-2y_0u_0^2 \neq 0$, that is, $y_0 \neq 0 \ and \ u_0 \neq 0$.
Notice that under this condition, the first eq. in (1) implies that $x_0 \neq 0$ and that x_0 and y_0 have he same sign .
The second eq. then implies that $v_0 \neq 0$ and that u_0 and v_0 have opposite signs .It's not hard to find explicitly $u = \pm\sqrt{\dfrac{x}{y}} \quad , \ v = \pm\sqrt{xy^3}$

The sign of u and v being the same as the signs of u_0 and v_0 resp. This solution is valid for all (x,y) in the same quadrants as (x_0, y_0).

EX. 5) Suppose $F(x,y) \in C^1$ is a function such that F(0,0) = 0**.**
What conditions on F will guarantee that the eq. F(F(x,y),y) = 0 can be solved for y as a C^1 function of x near (0,0) ?

Solution : 1) F(F(0,0),0) = 0

2) $F(F(x,y),y) \in C^1$

3) $\dfrac{\partial F(F(x,y),y)}{\partial y} = F_1F_2 + F_2 = F_2(F_1 + 1)$ at (0,0)

$$F_2(F_1 + 1) \neq 0 \ \ iff \ \ F_2(0,0) \neq 0 \ \ , F_1(0,0) \neq -1.$$

EX. 6) Investigate the possibility of solving the eqs, $xy + 2yz - 3xz = 0$, $xyz + x - y = 1$ for two of the variables as a function of the third near the point (x,y,z) = (1,1,1).

Solution : Let

$$F(x,y,z) = xy + 2yz - 3xz = 0$$
$$G(x,y,z) = xyz + x - y - 1 = 0$$

1) $F(1,1,1)=0$

2) $G(1,1,1)=0$

$$\frac{\partial(F,G)}{\partial(y,z)}=\det\begin{pmatrix} x+2z & 2y-3x \\ xz-1 & xy \end{pmatrix}\Big|_{(1,1,1)}=\det\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}=3\neq 0 .$$

Then F and G can be solved for y and z interms of x .

$$\frac{\partial(F,G)}{\partial(x,y)}=\det\begin{pmatrix} y-3z & x+2z \\ yz+1 & xz-1 \end{pmatrix}\Big|_{(1,1,1)}=\det\begin{pmatrix} -2 & 3 \\ 2 & 0 \end{pmatrix}=-6\neq 0$$

F and G can be solved for x and y interms of z

$$\frac{\partial(F,G)}{\partial(x,z)}=\det\begin{pmatrix} y-3z & 2y-3x \\ yz+1 & xy \end{pmatrix}\Big|_{(1,1,1)}=\det\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}=0$$

F and G can not be solved for x and z interms of y

Sec 3.1) 2 ,3 , 7 , 8 , 9.

CH4 sec 4.6 Improper integrals

The two most basic Types of improper integrals are as follows :

I) $\int_a^\infty f(x)\,dx$ where f is integrable over every finite subinterval [a,b]

II) $\int_a^b f(x)\,dx$ where f is integrable over [c,b] for every c > a ,but is unbounded near x =a

Improper integral of Type I

If f is defined on [a, ∞] and integrable on [a,b] ,for every b>a

Then $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$

If the limit exist the integral conv. . If the limit does not exist the integral div

Example : 1) $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{b\to\infty} -e^{-b} + 1 = 0 + 1 = 1$

2) $\int_0^\infty \cos x\,dx = \lim_{b\to\infty}\int_0^b \cos x\,dx = \lim_{b\to\infty}\sin x\Big|_0^b = \lim_{b\to\infty}\sin b - 0 = doesnot \quad exist$

TH: Suppose that $0 \le f(x) \le g(x)$ for all suffeicently large x . If $\int_a^\infty g(x)\,dx$ conv. So does $\int_a^\infty f(x)\,dx$. If $\int_a^\infty f(x)\,dx$ div so does $\int_a^\infty g(x)\,dx$.

Proof : Assume that $0 \le f(x) \le g(x)$ for all $x \ge a$

If $\int_a^\infty g(x)\,dx$ conv., then there exists B>0 such that $\int_a^\infty g(x)dx = B$. It implies that

$\phi(b) = \int_a^b f(x)\,dx$ has an upper bound as $b \to \infty$, because

$\phi(b) = \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \le \int_a^\infty g(x)\,dx$ =B. Also $\phi'(b) = f(b) \ge 0,$ so

$\phi(b)$ is increasing on $[b,\infty] \Rightarrow \int_a^\infty f(x)\,dx$ conv. And $\lim_{b\to\infty}\int_a^b f(x)dx \le B.$

Corollary : Suppose f > 0 , g > 0 and $\frac{f(x)}{g(x)} \to l$ as $x \to \infty$. If $0 < l < \infty$, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both conv. or both div.

If l = 0, the convergence of $\int_a^\infty g(x)\,dx$ implies the convergence of $\int_a^\infty f(x)\,dx$.

If $l = \infty$, the divergence of $\int_a^\infty g(x)\,dx$ implies the divergence of $\int_a^\infty f(x)\,dx$.

Proof : If $0 < l < \infty$ and $\frac{f(x)}{g(x)} \to l \Rightarrow \frac{f(x)}{g(x)} \le 2l \Rightarrow f(x) \le 2l\, g(x)$

And $f(x) \ge 0.5l\, g(x)$ for sufficiently large x

If $\int_a^\infty g(x)\,dx \;\; conv \Rightarrow 2l\int_a^\infty g(x)\,dx \;\; conv. \;\; \Rightarrow \int_a^\infty f(x)\,dx \;\; conv.$

Also , If $\int_a^\infty g(x)dx \;\; div.,$ then $0.5l\int_a^\infty g(x)dx \;\; div. \;\; \Rightarrow \int_a^\infty f(x)dx \;\; div.$

If $l = 0 \Rightarrow f(x) \le g(x)$ and if $l = \infty \;\; g(x) \le f(x)$ for suffeicently large x

If $\int_a^\infty g(x)\,dx \;\; conv \Rightarrow \int_a^\infty f(x)\,dx \;\; conv \;\; when \;\; l = 0$

If $\int_a^\infty g(x)\,dx \;\; div. \Rightarrow \int_a^\infty f(x)\,dx \;\; div \;\; when \;\; l = \infty$

Example: If $\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \lim_{b\to\infty}\frac{b^{1-p}-1}{1-p} = \begin{cases} \infty & if \;\; p<1, \text{ so the integration div.} \\ (p-1)^{-1} & if \;\; p>1, \text{ so the integration } conv. \end{cases}$

If $p = 1 \Rightarrow \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\ln b = \infty,$ so the integration div.

Corollary : If $0 \le f(x) \le Cx^{-p}$ for all sufficiently large x where P>1 ,then for a>0, $\int_a^\infty f(x)\,dx$ conv. . If $f(x) \ge Cx^{-1}$ $(C > 0)$ for all sufficiently large x , then $\int_a^\infty f(x)\,dx$ divergese.

Example 2:
Determine whether the integral conv. or div.

$$\int_0^\infty \frac{2x + 14}{x^3 + 1}\,dx$$

Solution: $\int_0^\infty \frac{2x+14}{x^3+1}dx = \int_0^1 \frac{2x+14}{x^3+1}dx + \int_1^\infty \frac{2x+14}{x^3+1}dx$.

$f(x) = \frac{2x+14}{x^3+1}$ behaves like $\frac{2x}{x^3} = \frac{2}{x^2}$

Let $g(x) = \frac{1}{x^2} \Rightarrow \int_1^\infty \frac{dx}{x^2}$ conv.

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{2x+14}{x^3+1} / \frac{1}{x^2} = \lim_{x\to\infty} \frac{2x^3+14x^2}{x^3+1} = 2$, so by the limit comparison test,

the integration $\int_1^\infty f(x)\,dx$ conv., so

$\int_0^\infty \frac{2x+14}{x^3+1}dx = \int_0^1 \frac{2x+14}{x^3+1}dx + \int_1^\infty \frac{2x+14}{x^3+1}dx$ conv., because $\int_0^1 \frac{2x+14}{x^3+1}dx$ is proper integration which converges and $\int_1^\infty \frac{2x+14}{x^3+1}dx$, conv.

TH: If $\int_a^\infty |f(x)|dx$ *conv.*, *then* $\int_a^\infty f(x)dx$ *conv.*

Proof : If f is a real valued function

Let $f^+(x) = \max[f(x),0]$ *and* $f^-(x) = \max[-f(x),0]$

Then $0 \le f^+(x) \le |f(x)|$ and $0 \le f^-(x) \le |f(x)|$ so $\int_a^\infty f^+(x)dx$ *and* $\int_a^\infty f^-(x)dx$ *conv.*. but

$f = f^+ - f^-$ so $\int_a^\infty f(x)dx$ *conv.*

If f is complex valued function $\Rightarrow |\operatorname{Re} f(x)| \le |f(x)|$ *and* $|\operatorname{Im} f(x)| \le |f(x)|$ So the convergence of $\int_a^\infty |f(x)|dx$ Implies the conv. of $\int_a^\infty |\operatorname{Re} f(x)|dx$ *and* $\int_a^\infty |\operatorname{Im} f(x)|dx$,

and hence the conv. of the real and imaginary parts of $\int_a^\infty f(x)dx$

Def: The integral $\int_a^\infty f(x)dx$ is called abs.convergent if $\int_a^\infty |f(x)|dx$ conv

Example: $\int_1^\infty \frac{\sin x}{x}dx$ is conv. but not abs. conv.

Solution : $\int_1^\infty \frac{\sin x}{x}dx = \lim_{b\to\infty}\int_1^b \frac{\sin x}{x}dx$

By integrating by parts let $u = \frac{1}{x}$ $\qquad dv = \sin x\, dx$

$$du = \frac{-1}{x^2}dx \qquad v = -\cos x$$

$$\int_1^\infty \frac{\sin x}{x}dx = \lim_{b\to\infty}\frac{-\cos x}{x}\Big|_1^b - \lim_{b\to\infty}\int_1^b \frac{\cos x}{x^2}dx$$

$\int_1^\infty \frac{\cos x}{x^2}dx$ conv. Since $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ conv. and $\lim_{b\to\infty}\frac{-\cos x}{x}\Big|_1^b = 0 + \cos 1$

$\int_1^\infty \frac{\sin x}{x}dx = 0+\cos 1-\int_1^\infty \frac{\cos x}{x^2}dx = conv.$, to show $\int_1^\infty |\frac{\sin x}{x}|dx$, div., by ex(8)

$\exists$ a constant positive number $c>0$, $\ni \frac{c}{x} \le |\frac{\sin x}{x}|$ for $x \in (n\pi,(n+1)\pi)$ and for all $n \ge 1$. Let $m\pi > n\pi > 1$, then

$$\int_1^{m\pi} |\frac{\sin x}{x}|dx = \int_1^{n\pi} |\frac{\sin x}{x}|dx + \int_{n\pi}^{(n+1)\pi} |\frac{\sin x}{x}|dx + \cdots\cdots + \int_{(m-1)\pi}^{m\pi} |\frac{\sin x}{x}|dx \ge$$

$$c\int_1^{n\pi} \frac{1}{x}dx + c\int_{n\pi}^{(n+1)\pi} \frac{1}{x}dx + \cdots\cdots + c\int_{(m-1)\pi}^{m\pi} \frac{1}{x}dx = c\int_1^{m\pi} \frac{1}{x}dx .$$

$\Rightarrow \lim_{m\to\infty} \int_1^{m\pi} |\frac{\sin x}{x}|dx \ge c \lim_{m\to\infty} \int_1^{m\pi} \frac{1}{x}dx = \lim_{m\to\infty} \ln m\pi = \infty$. So the integration diverges.

$\int_1^\infty |\frac{\sin x}{x}|dx \; div. \Rightarrow \int_1^\infty \frac{\sin x}{x}dx \; is \; conv. \; but \; not \; abs.conv.$

Improper Integral Type II

If f is defined on (a,b] and integrable over[c,b] for every c>a the improper integral

$$\int_a^b f(x)dx = \lim_{\substack{c\to a^+ \\ c>a}} \int_c^b f(x)dx$$

If the limit exist then the improper integral conv. and if the limit deos not exist the improper integral div.

TH: suppose that $0 \le f(x) \le g(x)$ for all x sufficiently close to a . If $\int_a^b g(x)dx$ conv. .so does $\int_a^b f(x)dx$. If $\int_a^b f(x)dx$ div. ,so does $\int_a^b g(x)dx$

Example: let $f(x) = \frac{1}{(x-a)^p}$

$$\int_a^b (x-a)^{-p}dx = \lim_{c\to a^+} \int_c^b (x-a)^{-p} = \lim_{c\to a^+} \frac{(x-a)^{1-p}}{1-p} |_c^b = \begin{cases} (1-p)^{-1}(b-a)^{1-p} & if \;\; p<1, \text{ so the integration conv.} \\ \infty & if \;\; p>1, \text{ so the integration div.} \end{cases}$$

For p =1 $\int_a^b (x-a)^{-1}dx = \lim_{c\to a^+} \log(x-a) |_c^b \to \infty \; as \; c \to a^+$, so the integration div.

Corollary : If $0 \le f(x) \le C(x-a)^{-p}$ for all x near a ,where p<1, then $\int_a^b f(x)dx$ conv. .If $f(x) > C(x-a)^{-1}$ $(C>0)$ for all x near a , then $\int_a^b f(x)dx$ divergese.

Example : Show that $\int_0^1 x^{-2}\sin 3x dx$ divergese.

Solution : $\lim_{x\to 0}\frac{\sin 3x}{x} = 3$, so $\frac{\sin 3x}{x^2} = \frac{1}{x}\frac{\sin 3x}{x} > \frac{2}{x}$ for all x near 0

$$\int_0^1 x^{-2}\sin 3x dx > \int_0^1 2x^{-1}dx = 2\lim_{c\to 0^+}\ln x \,|_c^1 \to \infty$$

So the integration diverges.

Example : Show $\int_0^1 x^{-0.5}\sin(x^{-1})dx$ that is abs. conv.

Solution : $\left|\frac{\sin(x^{-1})}{x^{0.5}}\right| \le \frac{1}{x^{0.5}}$

$\int_0^1 x^{-0.5}dx = \lim_{c\to 0^+} 2x^{0.5}\,|_c^1 = 2$ conv.

$\int_0^1 \frac{|\sin(x^{-1})|}{x^{0.5}}dx$ conv.

So, $\int_0^1 x^{-0.5}\sin(x^{-1})dx$ conv. absolutely

Other Types of Improper integrals:
Various other kinds of improper integrals can be built up out of those of types I and II.

For example : $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$.

If both integrals on the right conv. Then the original integral $\int_{-\infty}^{\infty} f(x)dx$ conv.

Otherwise it div.

Example :

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\lim_{b\to-\infty}\int_{b}^{0}\frac{dx}{1+x^2}+\lim_{a\to\infty}\int_{0}^{a}\frac{dx}{1+x^2}=\lim_{b\to-\infty}\tan^{-1}x\,|_b^0+\lim_{a\to\infty}\tan^{-1}x\,|_0^a=0-(-\frac{\pi}{2})+\frac{\pi}{2}-0=\pi$$

Another way: the function in the integration above is even function so,

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=2\lim_{b\to\infty}\int_{0}^{b}\frac{dx}{1+x^2}=2\lim_{b\to\infty}\tan^{-1}x\,|_0^b=\pi$$

Example: $\int_0^\infty x^{-p}dx$ is divergent for every p

Solution : $\int_0^\infty x^{-p}dx=\int_0^1 x^{-p}dx+\int_1^\infty x^{-p}dx$

For p<1 , $\int_0^1 x^{-p}dx$ conv. but $\int_1^\infty x^{-p}dx$ div.

For p >1 $\int_0^1 x^{-p}dx$ div. but $\int_1^\infty x^{-p}dx$ conv.

For p=1 ,both $\int_0^1 x^{-p}dx$ and $\int_1^\infty x^{-p}dx$ div.

So in all cases above $\int_0^\infty x^{-p}dx$ div.

Example : $\int_0^\infty \frac{dx}{x^{0.5}+x^{1.5}}$ determine whether the integral conv. or div.

Solution : $\int_0^\infty \frac{dx}{x^{0.5}+x^{1.5}}=\int_0^1 \frac{dx}{x^{0.5}+x^{1.5}}+\int_1^\infty \frac{dx}{x^{0.5}+x^{1.5}}$

$0<\frac{1}{x^{0.5}+x^{1.5}}<x^{-0.5}\Rightarrow\int_0^1\frac{dx}{x^{0.5}+x^{1.5}}$ *conv.*

$\frac{1}{x^{0.5}+x^{1.5}}<x^{-1.5}\Rightarrow\int_1^\infty\frac{dx}{x^{0.5}+x^{1.5}}$ *conv.*, because $\int_0^1 x^{-0.5}dx$ conv.

$\Rightarrow\int_0^\infty\frac{dx}{x^{0.5}+x^{1.5}}$ *conv.*, because $\int_0^\infty x^{-1.5}dx$ conv.

Improper integration where $\int_a^b f(x)\,dx$ where f is unbounded near one or more interior points of [a,b].

Example: Consider $I = \int_0^9 (x^3 - 8x^2)^{\frac{-1}{3}}\, dx$, $\int_0^\infty (x^3 - 8x^2)^{\frac{-1}{3}}\, dx$.

$f(x) = (x^3 - 8x^2)^{\frac{-1}{3}}$ is not defined at 0 and at x = 8.

$I = \int_0^c f(x)\,dx + \int_c^8 f(x)\,dx + \int_8^9 f(x)\,dx$, where (0<c<8).

$|f(x)| = |x^3 - 8x^2|^{\frac{-1}{3}} = x^{\frac{-2}{3}} |x-8|^{\frac{-1}{3}} \le \frac{1}{2} x^{\frac{-2}{3}}$ for x near 0.

$|f(x)| = |(x^3 - 8x^2)^{\frac{-1}{3}}| = |x|^{\frac{-2}{3}} |x-8| \le \frac{1}{4} |x-8|^{\frac{-1}{3}}$ for x near 8.

and $\int_0^c \frac{1}{2} x^{\frac{-2}{3}} dx = \frac{1}{2} 3x^{\frac{1}{3}} |_0^c = \frac{3}{2} c^{\frac{1}{3}}$ conv.

$\int_c^8 \frac{1}{4} |x-8|^{\frac{-1}{3}} dx = \frac{1}{4}\frac{3}{2}(x-8)^{\frac{2}{3}} |_c^8 = -\frac{3}{8}(c-8)^{\frac{2}{3}}$ conv. And $\int_8^9 \frac{1}{4} |x-8|^{\frac{-1}{3}} dx = \frac{1}{4}.\frac{3}{2}(x-8)^{\frac{2}{3}} |_8^9 = \frac{3}{8}$ conv.

So, $\int_8^9 |f(x)|dx$ conv. so, $\int_8^9 f(x)\,dx$ conv. abs.

On the other hand $f(x) > 0 \;\; for \;\; x > 8$ and $\frac{f(x)}{x^{-1}} = (1 - 8x^{-1})^{\frac{-1}{3}} \to 1 \;\; as \;\; x \to \infty$

So , $\int_9^\infty (x^3 - 8x^2)^{\frac{-1}{3}} dx$ div. by limit comparison test if $g(x) = \frac{1}{x} \Rightarrow \int_9^\infty \frac{1}{x} dx$ div.

So $\int_0^\infty f(x)dx = \int_0^9 f(x)dx + \int_9^\infty f(x)dx$, div.

Example : $\int_{-1}^{1} \frac{1}{x} dx$ Improper Integral does not exist

$\int_{-1}^{1} \frac{1}{x}dx = \int_{-1}^{0} \frac{1}{x}dx + \int_0^1 \frac{1}{x}dx = \lim_{c\to 0^-} \ln|x| \,|_{-1}^{c} + \lim_{a\to 0^+} \ln|x| \,|_a^1 = \lim_{c\to 0^-} \log|c| - \lim_{a\to 0^+} \log a = -\infty + \infty$ indeterminate value,

in this case the improper integral does not exists.

$(if \;\; a = c \Rightarrow \log|c| - \log|a| = \log\frac{c}{a} = 0)$.

Since $f(x)=\frac{1}{x}$ is odd function, then the Cauchy principal value of $\int_{-1}^{1}\frac{1}{x}dx$, p.v. $\int_{-1}^{1}\frac{1}{x}dx=0$.

Def: Suppose a < c < b and suppose f is integrable on $[a,c-\varepsilon]$ and on $[c+\varepsilon,b]$ for all $\varepsilon>0$. The Cauchy principal value of the integral $\int_{a}^{b}f(x)\,dx$ is

p.v. $\int_{a}^{b}f(x)dx=\lim_{\varepsilon\to 0}[\int_{a}^{c-\varepsilon}f(x)dx+\int_{c+\varepsilon}^{b}f(x)dx]$.

Provided that the limit exists of course if $\int_{a}^{b}f(x)\,dx$ conv. and its Cauchy principles value is its ordinary value, (i,e p.v. $\int_{a}^{b}f(x)dx=\int_{a}^{b}f(x)dx$.

Proposition : Suppose a < 0 < b . If ϕ is cont. on [a,b] and differentiable at 0 then p.v. $\int_{a}^{b}x^{-1}\phi(x)\,dx$ exists .

Proof : Let $\phi=1$

p.v. $\int_{a}^{b}\frac{1}{x}dx=\lim_{\varepsilon\to 0}[\int_{a}^{-\varepsilon}\frac{dx}{x}+\int_{+\varepsilon}^{b}\frac{dx}{x}]=\lim_{\varepsilon\to 0}\log|x|\Big|_{a}^{-\varepsilon}+\log|x|\Big|_{\varepsilon}^{b}=\log\frac{b}{|a|}$

For general case , we write $\phi(x)=\phi(0)+[\phi(x)-\phi(0)]$,

obtaining p.v . $\int_{a}^{b}\frac{\phi(x)}{x}dx=\phi(0)\;p.v.\int_{a}^{b}\frac{dx}{x}+\int_{a}^{b}\frac{\phi(x)-\phi(0)}{x}dx$

The first quantity on the right exists and the second one is a proper integral

If $\frac{\phi(x)-\phi(0)}{x}=\phi'(0)$, then

$\int_{a}^{b}\phi'(0)dx=\phi'(0)(b-a)$ exists

$\Rightarrow p.v.\int_{a}^{b}\frac{\phi(x)}{x}dx$ exists.

** The $p.v.\int_{-\infty}^{\infty}f(x)\,dx=\lim_{R\to\infty}\int_{-R}^{R}f(x)\,dx$.

Example : $\int_{-\infty}^{\infty}\frac{x}{(1+x^2)}dx$

As improper integral, $\int_{-\infty}^{\infty} \frac{x}{(1+x^2)} dx = \int_{-\infty}^{1} \frac{x}{(1+x^2)} dx + \int_{1}^{\infty} \frac{x}{(1+x^2)} dx$

f(x)= $\frac{x}{(1+x^2)}$ behaves like $\frac{1}{x}$, so if $g(x) = \frac{1}{x}$, then by limit comparison test,

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$ and

$\int_{1}^{\infty} \frac{1}{x} dx \ div. \Rightarrow \int_{1}^{\infty} \frac{x}{(1+x^2)} dx \ div.$

But p.v. $\int_{-\infty}^{\infty} \frac{x}{(1+x^2)} dx = 0$ because f is odd function .

Ex. 1,2,3,4,5,10,11

CH5 5.4 Vector derivatives

Let ∇ denote the n-tuples partial diff. operator $\partial_j = \dfrac{\partial}{\partial x_j}$

$\nabla = (\partial_1, \partial_2, \ldots\ldots\ldots\ldots, \partial_n)$

If f is a scalar function on R^n, $f \in C^1$, then $grad\ f = \nabla f = (\partial_1 f, \partial_2 f, \ldots\ldots\ldots\ldots, \partial_n f)$

If $F\ is\ a\ C^1$ vector valued function on an open subset of R^n, then the divergence of F is the function defined by

$$div\ F = \nabla . F = \partial_1 F + \partial_2 F + \ldots\ldots\ldots . + \partial_n F$$

$$\nabla . F = (\partial_1, \partial_2, \ldots\ldots, \partial_n).(F_1, F_2, \ldots\ldots\ldots\ldots, F_n) = \partial_1 F + \partial_2 F + \ldots\ldots\ldots . + \partial_n F$$

Let n = 3 . If $F\ is\ aC^1$ vector valued function on an open subset of R^3 , the curl of F is the vector defined

$$Curl\ f = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = (\partial_2 F_3 - \partial_3 F_2)\, i - (\partial_1 F_3 - \partial_3 F_1)\, j + (\partial_1 F_2 - \partial_2 F_1)\, k$$

Properties : 1) $grad(fg) = f\ grad\,(g) + g\ grad(f)$

$\nabla(fg) = f\ \nabla g + g \nabla f$

Where f, g are scalar real valued functions

2) If F, G are vector valued function ,then

$$grad(F.G) = (F.\nabla)G + F \times (curl\ G) + (G.\nabla)F + G \times (curl\ F)$$
$$= (F.\nabla)G + F \times (\nabla \times G) + (G.\nabla)F + G \times (\nabla \times F)$$

3) If f is a scalar real valued function and G is a vector valued function ,then

$$Curl\,(fG) = f\ CurlG + (grad\ f) \times G .$$
$$\nabla \times (fG) = f\,(\nabla \times G) + (\nabla\ f) \times G .$$

4) If F, G are vector valued functions ,then

$$Curl\,(F \times G) = (G.\nabla)F + (divG)F - (F.\nabla)G - (divF)G$$
$$\nabla \times (F \times G) = (G.\nabla)F + (\nabla.G)F - (F.\nabla)G - (\nabla.F)G$$

5) If f is a real function , then

$$div\,(fG) = f\ divG + (gradf).G$$
$$\nabla.(fG) = f\,(\nabla.G) + (\nabla f).G$$

6) If F, G are vector valued functions , then

$$div\,(F \times G) = G.(Curl\,F) - F.(curl\,G)$$

$$\nabla.(F \times G) = G.(\nabla \times F) - F.(\nabla \times G)$$

Note 1) : $F.\nabla = \sum_{j=1}^{n} F_j \partial_j = F_1 \frac{\partial}{\partial x_1} + F_2 \frac{\partial}{\partial x_2} + ... + F_n \frac{\partial}{\partial x_n}$

And $(F.\nabla)G = F_1 \frac{\partial G}{\partial x_1} + F_2 \frac{\partial G}{\partial x_2} + ... + F_n \frac{\partial G}{\partial x_n}$

Note 2) : **Properties** 1, 5 are valid in R^n for any n , and properties 2, 3, 4, 6 which involve cross products and Curls are valid in R^3

Proof of property 1) : $\nabla(fg) = f\,\nabla g + g\nabla f$

Proof: $\nabla(fg) = (\frac{\partial fg}{\partial x_1}, \frac{\partial fg}{\partial x_2}, ..., \frac{\partial fg}{\partial x_n})$

$$\nabla(fg) = (f\,\frac{\partial g}{\partial x_1} + g\,\frac{\partial f}{\partial x_1},\ f\,\frac{\partial g}{\partial x_2} + g\,\frac{\partial f}{\partial x_2}, ..., f\,\frac{\partial g}{\partial x_n} + g\,\frac{\partial f}{\partial x_n})$$

$$\nabla(fg) = (f\,\frac{\partial g}{\partial x_1}, f\,\frac{\partial g}{\partial x_2}, ..., f\,\frac{\partial g}{\partial x_n}) + (g\,\frac{\partial f}{\partial x_1}, g\,\frac{\partial f}{\partial x_2}, ..., g\,\frac{\partial f}{\partial x_n})$$

$$\nabla(fg) = f\,(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, ..., \frac{\partial g}{\partial x_n}) + g\,(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n})$$

$$\nabla(fg) = f\,\nabla g + g\nabla f$$

Proof of property 6) : $\nabla.(F \times G) = G.(\nabla \times F) - F.(\nabla \times G)$

Proof : $F, G \in R^3$

$$F \times G = \begin{vmatrix} i & j & k \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = (F_2G_3 - F_3G_2)\,i - (F_1G_3 - F_3G_1)\,j + (F_1G_2 - F_2G_1)\,k$$

$$\nabla = (\partial_1, \partial_2, ..., \partial_n)$$

Left side = $\nabla.(F \times G) = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).(F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1)$

$$= \frac{\partial(F_2G_3 - F_3G_2)}{\partial x_1} + \frac{\partial(F_3G_1 - F_1G_3)}{\partial x_2} + \frac{\partial(F_1G_2 - F_2G_1)}{\partial x_3}$$

$$= \frac{\partial F_2}{\partial x_1}G_3 + F_2\frac{\partial G_3}{\partial x_1} - \frac{\partial F_3}{\partial x_1}G_2 - F_3\frac{\partial G_2}{\partial x_1} + \frac{\partial F_3}{\partial x_2}G_1 + F_3\frac{\partial G_1}{\partial x_2} - \frac{\partial F_1}{\partial x_2}G_3 - F_1\frac{\partial G_3}{\partial x_2}$$

$$+ \frac{\partial F_1}{\partial x_3}G_2 + F_1\frac{\partial G_2}{\partial x_3} - \frac{\partial F_2}{\partial x_3}G_1 - F_2\frac{\partial G_1}{\partial x_3}$$

$$= G.Curl\,F - F(Curl G)$$

Right side $= (G_1 i + G_2 j + G_3 k).[(\frac{\partial}{\partial x_2}F_3 - \frac{\partial}{\partial x_3}F_2)i + (\frac{\partial}{\partial x_3}F_1 - \frac{\partial}{\partial x_1}F_3)j + (\frac{\partial}{\partial x_1}F_2 - \frac{\partial}{\partial x_2}F_1)k]$

$- (F_1 i + F_2 j + F_3 k).[(\frac{\partial}{\partial x_2}G_3 - \frac{\partial}{\partial x_3}G_2)i + (\frac{\partial}{\partial x_3}G_1 - \frac{\partial}{\partial x_1}G_3)j + (\frac{\partial}{\partial x_1}G_2 - \frac{\partial}{\partial x_2}G_1)k]$

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If $f \in C^2$ is a scalar real valued function in R^3 , and F is vector valued function on R^3 ,then

$$Curl\,(grad\,f) = \nabla \times (\nabla f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{vmatrix} =$$

$$(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_2})i - (\frac{\partial^2 f}{\partial x_1 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_1})j + (\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1})k = 0,$$

because the mixed partial derivatives are equal $(f \in C^2)$ also,

$$div\ (Curl\ F) = \nabla.(\nabla \times F) == (\frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_j} k).\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} =$$

$$\frac{\partial}{\partial x_1}[\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}] + \frac{\partial}{\partial x_2}[\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}] + \frac{\partial}{\partial x_3}[\frac{\partial F_3}{\partial x_1} - \frac{\partial F_2}{\partial x_2}] = 0$$

$$Scalar\ function \xrightarrow{grad} vector\ function \xrightarrow{Curl} vector\ function \xrightarrow{div} scalar\ function.$$

If f is a scalar real valued function on R^n, then The Laplacian of f is denoted by $\nabla^2 f\ or\ \Delta f$

$$\nabla^2 f = div(grad\ f) = \nabla.(\nabla\ f)$$

$$= (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}).(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n})$$

$$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + ... + \frac{\partial^2 f}{\partial x_n^2}$$

The Laplacian for a vector function F where $F \in R^3$

$$\begin{aligned} \nabla^2 F &= grad\,(div\,F) - Curl\,(Curl\,F) \\ &= \nabla(\nabla.F) - \nabla \times (\nabla \times F) \\ &= \nabla^2 F_1 i + \nabla^2 F_2 j + \nabla^2 F_3 k \end{aligned}$$

Proof : $grad\,(div\,F) = \nabla(\nabla.F) = \nabla(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3})$

$$= (\frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_1 \partial x_2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3})i + (\frac{\partial^2 F_1}{\partial x_1 \partial x_2} + \frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_3}{\partial x_2 \partial x_3})\ j + (\frac{\partial^2 F_1}{\partial x_1 \partial x_3} + \frac{\partial^2 F_2}{\partial x_2 \partial x_3} + \frac{\partial^2 F_3}{\partial x_3^2})k$$

$$Curl\ (Curl\ F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} & \frac{\partial F_3}{\partial x_3} - \frac{\partial F_3}{\partial x_1} & \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{vmatrix}$$

و بعد التجميع ينتج أن

$$\nabla^2 F = \nabla(\nabla . F) - \nabla \times (\nabla \times F) = \nabla^2 F_1 i + \nabla^2 F_2 j + \nabla^2 F_3 k .$$

Ex. 1,2,3,….,9

CH6 infinite series

6.1 Definitions and examples

$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \dots\dots\dots\dots$ (infinite series)

Where a_n can be real no. , complex no. ,vectors , …..

$s_0 = a_0 , s_1 = a_0 + a_1$, ..., $s_k = a_0 + a_1 + \dots\dots + a_k$ are called partial sums

The seq. $\{ S_n \}$ is called a seq. of partial sums

The series $\sum_{n=0}^{\infty} a_n$ conv. if the seq. $\{ S_n \}$ of partial sums conv.

If $\lim_{n \to \infty} S_n = s \Rightarrow \sum_{n=0}^{\infty} a_n = s$

TH: a) If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are conv. with sums S and T , then $\sum_{n=0}^{\infty} (a_n + b_n)$ is conv. with sums S+T

b) If the series $\sum_{n=0}^{\infty} a_n$ is convergent , with sum S, then for any $c \in R$ the series $\sum_{n=0}^{\infty} c a_n$ is conv. with sum cS .

c) If the series $\sum_{n=0}^{\infty} a_n$ is conv. then $\lim_{n \to \infty} a_n = 0$ equivalenty , if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=0}^{\infty} a_n$ divergent .

Proof: let $\{s_k\}$ and $\{t_k\}$ be the seq. of partial sums of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ resp. if $\lim_{k\to\infty} s_k = S$ and $\lim_{k\to\infty} t_k = T$ then $\lim_{k\to\infty} s_k + t_k = S + T$ and $\lim_{k\to\infty} cs_k = cS$

$\Rightarrow a$ *and* b *are follows*

From c) we observe that $a_n = S_n - S_{n-1}$ .If the series converges to the sum S ,it follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = S - S = 0$

Geometric series $\sum_{n=0}^{\infty} a\,x^n = a + ax + ax^2 + \ldots\ldots\ldots\ldots$ is called G.S with first term a , and x is the base or the ratio of the series.

The k-th partial sums of $\sum_{n=0}^{\infty} a\,x^n$

$S_k = a + ax + ax^2 + \ldots\ldots\ldots + ax^k$

$xS_k = ax + ax^2 + ax^3 + \ldots\ldots\ldots + ax^{k+1}$

$(1-x)S_k = a - ax^{k+1} = a(1-x^{k+1})$

$S_k = \dfrac{a(1-x^{k+1})}{(1-x)} \quad x \neq 1$

If $|x| < 1$,then $\lim_{k\to\infty} x^{k+1} = 0$

$\lim_{k\to\infty} S^k = \dfrac{a}{1-x}$ it follows that the series $\sum_{n=0}^{\infty} a\,x^n$ conv. to $\dfrac{a}{1-x}$

If $|x| \geq 1$, the series div.

TH : The geometric series $\sum_{n=0}^{\infty} a\,x^n$ conv. iff $|x| < 1$ in which case its sum is $\dfrac{a}{1-x}$

Taylor series : If $f \in C^{\infty}$ on $(-C, C)$

$f(x) = f(0) + f'(0)x + \dfrac{f''(0)x^2}{2!} + \ldots\ldots\ldots\ldots + \dfrac{f^{(k)}(0)x^k}{k!} + R_k(x)$

If $R_k(x) \to 0$ *as* $k \to \infty$, |x|<c.

The Taylor series of $f(x)$ at x = 0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)\,x^n}{n!}$$

$R_k(x) \to 0$ *as* $k \to \infty$ follows from the estimate for the taylor remainder

$$|R_k(x)| \leq \sup_{|t|<|x|} |f^{k+1}(t)| \frac{|x|^{k+1}}{(k+1)!}$$

TH : Let f be a function of class C^{∞} on the interval (-c,c) ,where $0 < c < \infty$

a) If there exist constants a,b>0 such that $|f^{k}(x)| \le ab^{k}k!$ for all |x|< c and $k \ge 0$,then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)\, x^{n}}{n!}$$ holds for $|x| < \min(c, \frac{1}{b})$

b) If there exist constants A ,B >0 such that $|f^{k}(x)| \le AB^{k}$ for all |x|< c and $k \ge 0$,then

$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)\, x^{n}}{n!}$ holds for |x|< c

Proof :a) If $|f^{k}(x)| \le ab^{k}k! \Rightarrow |R_{k}(x)| \le \frac{ab^{k+1}(k+1)!\,|x|^{k+1}}{(k+1)!} \le a|bx|^{k+1}$

For |x| < c , If $|x| < b^{-1} \Rightarrow |xb| < 1 \Rightarrow |xb|^{k+1} \to 0 \quad as \quad k \to \infty$

$\Rightarrow \lim_{k\to\infty} R_{k}(x) = 0$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)\, x^{n}}{n!}$

b) $\frac{C^{k}}{k!} \to 0 \quad as \quad k \to \infty$ so ,for any positive A ,B and b ,the seq. $\frac{A(B/b)^{k}}{k!} \to 0 \quad as \quad k \to \infty$

Let a be the largest term of the set

$\Rightarrow AB^{k} = [\frac{A(B/b)^{k}}{k!}]\, b^{k} k! \le ab^{k}k!$

So the estimate $|f^{k}(x)| \le AB^{k}$, for a given A and B implies the estimate $|f^{k}(x)| \le ab^{k}k!$ for every b > 0 (with a depending on b). Hence (b) follows from (a).

Example 1: $f(x) = \cos x$

$f^{k}(x) = \pm \cos x \quad or \quad \pm \sin x$

$t \in (0, x)$ so $|f^{k}(x)| \le 1$ for all x

$$R_{0,k}(x)=\frac{f^{k+1}(t)\,x^{2k+1}}{(2k+1)!}\Rightarrow |R_{0,k}(x)|\le\frac{x^{2k+1}}{(2k+1)!}\Rightarrow R_{0,k}(x)\to 0$$

And cos x conv. to its taylor series $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

Similarly for $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Example 2: $f(x)=e^x$

$$f^k(x)=e^x \quad \text{for all } k$$
$$\text{for } |x|<c \Rightarrow |f^k(x)|<e^c$$

$$R_k(x)=\frac{f^{k+1}(t)\,x^{k+1}}{(k+1)!}\le\frac{e^c x^{k+1}}{(k+1)!}\Rightarrow R_k(x)\to 0 \ \text{ as } \ k\to\infty$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{for } |x|<c \quad \text{but } c \text{ is arbitrary for all } x$$

Ex. 1,2,3

EX :1 d) Find the values of x for which each of the following series converges and compute its sum

$$\log x+(\log x)^2+(\log x)^3+\ldots\ldots\ldots\ldots+(\log x)^n$$

Solution : $|\log x|<1\Rightarrow -1<\log x<1\Rightarrow e^{-1}<x<e^{1}$

2) Tell whether each of the following series converges if it does , Find its sum

a) $1+\frac{3}{4}+\frac{5}{8}+\frac{9}{16}+\frac{17}{32}\ldots\ldots\ldots\ldots=\sum_{n-0}^{\infty}\frac{2^n+1}{2.2^n}$

$a_n=\frac{2^n+1}{2.2^n}\to\frac{1}{2}\ne 0$ so the series div.

c) $(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})\ldots\ldots\ldots\ldots\ldots\ldots$

$S_n=\sqrt{n+1}-\sqrt{1}\to\infty$ so the series div.

3) Let $f(x)=\log(1+x)$ show that the Taylor Lagrange remainder $R_{0,k}(x)$ tends to zero as $k\to\infty$ for $-1<x\le 1$, and conclude that $\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}$ for $-1<x\le 1$

Solution : we use the lagrange Remainder formula for $R_{0,k}(x)$

$$R_{0,k}(x)=\frac{f^{k+1}(c)\,x^{k+1}}{(k+1)!}\quad c\in(0,x)\ \textit{when}\ -0.5<x\le 1$$

$$f(x)=\log(1+x),\ f^{'}(x)=\frac{1}{1+x}\quad ,f^{'}(0)=1$$

$$f^{''}(x)=-(x+1)^{-2}\Rightarrow f^{''}(0)=-1$$

$$f^{'''}(x)=2(x+1)^{-3}\Rightarrow f^{'''}(0)=2$$

$$f^{(4)}(x)=(-1)^3 3!(x+1)^{-4}\Rightarrow f^{(4)}(0)=6$$

$$f^{(n+1)}(x)=(-1)^n n!(x+1)^{-(n+1)}\Rightarrow f^{(k+1)}(c)=(-1)^k k!(c+1)^{-(k+1)}$$

$$|R_{0,k}(x)|\le\frac{k!\,|x|^{k+1}}{(c+1)^{k+1}(k+1)!}\quad ,\left|\frac{x}{c+1}\right|<1\quad \textit{for}\ -0.5<x\le 1\Rightarrow\left|\frac{x}{c+1}\right|^{k+1}\to 0$$

$$|R_{0,k}(x)|\to 0$$

for $-1<x\le -0.5$ we use the formula

$$R_{0,k}(h)=\frac{h^{k+1}}{k!}\int_0^1(1-t)^k f^{(k+1)}(th)dt$$

Let u =th , $0<t\le 1$

$$\int_0^1(1-t)^k f^{(k+1)}(th)dt=\int_0^h(1-\frac{u}{h})^k f^{(k+1)}(u)\frac{du}{h}$$

By the mean value theorem of integration $\exists$ a number

$$u' \in (u,0) \ni \int_0^h (1-\frac{u}{h})^k f^{(k+1)}(u)\frac{du}{h} = h.(1-\frac{u'}{h})^k \frac{(-1)^k k!(u'+1)^{-k-1}}{h} = \frac{(h-u')^k (-1)^k k!(u'+1)^{-k-1}}{h^k}$$

$$|R_{0,k}(x)| \le \frac{x^{k+1}(x-x')^k (x'+1)^{-k-1}}{x^k} = |x||x-x'||x'+1|^{-k-1} = \frac{|x||x-x'|^k}{|x'+1|^{k+1}}$$

$$\le \frac{|x||x-x'|^k}{|x'+1||x'+1|^k}, |x|<1 \Rightarrow \frac{|x-x'|}{|x'+1|} < |x| \Rightarrow \frac{|x-x'|}{|x+x|} \le |x|^k \Rightarrow$$

$$|R_{0,k}(x)| \le \frac{|x|^k|x|}{|x'+1|} \text{ for } (x,0),\ x' \in (x,0),\ x<x' \Rightarrow x+1<x'+1 \Rightarrow \left|\frac{1}{x'+1}\right| < \left|\frac{1}{1+x}\right|$$

$$|R_{0,k}(x)| \le \frac{|x|^{k+1}}{|x'+1|} \le \frac{|x|^{k+1}}{|1+x|} \to 0 \text{ as } k \to \infty.$$

6.2 Series with Non negative terms

The integral test

If $a_n = f(n)$ where f is a function of a real variable , a sum $\sum_j^k a_n$ can be compared to an integral $\int_j^k f(x)dx$

Theorem : Suppose f is a positive decreasing function on the half – line $[a,\infty)$, Then for any integers j, k with $a \le j \le k$,

$$\sum_{n=j}^{k-1} f(n) \ge \int_j^k f(x)dx \ge \sum_{n=j+1}^{k} f(n)$$

Proof: Since f is decreasing , for $n \le x \le n+1$ we have $f(n) \ge f(x) \ge f(n+1)$

And hence $f(n) = \int_n^{n+1} f(n)dx \ge \int_n^{n+1} f(x)dx \ge \int_n^{n+1} f(n+1)dx = f(n+1)$ adding up these inequalities from

n = j to n= k-1 we obtain the asserted $\sum_{n=j}^{k-1} f(n) \ge \int_j^k f(x)dx \ge \sum_{n=j+1}^{k} f(n)$

$$f(j) \geq \int_{j}^{j+1} f(x)dx \geq f(j+1)$$

$$f(j+1) \geq \int_{j+1}^{j+2} f(x)dx \geq f(j+2)$$

$$f(j+2) \geq \int_{j+2}^{j+3} f(x)dx \geq f(j+3)$$

$\vdots$

$$\sum_{n=j}^{k-1} f(n) \geq \int_{j}^{k} f(x)dx \geq \sum_{n=j+1}^{k} f(n).$$

$$\Rightarrow \sum_{n=2}^{k+1} f(n) \leq \int_{1}^{k+1} f(x)dx \leq \sum_{n=1}^{k} f(n).$$

Corollary : (The integral test) Suppose f is a positive decreasing function on the half -line $[1,\infty)$,Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral $\int_{1}^{\infty} f(x)dx$ converges.

Proof : Let $S_k = \sum_{n=1}^{k} f(n)$. If $\int_{1}^{\infty} f(x)dx < \infty$ we have

$$S_k = f(1) + \sum_{n=2}^{k} f(n) \leq f(1) + \int_{1}^{k} f(x)dx \leq f(1) + \int_{1}^{\infty} f(x)dx.$$

So the partial sums are bounded above and hence the series converges. On the other hand if $\int_{1}^{\infty} f(x)dx = \infty$ we have $S_k = \sum_{n=1}^{k-1} f(n) + f(k) \geq \int_{1}^{k} f(x)dx + f(k) \to \infty \ as \quad k \to \infty$

So the series div.

Theorem : The series $\sum_{n=1}^{\infty} n^{-p}$ converges if $p>1$ and div. if $p \le 1$

$$\int_1^{\infty} x^{-p} dx = \lim_{k\to\infty} \frac{x^{1-p}}{1-p} = \begin{cases} (p-1)^{-1} & if \quad p>1 \\ \infty & if \quad p<1 \end{cases}$$

And $\int_1^{\infty} x^{-1} dx = \lim_{k\to\infty} \log x |_1^k = \infty$

General Comparasion tests

Theorem : Suppose $0 \le a_n \le b_n$ for $n \ge 0$

If $\sum_{n=0}^{\infty} b_n$ *conv.* , *then so does* $\sum_{n=0}^{\infty} a_n$

If $\sum_{n=0}^{\infty} a_n$ *div.* , *then so does* $\sum_{n=0}^{\infty} b_n$

Proof: Let $S_k = \sum_{n=0}^{k} a_n$ *and* $t_k = \sum_{n=0}^{k} b_n$ thus $0 \le S_k \le t_k$ for all k . If $\sum_{n=0}^{\infty} b_n$ conv. Then the seq.{ t_k } is bounded set , hence so the seq. { S_k } ⇒ The seq . { S_k } conv..
By monotone seq. theorem this proves this first assertion , to which the second one is logically equivalence .

Example : The series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ div.

Because $\frac{1}{2n-1} \ge \frac{1}{2n}$ *for* $n>1$ because $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ *div.*

Th.(The limit comparison test) : suppose { a_n } and { b_n } are seq of positive numbers and that $\frac{a_n}{b_n}$ approaches a positive , finite limit as $n \to \infty$, then the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are either both convergent or both divergent .

Proof: If $\frac{a_n}{b_n} \to l$ as $n \to \infty$,where $0 < l < \infty$, we have $\frac{1}{2} l < \frac{a_n}{b_n} < 2l$ when n is large.
That is $a_n < 2l\, b_n$ and $b_n < (2/l)\, a_n$
The result therefore follows from previous Th. and the remark s following it .

Example 2) $\sum_{n=1}^{\infty}(n^2 - 6n + 10)^{-1}$

$a_n = \frac{1}{n^2 - 6n + 10}$ *behave like* $\frac{1}{n^2}$

$b_n = \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$ *conv. and* $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$

The series $\sum_{n=1}^{\infty} a_n$ conv. by limit comparison test.

Extension of previous theorem :

If $\frac{a_n}{b_n} \to 0$ *as* $n \to \infty$ *then* $a_n < b_n$ for large n , so the convergence of $\sum_{n=0}^{\infty} b_n$ will imply the conv. of $\sum_{n=0}^{\infty} a_n$, also if $\frac{a_n}{b_n} \to \infty$ *as* $n \to \infty$ *then* $a_n > b_n$ for large n , so the divergence of $\sum_{n=0}^{\infty} a_n$ will imply the div. of $\sum_{n=0}^{\infty} b_n$

Th: (The ratio test) Suppose $\{a_n\}$ is a sequence of positive numbers

a) If $\frac{a_{n+1}}{a_n} < r$ for all sufficiently large n , where r < 1 ,then the series $\sum_{n=0}^{\infty} a_n$ converges. On the other hand , if $\frac{a_{n+1}}{a_n} \geq 1$,for all sufficiently large n, then the series $\sum_{n=0}^{\infty} a_n$ diverges

b) Suppose that $l = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ exists . Then the series $\sum_{n=0}^{\infty} a_n$ converges if $l < 1$ and diverges if $l > 1$, No conclusion can be drawn if $l = 1$

Proof: Suppose $\frac{a_{n+1}}{a_n} < r < 1$ for all $n \geq N$, Then

$a_{N+1} < r\,a_N$, $a_{N+2} < r\,a_{N+1} < r^2 a_N$, $a_{N+3} < r\,a_{N+2} < r^2 a_{N+1} < r^3 a_N$.

So $a_{N+m} < r^m a_N$ for all $m \geq 0$,The series $\sum_{n=0}^{\infty} a_n$ therefore converges by comparison to the Geometric series $\sum_{n=0}^{\infty} r^m$

$$\sum_{n=0}^{\infty} a_n < a_0 + \ldots\ldots\ldots\ldots + a_{N-1} + a_N(1 + r + r^2 + \ldots\ldots\ldots\ldots\ldots\ldots) < \infty$$

On the other hand , if $\frac{a_{n+1}}{a_n} \geq 1$ *then* $a_{n+1} \geq a_n$ if this is , So for all $n \geq N$ *then* $\lim_{n\to\infty} a_n \neq 0$.

So $\sum_{n=0}^{\infty} a_n$ can not converges , This prove (a) .

b) If $l < 1$, choose r with $l < r < 1$, if $l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, then $\frac{a_{n+1}}{a_n} < r$ for large n , so $\sum_{n=0}^{\infty} a_n$ converges by part (a) If $l > 1$, then $\frac{a_{n+1}}{a_n} \geq 1$ for large n , so $\sum_{n=0}^{\infty} a_n$ div.

Finally , if we take $a_n = n^{-p}$, we know that $\sum_{n=0}^{\infty} a_n$ converges if $p > 1$ and diverges if $p \leq 1$ but $\frac{a_{n+1}}{a_n} = [\frac{n}{n+1}]^p \to 1$, no matter what p is

Hence the test is inconclusive if $l = 1$.

TH : (The root test) Suppose $\{a_n\}$ is a seq. of positive numbers

a) If $a_n^{1/n} < r$ for all sufficiently large n , where $r < 1$, then the series $\sum_{n=0}^{\infty} a_n$ converges . On the other hand if $a_n^{1/n} \geq 1$ for all sufficiently large n , then the series $\sum_{n=0}^{\infty} a_n$ diverges .

b) Suppose that $l = \lim_{n\to\infty} a_n^{1/n}$ exists . Then the series $\sum_{n=0}^{\infty} a_n$ conv. if $l < 1$ and diverges if $l > 1$. No conclusion can be drawn if $l = 1$

Proof : If $a_n^{1/n} < r$, we have $a_n < r^n$, So we have an immediate comp to G.S $\sum_{n=0}^{\infty} r^m$ that given the convergence of $\sum_{n=0}^{\infty} a_n$ where r<1.

If $a_n^{1/n} \geq 1$, then $a_n \geq 1$ So, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=0}^{\infty} a_n$ div. (This prove (a)) .

Part(b) follows as in the proof of the ratio test

If $a_n^{1/n} \to l < 1$, let $r \in (l,1)$ So for large n , $a_n^{1/n} < r < 1$, so $\sum_{n=0}^{\infty} a_n$ conv.

If $a_n^{1/n} \to l > 1$, then $a_n^{1/n} \geq 1$ for large n and $\sum_{n=0}^{\infty} a_n$ div.

Finally . for $a_n = n^{-p}$ we have $a_n^{1/n} = n^{\frac{-p}{n}} \to 1$ for any p as $n \to \infty$, so the test is in conclusive when $l = 1$

Ex 7) : $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ Determine whether the series. conv. or div.

$$l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} . \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n+1}{10} = \infty > 1$$

So the series div. by ratio test .

Ex.12) $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$

By root test

$$l = \lim_{n\to\infty} ((\frac{n}{n+1})^{n^2})^{1/n} = \lim_{n\to\infty} (\frac{n}{n+1})^n = \lim_{n\to\infty} \frac{1}{(\frac{n+1}{n})^n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$$

So the series conv. by root test.

TH : (Raabe's Test) Let $\{a_n\}$ be a seq. of positive numbers suppose that

$$\frac{a_{n+1}}{a_n} \to 1 \quad and \quad n[1 - \frac{a_{n+1}}{a_n}] \to L \ as \ n \to \infty$$

If $l>1$, then the series $\sum_{n=0}^{\infty} a_n$ conv. and if $l<1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges (If $L=1$, no Conclusion can be drawn)

Proof : If $l>1$, choose a number p with $1<p<L$,then when n is large , we have $n[1-\frac{a_{n+1}}{a_n}]>p$, that is $\frac{a_{n+1}}{a_n}<1-\frac{p}{n}$.

Since, $\frac{(n+1)^{-p}}{n^{-p}}=[1+\frac{1}{n}]^{-p}=1-\frac{p}{n}+E_n$, where $0<E_n<\frac{p(p+1)}{2n^2}$ …..(1)

Then $\frac{a_{n+1}}{a_n}<1-\frac{p}{n}<\frac{(n+1)^{-p}}{n^{-p}}$ or $\frac{a_{n+1}}{(n+1)^{-p}}<\frac{a_n}{n^{-p}}$.

Thus the seq. $\{\frac{a_n}{n^{-p}}\}$ is dec. , so it is bounded above by a constant C . In other words , $a_n \le C n^{-p}$, So Since p >1 , $\sum_{n=0}^{\infty} a_n$ converges by comparison to $\sum_{n=0}^{\infty} n^{-p}$

On the other hand , if $l<1$, choose numbers p and q with L<q<p<1.

Then , when n is large , we have $n[1-\frac{a_{n+1}}{a_n}]<q$ that is $\frac{a_{n+1}}{a_n}>1-\frac{q}{n}$

If also $n>\frac{p(p+1)}{2(p-q)}$, we have $\frac{p(p+1)}{2n^2}<\frac{p-q}{n}$.

So by (1) $\frac{a_{n+1}}{a_n}>1-\frac{q}{n}=1-\frac{q}{n}+\frac{p}{n}-\frac{p}{n}=1-\frac{p}{n}+\frac{p-q}{n}>1-\frac{p}{n}+E_n=\frac{(n+1)^{-p}}{n^{-p}}$, since $\frac{p-q}{n}>\frac{p(p+1)}{2n^2}>E_n$.

Thus $\frac{(n+1)^{-p}}{a_{n+1}}<\frac{n^{-p}}{a_n}$ So the seq. $\{\frac{n^{-p}}{a_n}\}$ is dec.

As before , this gives $n^{-p}\le C\,a_n$ and p <1 in this case , so $\sum_{n=0}^{\infty} a_n$ diverges by comparison to $\sum_{n=0} n^{-p}$.

Ex.17) $\sum_{n=1}^{\infty}\frac{1.3..........(2n-1)}{4.6............(2n+2)}$

Solution : $l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1.3...............(2n-1).(2n+1)}{4.6...............(2n+2).(2n+4)} \cdot \frac{4.6.................(2n+2)}{1.3.................(2n-1)}$

$= \lim_{n\to\infty} \frac{2n+1}{2n+4} = 1$

and $n[1 - \frac{a_{n+1}}{a_n}] = n[1 - \frac{2n+1}{2n+4}] = n[\frac{2n+4-2n-1}{2n+4}] = \frac{n[3]}{2n+4} \to \frac{3}{2} > 1$

By raabe's test , the S. conv.

Ex.19) Suppose $a_n > 0$ Show that if $\sum_{n=0}^{\infty} a_n$ conv. ,then so does $\sum_{n=0}^{\infty} a_n^p$ for any p >1

Let $b_n = a_n^p$

$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{a_n^p}{a_n} = \lim_{n\to\infty} a_n^{p-1} \to 0$

So by limit comp. test The S .conv.

6.3 Absolute and Conditional convergence

Def: A series $\sum_{n=0}^{\infty} a_n$ is called abs. conv. if the series $\sum_{n=0}^{\infty} |a_n|$ conv**.**

Theorem : Every absolutely convergent series is convergent .

Proof: Let $s_k = \sum_{n=0}^{k} a_n$, and $S_k^{'} = \sum_{n=0}^{k} |a_n|$,The seq. $\{S_k^{'}\}$ is conv. and hence Cauchy.

So given $\varepsilon > 0$, there exist an integer K such that $|a_{j+1}| + \ldots\ldots\ldots + |a_k| = S_k^{'} - S_j^{'} < \varepsilon$ whenever k > j>K.

But then

$|s_k - s_j| = |a_{j+1} + \ldots\ldots\ldots + a_k| \leq |a_{j+1}| + \ldots\ldots\ldots + |a_k| < \varepsilon$ whenever $k > j \geq K$

So the seq. $\{s_k\}$ is cuachy seq., so conv. and hence the series $\sum_{n=0}^{\infty} a_n$ is conv.

Remark : The converse of the above theorem is false

Def: A series that conv. but does not conv. absolutely is said to be convergent conditionally

Example : $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

This series is conditionally conv. because $\sum_{n=1}^{\infty} \frac{1}{n}$ div.

Example: Let

$f(x) = \log(1+x)$, for n>0, $f^{(n)}(x) = (-1)^n (n-1)!(1+x)^n$ and

$f^{(n+1)}(x) = (-1)^{(n+1)} n!(1+x)^{-n-1}$, so $f^{(n)}(0) = (-1)^n (n-1)!$ and the Taylor series of $f(x)$ is given by,

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n + R_k(x)$$, where

$$|R_k(x)| \leq \frac{1}{(k+1)!} \sup_{0 \leq t \leq 1} \left| \frac{(-1)^k k!}{(1+t)^{k+1}} \right| = \frac{1}{1+k} \to 0 \quad as \quad k \to \infty \quad for \quad -1 < t \leq 1,$$ so

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n .$$

$$\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1^n$$

It follows that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ conv. to log (2)

Ex. $\sum_{n=1}^{\infty} \frac{\sin nt}{n^2}$ show the series conv absolutely

Sol: $\left|\frac{\sin nt}{n^2}\right| \leq \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv., so $\sum_{n=1}^{\infty} \frac{\sin nt}{n^2}$ abs. conv.

Let $a_n^+ = \max(a_n, 0),\ a_n^- = \min(-a_n, 0)$

That is $a_n^+ = a_n$ if a_n is positive , and $a_n^+ = 0$ otherwise and $a_n^- = |a_n|$ if a_n is negative and $a_n^- = 0$ otherwise, the nonzero a_n^+'s are the positive terms of $\sum_{n=0}^{\infty} a_n$ and the non zero a_n^- are the absolute values of the negative terms. So $a_n^+ - a_n^- = a_n$ and $a_n^+ + a_n^- = |a_n|$

TH : If $\sum_{n=0}^{\infty} a_n$ is abs. conv. , the series $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ are both conv. If $\sum_{n=0}^{\infty} a_n$ is conditionally conv. then the series $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ are both divergent

Proof: The theorem follows from the followeing three facts

1) The conv. of $\sum_{n=0}^{\infty} |a_n| \Rightarrow$ The conv. of $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$
2) The div. of $\sum_{n=0}^{\infty} |a_n| \Rightarrow$ The div. of at least one of $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$
3) If $\sum_{n=0}^{\infty} a_n$ conv. conditionally it can not happen that one of $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ conv. while the other one div.

Proof : 1) Since $0 \le a_n^+ \le |a_n|$ and $0 \le a_n^- \le |a_n|$

If $\sum_{n=0}^{\infty} |a_n|\ Conv. \Rightarrow$ both $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ conv.

Proof : 2)since $a_n^+ + a_n^- = |a_n|$ if $\sum_{n=0}^{\infty} |a_n|$ div. $\Rightarrow$ at least one of $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ div.

Proof : 3) Let $S_k = \sum_{n=1}^{k} a_n$, $S_k^{\pm} = \sum_{n=1}^{k} a_n^{\pm}$ be the kth partial sums $\Rightarrow S_k^+ - S_k^- = S_k$.

Suppose that $\sum_{n=1}^{\infty} a_n^+ = \infty$ while $\sum_{n=1}^{\infty} a_n^- = S < \infty$, then for any C>0 for large k we have $S_k^+ > C + S$ while $S_k^- \le S$, so that $S_k > C + s - s = C \Rightarrow S_k \to \infty$

So $\sum_{n=0}^{\infty} a_n$ div.

Rearrangement of $\sum_{n=0}^{\infty} a_n$ **:**

If $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \ldots\ldots\ldots\ldots\ldots\ldots$ then if we forming a new series by writing the terms in a different order such as $a_0, a_2, a_1, a_4, a_6, a_3, a_8, a_{10}, a_5$, this is called a rearrangement of $\sum_{n=0}^{\infty} a_n$.

In general if σ is any one to one mapping from the set of nonnegative integers onto it self , we can form the series $\sum_{n=0}^{\infty} a_{\sigma(n)}$, which we call a rearrangement of $\sum_{n=0}^{\infty} a_n$.

TH: If $\sum_{n=0}^{\infty} a_n$ is abs. conv. with sum S , then every rearrangement $\sum_{n=0}^{\infty} a_{\sigma(n)}$ is also abs. conv. with sum S.

TH : Suppose $\sum_{n=0}^{\infty} a_n$ is conditionally conv. Given any real numbers S , there is a rearrangement $\sum_{n=0}^{\infty} a_{\sigma(n)}$ that conv. to S .

Ex. 1,2,3,4

Ex 3) Consider the rearrangement of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ obtained by taking two positive terms , one negative term ,two positive terms , one negative term and so forth

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+..............$$

Show that the sum of this series is $\frac{3}{2}\log(2)$

(Hint: Deduce from Example 2 that $0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+..........=\frac{1}{2}\log(2)$)

Solution : $(0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+..........)=\frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}+...............)=\frac{1}{2}\log(2)$

Since $\log(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}.......$

$$\frac{1}{2}\log(2)=0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\frac{1}{10}-..........$$

$$\log(2)+\frac{1}{2}\log(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+..........=\frac{3}{2}\log(2).$$

6.4 More convergence Tests

TH : a) If $|a_n| \le C\, n^{-1-\varepsilon}$ for some $C, \varepsilon > 0$, then $\sum_{n=0}^{\infty} a_n$ conv. abs.

If $|a_n| \ge C\, n^{-1}$ for some $C > 0$, then $\sum_{n=0}^{\infty} a_n$ either converges conditionally or div .

b) (The ratio test) if $\left|\frac{a_{n+1}}{a_n}\right| \to l \;\; as \;\; n \to \infty$ then $\sum_{n=0}^{\infty} a_n$ converges abs. if $l < 1$ and div. if $l > 1$

c) (The root test) If $|a_n|^{\frac{1}{n}} \to l \quad as \;\; n \to \infty$, then $\sum_{n=0}^{\infty} a_n$ conv. abs. if $l < 1$ and div. if $l > 1$

Example : $\sum_{n=0}^{\infty} (-2)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1}}{(-2)^n}\right| = 2 > 1 \Rightarrow The \;\; Series.\, div.$$

Example : $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

$\ln n < n^{\varepsilon} \;\; for \;\; \varepsilon > 0$ and large n $\Rightarrow \frac{\ln n}{n^2} < \frac{n^{0.5}}{n^2} = \frac{1}{n^{3/2}}$ and

$$\lim_{n\to\infty} \frac{\ln n}{n^2} \Big/ \frac{1}{n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{1}{n} \Big/ \frac{1}{2} n^{-\frac{1}{2}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{2}}} = 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conv. so by limit comp. test $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ conv.

TH : (The alternating series test)

Suppose the seq $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent . Moreover , If S_k and S denote the kth partial sum and the full sum of the series , we have $S_k > S$ for k even , $S_k < S$ for k odd and $|S_k - S| < a_{k+1}$ for all k

The summery of the test :

1) $\lim_{n\to\infty} a_n = 0$ 2) $a_n \ge a_{n+1} \;\; for \;\; n > N$ 3) $a_n \ge 0 \;\; for \;\; all \;\; n > N$

Proof: Since $a_k \ge a_{k+1} \;\; \forall all \;\; k$, we have

$S_{2m+1} = S_{2m-1} + a_{2m} - a_{2m-1} \geq S_{2m-1}$ *because* $(a_{2m} - a_{2m+1} \geq 0)$,

$S_{2m+2} = S_{2m} - a_{2m+1} + a_{2m+2} \leq S_{2m}$ because $(-a_{2m+1} + a_{2m+2} \leq 0)$.

Thus the seq. $\{S_{2m-1}\}$ of odd numbered partial sums is increasing and the seq. $\{S_{2m}\}$ of even numbered partial sums is decreasing . This monotonicity implies that

$S_{2m-1} = S_{2m-2} - a_{2m-1} \leq S_{2m-2} \leq S_0$ *because* $(-a_{2m-1} \leq 0$ *and* $\{S_{2m}\}$ *is* decreasing seq.)

Also $S_{2m} = S_{2m-1} + a_{2m} \geq S_{2m-1} \geq S_1$, *because* $(a_{2m} \geq 0)$.

So , $\{S_{2m-1}\}$ and $\{S_{2m}\}$ are bounded above and below resp. . So by the monotone seq. theorem these seq. both conv. and since $S_{2m} - S_{2m-1} = a_{2m} \to 0 \Rightarrow$ $\lim_{m\to\infty} S_{2m} = \lim_{m\to\infty} S_{2m-1}$ are equal, so the whole seq. $\{S_k\}$ also conv. and hence the series $\sum_{n=0}^{\infty} (-1)^n a_n$ conv.

The even – numbered partial sums decrease to the full sum S , while the odd-numbered ones increase . So $S < S_{2m}$ *and* $S > S_{2m-1}$ for all m. In particular ,

$0 < S - S_{2m-1} < S_{2m} - S_{2m-1} = a_{2m}$,

And $0 < S_{2m} - S < S_{2m} - S_{2m+1} = a_{2m}$, so $|S_k - S| < a_{k+1}$ where k is even or odd

Remark : $\{a_n\}$ is called monotone seq. If $a_n \geq a_{n+1} \;\; \forall n \geq N$ *or* $a_n \leq a_{n+1} \;\; \forall n \geq N$

Example : $\sum_{n=1}^{\infty} (-1)^n (e^{1/n} - 1)$ conv .by alternating series test .

Because 1) $\lim_{n\to\infty} (e^{1/n} - 1) \to 0$

2) $(e^{1/n} - 1) \geq (e^{1/n+1} - 1)$ *for* $n \geq 1$

3) $(e^{1/n} - 1) > 0$ *for all* $n \geq 1$

The conv. is conditionally , because

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

$e^{1/n} = 1 + \frac{1}{n} + \frac{1}{2!n^2} + ... \approx 1 + \frac{1}{n}$ for large n

$e^{1/n} - 1 = \frac{1}{n} + R(\frac{1}{n})$ for large n

$\sum_{n=1}^{\infty} (e^{1/n} - 1) = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} R(\frac{1}{n})$

$\sum_{n=1}^{\infty} \frac{1}{n}$ div. and $\sum_{n=1}^{\infty} R(\frac{1}{n})$ conv . by comp. test

$\sum_{n=1}^{\infty} (e^{1/n} - 1)$ div.

But $\sum_{n=1}^{\infty} (-1)^n (e^{1/n} - 1)$ conv. conditionally by alternating series test.

Interval of conv. for power series:

$\sum_{n=}^{\infty} C_n (x-a)^n$ we use the ratio or nth root test

$l = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{C_{n+1}}{C_n}| |x-a| < 1$

Example: $\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{(n+1)\, 2^{2n+1}}$

$l = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-1)^{n+1} (x-3)^{n+1} / (n+2)\, 2^{2n+3}}{(-1)^n (x-3)^n / (n+1)\, 2^{2n+1}}| = \frac{n+1\,|x-3|}{(n+2)\; 4} \to |\frac{x-3}{4}|$ as n $\to \infty$.

If $\frac{|x-3|}{4} < 1 \Rightarrow |x-3| < 4$, then the series conv. abs . If $|x-3| > 4$, then the series div.

$-4 < x-3 < 4 \Rightarrow -1 < x < 7$

At end points

$x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (-4)^n}{(n+1)\, 2^{2n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n+1}$ div.

$x = 7 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (4)^n}{(n+1)\, 2^{2n+1}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ conv.

$\Rightarrow -1 < x \le 7$ the interval of conv. and radius of conv. is $R = 4$

Ex 6.4 1,2,3,…..,14, 16-18.

EX 6.4) Determine the values of x at which the series converges absolutely or conditionally .

1) $\sum_{n=0}^{\infty} \frac{(x+2)^n}{n^2+1}$

By ratio test

$$l = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x+2)^n} \right| = |x+2| \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = |x+2| < 1$$

$-1 < x+2 < 1 \Rightarrow -3 < x < -1$

$x = -3 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ conv. abs.

$x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n^2+1}$ conv. abs.

$\Rightarrow -3 \le x \le -1$ conv. abs.

3) $\sum_{n=0}^{\infty} \frac{x^{2n}}{1.3..........(2n+1)}$

$$l = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{1.3........(2n+3)} \cdot \frac{1.3............(2n+1)}{x^{2n}} \right| = x^2 \lim_{n\to\infty} \frac{1}{2n+3} = 0$$

$\Rightarrow$ The S . conv. abs. for all x

5) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{(2^n-3)\log(n+3)}$

$$l = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-4)^{n+1}}{(2^{n+1}-3)\log(n+4)} \cdot \frac{(2^n-3)\log(n+3)}{(-1)^n (x-4)^n} \right|$$

$$= \lim_{n\to\infty} |x-4| \frac{2^n-3}{2.2^n-3} . \lim_{n\to\infty} \frac{\log(n+3)}{\log(n+4)} = \frac{|x-4|}{2} . \lim_{n\to\infty} \frac{(n+3)}{(n+4)} = \frac{|x-4|}{2} < 1$$

$\frac{|x-4|}{2} < 1 \Rightarrow |x-4| < 2 \Rightarrow -2 < x-4 < 2 \Rightarrow 2 < x < 6$.

At $x = 2 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{(2^n-3)\log((n+3)} = \sum_{n=0}^{\infty} \frac{(2)^n}{(2^n-3)\log((n+3)}$

$$\frac{1}{n+3} < \frac{2^n}{2^n} . \frac{1}{\log((n+3)} < \frac{2^n}{(2^n-3)} . \frac{1}{\log((n+3)}$$

$\sum_{n=0}^{\infty} \frac{1}{n+3}$ div.

$x = 6 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (2)^n}{(2^n - 3)\log((n+3)}$ alternating

$\lim_{n\to\infty} \frac{(2)^n}{(2^n - 3)\log((n+3)} = 0$, $a_n \geq a_{n+1}$

The series conv.

$\Rightarrow 2 < x < 6$ conv. absolutely , conditionally at x = 6

6) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (\frac{x-1}{x+1})^n$

By the nth root test

$\lim_{n\to\infty} \frac{1}{(\sqrt{n})^{\frac{1}{n}}} |\frac{x-1}{x+1}| = \frac{|x-1|}{|x+1|}$, then the series conv. for |x-1|<|x+1| then,

$x^2 - 2x + 1 < x^2 + 2x + 1 \Rightarrow 2x > 0 \Rightarrow x > 0,$ so the series conv. abs. for x>0,

at x=0, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv. conditionaly.

Lemma : (Summation by parts) Given two numerical sequence {a_n} and {b_n} ,
Let $a_n^{'} = a_n - a_{n-1}$, $B_n = b_0 + b_1 + \ldots\ldots\ldots + b_n$

Then $\sum_{n=0}^{k} a_n b_n = a_k B_k - \sum_{n=1}^{k} a_n^{'} B_{n-1}$

Proof : we have $b_0 = B_0$, and $b_n = -B_{n-1} + B_n$ *for* $n \geq 1$ so,

$a_0 b_0 + a_1 b_1 + \ldots\ldots\ldots\ldots\ldots\ldots + a_k b_k = a_0 B_0 + a_1(-B_0 + B_1) + a_2(-B_1 + B_2) + \ldots\ldots\ldots\ldots +$
$a_k(-B_{k-1} + B_k)$

$= a_0 B_0 - a_1 B_0 + a_1 B_1 - a_2 B_1 + a_2 B_2 + \ldots\ldots\ldots\ldots - a_k B_{k-1} + a_k B_k$

$= (a_0 - a_1) B_0 + (a_1 - a_2) B_1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_k B_k$

$= -a_1^{'} B_0 - a_2^{'} B_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots - a_k^{'} B_{k-1} + a_k B_k$

$= a_k B_k - \sum_{n=1}^{k} a_n^{'} B_{n-1}$

TH : (Dirichlet Test) Let { a_n } and { b_n } be numerical seq. Suppose that the seq. { a_n } is dec. and tends to zero as $n \to \infty$, and that the sums $B_n = b_0 + b_1 + \dots\dots\dots + b_n$ are bounded in absolute value by a constant C independent of n . Then the series $\sum_{n=0}^{\infty} a_n b_n$ converges

Proof: By previous lemma $\sum_{n=0}^{k} a_n b_n = a_k B_k - \sum_{n=1}^{k} a_n^{'} B_{n-1}$, so it is enough to show that $\lim_{k\to\infty} a_k B_k$ exist and that the series $\sum_{n=0}^{\infty} a_n^{'} B_{n-1}$ conv. Since $|B_k| \le C$ and $\lim_{k\to\infty} a_k = 0$,then $|a_k B_k| \le C |a_k| \to 0 \ \ as \ \ k \to \infty$,s since the seq. { a_n } is dec. ,then we have $a_n^{'} \le 0 \quad \forall n$, so

$$|\sum_{n=1}^{k} a_n^{'} B_{n-1}| = \sum_{n=1}^{k} |a_n^{'}| \, |B_{n-1}| \le C \sum_{n=1}^{k} |a_n^{'}|$$
$$= C \, [(a_0 - a_1) + (a_1 - a_2) + \dots\dots\dots\dots + (a_{k-1} - a_k)]$$
$$= C \, (a_0 - a_k) \le C a_0 \quad \forall k$$

So , $\sum_{n=0}^{\infty} a_n^{'} B_{n-1}$ is abs. conv. and hence conv.

So $\sum_{n=0}^{\infty} a_n b_n$ conv.

Lemma : If θ is not an integer multiple of 2π , then

$$\sum_{n=1}^{k} \cos n\theta = \frac{\cos \frac{1}{2}(k+1)\theta \ . \sin \frac{1}{2} k\theta}{\sin \frac{1}{2}\theta}$$

$$\sum_{n=1}^{k} \sin n\theta = \frac{\sin \frac{1}{2}(k+1)\theta \ . \sin \frac{1}{2} k\theta}{\sin \frac{1}{2}\theta}$$

Proof: $\sum_{n=1}^{k} e^{ni\theta} = \sum_{n=1}^{k} \cos n\theta + i\sum_{n=1}^{k} \sin n\theta$ ……..(1)

$$Left\ \ side = e^{i\theta} + e^{i2\theta} + e^{i3\theta} + ... + e^{ik\theta} = e^{i\theta}(1 + e^{i\theta} + ... + e^{i(k-1)\theta}$$

$$= e^{i\theta}\frac{(e^{ik\theta}-1)}{e^{i\theta}-1} = e^{i\theta}\frac{e^{ik\theta/2}[e^{ik\theta/2} - e^{-ik\theta/2}]}{e^{i\theta/2}[e^{i\theta/2} - e^{-i\theta/2}]}$$

$$= e^{i\frac{\theta}{2}(k+1)} . \frac{\sin\frac{1}{2}k\theta}{\sin\frac{1}{2}\theta}$$

$$= [\cos\frac{1}{2}(k+1)\theta + i\ \sin\frac{1}{2}(k+1)\theta]\frac{\sin\frac{1}{2}k\theta}{\sin\frac{1}{2}\theta}$$

=right side .

بمساواة الجزء الحقيقى والتخيلى فى الطرفين فى المعادلة 1 نحصل على

$$\sum_{n=1}^{k} \cos n\theta = \frac{\cos\frac{1}{2}(k+1)\theta\ .\sin\frac{1}{2}k\theta}{\sin\frac{1}{2}\theta}$$

$$\sum_{n=1}^{k} \sin n\theta = \frac{\sin\frac{1}{2}(k+1)\theta\ .\sin\frac{1}{2}k\theta}{\sin\frac{1}{2}\theta}$$

Corollary : Suppose that the seq. { a_n} decreases to zero as $k \to \infty$,then the series $\sum_{n=1}^{\infty} a_n \cos n\theta$ conv. for all θ except perhaps for integer multiples of 2π , and the series $\sum_{n=1}^{\infty} a_n \sin n\theta$ conv. for all θ .

Proof : For $\theta \neq 2\pi\ j$, for if $b_n = \cos n\theta\ \ or\ \ \sin n\theta$

$$|B_k| = |\sum_{n=1}^{k} \cos n\theta| = \frac{|\cos\frac{1}{2}(k+1)\theta \,.\sin\frac{1}{2}k\theta|}{|\sin\frac{1}{2}\theta|} \le |\csc\frac{1}{2}\theta| \text{ for all n.}$$

$$or \ |\sum_{n=1}^{k} \sin n\theta| = \frac{|\sin\frac{1}{2}(k+1)\theta \,.\sin\frac{1}{2}k\theta|}{|\sin\frac{1}{2}\theta|} \le |\csc\frac{1}{2}\theta| \text{ for all n.}$$

So , { B_k } is bounded and since { a_k } dec. to zero so by Dirichlet test

$\Rightarrow \sum_{n=1}^{\infty} a_n \cos n\theta \ and \ \sum_{n=1}^{\infty} a_n \sin n\theta,$ conv.

If $\theta = 2\pi j \Rightarrow \sum_{n=1}^{k} \sin n\theta \to 0 \ \ for \ all \ \ n$ since, $(\sin 2\pi j = 0)$, so $\sum_{n=1}^{k} \sin n\theta$ is bounded.,

Then $\sum_{n=1}^{\infty} a_n \sin n\theta$ conv.

$\sum_{n=1}^{\infty} a_n \cos n\theta$ perhaps conv. or div. on $\theta = 2\pi j$ because $\sum_{n=1}^{k} \cos n\theta$ unbounded.

***Thank you ***