دراسات عليا - ماجستير

تحليل دالي **Functional Analysis**

نظري : 3 مناقشة : 1 عدد الوحدات : 3

محاضرات الدكتور نوري فرحان المياحي

في

التحليل الدالي Functional Analysis

التحليل الدالي(Functional Analysis) هو فرع من فروع التحليل الرياضي، اللبنة الأساسية له تتم خلال دراسة الفضاءات الخطية مع نوعا من بنية ذات الصلة غاية ، (مثلا الضرب الداخلي ، المعيار، التبولوجي ، الخ) والمؤثرات الخطية بناء على هذه الفضاءات واحترام هذه الهياكل بمعنى مناسب . الجذور التاريخية للتحليل الدالي اكمن في دراسة فضاء الدوال وصياغة خواص التحويلات للدوال مثل تحويل فوربيه كما التحويلات تعرف الاستمرارية والوحدوية الخ. مؤثرات بين فضاءات الدوال. وجهة النظر هذه مفيدة جدا في دراسة المعادلات التفاضلية والتكاملية. استخدام كلمة دالية يعود إلى حساب التفاضل والتكامل للمتغيرات ، للدلالة على الدالة التي محاججتها هي دالة وأول استخدم هذا الاسم من قبل هادمارد(Hadamard)سنة 1910 في كتاب بشان هذا الموضوع. ومع ذلك سبق وان قدم المفهوم العام للدالية في عام 1887 من قبل عالم الرياضيات الايطالي والفيزيائي فيتو فولتيرا . نظرية الداليات اللاخطية استمرت من قبل هادمارد، وبشكل خاص كل من فريشيه(Fréchet) وليفي(Levy). كما أسس هادمارد المدرسة الحديثة للتحليل الدالي الخطي وطورت من قبل ريس(Riesz) ومجموعة علماء الرياضيات البولنديين حول ستيفان بناخ (Stefan Banach). في النصوص التمهيدية الحديثة للتحليل الدالي، يرى بأنه موضوع يهتم بدراسة الفضاءات الخطية مع التبولوجيا(أي دراسة التبولوجيا على فضاءات خطية) وبشكل هذه الفضاءات غير منتهية البعد. في المقابل الجبر الخطي يتعامل في الغالب مع الفضاءات الخطية منتهية البعد وعدم استخدام التبولوجيا. جزء مهم من التحليل الدالي هو توسيع لنظرية القياس ،التكامل والاحتمالية للفضاءات غير منتهية الأبعاد، المعروف أيضا بالتحليل غير منتهي البعد.

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1.Fundamental Concepts

1.1 Linear Spaces

The letters $\mathbb{R}$ and $\mathbb{C}$ will always denote the field of real numbers and the field of complex numbers, respectively. For the moment, let F stand for either $\mathbb{R}$ or $\mathbb{C}$.A scalar is a member of the scalar field F.

Definition(1.1.1)

A linear space over F is a set X, whose elements are called vector, and in which two operations, addition $(+ : X \times X \to X)$ and scalar multiplication $(. : F \times X \to X)$ such that

(1) $x + y \in X$ for all $x, y \in X$

(2) $x + y = y + x$ for all $x, y \in X$

(3) $x + (y + z) = (x + y) + z$ for all $x, y, z \in X$

(4) there exists $0 \in X$ such that $x + 0 = 0 + x = x$ for all $x \in X$ and 0 is the Zero vector or the origin.

(5) for all $x \in X$, there exists $-x \in X$ such that $x + (-x) = (-x) + x = 0$

(6) $\alpha . x \in X$ for all $\alpha \in F$ and for all $x \in X$

(7) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ for all $\alpha \in F$ and for all $x, y \in X$

(8) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$ for all $\lambda, \mu \in F$ and for all $x \in X$

(9) $(\lambda \cdot \mu) \cdot x = \lambda \cdot (\mu \cdot x)$ for all $\lambda, \mu \in F$ and for all $x \in X$

(10) $1 \cdot x = x$ for all $x \in X$ and 1 is the unity element of the field F.

Remark

A real linear space is one for which $F = \mathbb{R}$, a complex is linear space is one for which $F = \mathbb{C}$.

Theorem (1.1.2)

Let X be a linear space over F

(1) $0.x = 0$ for all $x \in X$

(2) $\alpha \cdot 0 = 0$ for all $\alpha \in F$

(3) $-(\alpha \cdot x) = (-\alpha) \cdot x = \alpha \cdot (-x)$ for all $\alpha \in F$ and for all $x \in X$

(4) If $x, y \in X$, there is a unique $z \in X$ such that $x + z = y$

(5) $\alpha (x - y) = \alpha x - \alpha y$

(6) If $\alpha x = 0$, then either $\alpha = 0$ or $x = 0$

(7) If $x \neq 0$, then $\alpha_1 x = \alpha_2 x \Rightarrow \alpha_1 = \alpha_2$

(8) If $x \neq 0, y \neq 0$, then $\alpha x = \alpha y$ with $\alpha \neq 0 \Rightarrow x = y$

Example(1.1.3)

(1) F^n **-Space** : If F is a field, then the set $F^n = \{(x_1, x_2, \cdots, x_n) : x_i \in F, \ i = 1, 2, \cdots, n\}$ is a linear space over F for the addition and scalar multiplications defined as

(a) $x + y = (x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$ for all $x, y \in F^n$

(b) $\alpha x = \alpha(x_1, x_2, \cdots, x_n) = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n)$ for all $x \in F^n$ and for all $\alpha \in F$

(2) ℓ^p-Space, $1 \le p < \infty$: If F is a field, then the set $\ell^p = \{(x_1, x_2, \cdots) : x_i \in F, \ \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ is a linear space over F for the addition and scalar multiplications defined as

(a) $x + y = (x_1, x_2, \cdots) + (y_1, y_2, \cdots) = (x_1 + y_1, x_2 + y_2, \cdots)$ for all $x, y \in \ell^p$

(b) $\alpha x = \alpha(x_1, x_2, \cdots) = (\alpha x_1, \alpha x_2, \cdots)$ for all $x \in \ell^p$ and for all $\alpha \in F$

(3) ℓ^∞-Space : If F is a field, then the set $\ell^\infty = \{(x_1, x_2, \cdots) : x_i \in F, \ |x_i| \le k_i, \ i = 1, 2, \cdots\}$ (k_i real number dependent on x but not dependent on i) is a linear space over F for the addition and scalar multiplications defined as

(a) $x + y = (x_1, x_2, \cdots) + (y_1, y_2, \cdots) = (x_1 + y_1, x_2 + y_2, \cdots)$ for all $x, y \in \ell^\infty$

(b) $\alpha x = \alpha(x_1, x_2, \cdots) = (\alpha x_1, \alpha x_2, \cdots)$ for all $x \in \ell^\infty$ and for all $\alpha \in F$

(4) $C[a,b]$-Space : If F is a field, then the set $C[a,b] = \{f : [a,b] \to F$ is continuous function$\}$ is a linear space over F for the addition and scalar multiplications defined as

(a) $(f + g)(x) = f(x) + g(x)$ for all $f, g \in C[a,b]$

(b) $(\alpha f)(x) = \alpha f(x)$ for all $f \in C[a,b]$ and for all $\alpha \in F$

Remark

If X be a linear space over F, and $A, B \subseteq X$, $G \subset F$, the following notations will be used.

$$A + B = \{x = a + b : a \in A, b \in B\}, \quad GA = \{x = \alpha a : \alpha \in G, a \in A\}$$

(1) If $A = \{a\}$, we write $a + B$ instate of $\{a\} + B$ and we say that $a + B$ is obtained by translating B by a

(2) If $0 \in A$, then $B \subset A + B$

(3) $A + B = \bigcup_{a \in A} (a + B)$

(4) If $G = \{\alpha\}$, we write αA instate of $\{\alpha\} A$ such that $\alpha A = \{x = \alpha a : a \in A\}$

In particular : $-A = (-1)A = \{-a : a \in A\}$

We say that A is symmetric if $-A = A$, so that $A \cap (-A)$ is symmetric for any subset A of X

Definition(1.1.4)

A subset A of a linear space X over F is said to be balanced if $\alpha A \subset A$ for every $\alpha \in F$ with $|\alpha| \le 1$

Theorem(1.1.5)

If A and B are balanced sets in a linear space X over F, then $A \cap B, A \cup B, A + B$ are also balanced in X.

Proof :

Let $\alpha \in F$ with $|\alpha| \le 1 \Rightarrow \alpha A \subset A$ and $\alpha B \subset B$

(1) Let $x \in \alpha(A \cap B) \Rightarrow x = \alpha y$ such that $y \in A \cap B$

$\Rightarrow y \in A$ and $y \in B \Rightarrow x \in \alpha A$ and $x \in \alpha B$

$\Rightarrow \quad x\in A, \quad x\in B \quad \Rightarrow \quad x\in A\cap B$

$ª(A\cap B)\subset A\cap B \quad \Rightarrow \quad A\cap B$ is balanced set. Similarly to prove $A\cup B$ is balanced

(2) Let $x\in ª(A+B) \quad \Rightarrow \quad x= ª(a+b)$ such that $a\in A, \quad b\in B \Rightarrow \quad x=ªa+ªb$

Since $ªa\in A$ because $ªA\subset A$ and also $ªb\in B$ because $ªB\subset B$

$\Rightarrow \quad x\in A+B \quad \Rightarrow \quad ª(A+B)\subseteq A+B \quad \Rightarrow \quad A+B$ is a balanced set.

Theorem(1.1.6)

If A are balanced sets in a linear space X over F and $ª\in F$ such that $|ª|=1$, then $ªA=A$, and hence every balanced set is symmetric.

Proof.

Since A is balanced $\Rightarrow \quad ªA\subseteq A$ for all $ª\in F$ with $|ª|\le 1$

$\Rightarrow \quad ªA\subseteq A$ when $|ª|=1$. We must to show that $A\subseteq ªA$

Let $x\in A$

Since $|ª|\neq 0 \quad \Rightarrow \quad ª\neq 0$. Put $\sim\, =\frac{1}{ª} \quad \Rightarrow \quad |\sim|=1$

Since A is balanced set $\Rightarrow \quad \sim A\subset A \quad \Rightarrow \quad \sim x\in A$

$\Rightarrow ª(\sim x)\in ªA \quad \Rightarrow \quad x\in ªA \quad \Rightarrow \quad A\subseteq ªA \quad \Rightarrow \quad ªA=A$

Now we show that A is symmetric. Put $ª=-1 \quad \Rightarrow \quad |ª|=1$

Since $ªA=A \quad \Rightarrow \quad -A=A \quad \Rightarrow \quad A$ is symmetric

Definition(1.1.7)

Let A and B be two subsets in a linear space X over F .We say that A is absorbs B if there exists $ª_0\in A$ such that $B\subset ªA$ for all $|ª|\ge|ª_0|$. And we say that A is an absorbing if for every $x\in X$, there exists $ª>0$ such that $x\in ªA$.

Definition(1.1.8)

Let M be a subset of a linear space X over F. We say that M is a subspace of X if M itself is a linear space over F with respect to the same operations in X .

It is clear to show that : A non-empty subset M of a linear space X over F is a subspace of X iff

(1) $x+y\in M$ for all $x,y\in M$ (2) $ªx\in M$ for all $ª\in F$ and for all $x\in M$

or equivalently, $\sim x+\ y\in M$ for all $\sim,\ \in F$ and for all $x,y\in M$. Also equivalent, if $0\in M$ and $\sim M+\ M\subset M$ for all $\sim,\ \in F$

Remark

Every linear space X has at least two trivial subspaces, namely X itself and the zero subspace $\{0\}$. Subspaces distinct from X are called proper subspace.

Theorem(1.1.9)

Let M_1 and M_2 be two subspaces of a linear space X over F

(1) $M_1 \cap M_2$ is a subspace of X

(2) $M_1 \cup M_2$ is a subspace of X iff $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$

(3) $M_1 + M_2$ is a subspace of X and $M_1 \subseteq M_1 + M_2$, $M_2 \subseteq M_1 + M_2$

Proof :

(1)

Since $0 \in M_1, 0 \in M_2 \Rightarrow 0 \in M_1 \cap M_2 \Rightarrow M_1 \cap M_2 \neq \Phi$

Let $x, y \in M_1 \cap M_2$ and $\alpha, \beta \in F$

$\Rightarrow x, y \in M_1$ and $x, y \in M_2$

Since M_1, M_2 are subspaces

$\Rightarrow \alpha x + \beta y \in M_2, \ \alpha x + \beta y \in M_1 \Rightarrow \alpha x + \beta y \in M_1 \cap M_2$

So that $M_1 \cap M_2$ is a subspace of X .

Definition(1.1.10)

Let A be a subset of a linear space X over F. The smallest subspace of X which contains A is called the subspace spanned (or generated) by A and denoted by $[A]$ or $Span(A)$.

It is clear to show that

(1) $A \subseteq [A]$

(2) $[A] =$ intersection of all subspaces of X which containing A

(3) A is a subspace iff $A = [A]$

(4) $[A] = \left\{ x = \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in F, x_i \in A, i = 1,..,n, n \in Z^+ \right\}$

Remarks

(1) If $A = \{x_0\}$, we write $[x_0]$, instate of $[\{x_0\}]$, so that $[x_0] = \{x = \alpha x_0 : \alpha \in F\}$

(2) If A is a subset of a set X and let $x_0 \notin A$, then $[A \cup \{x_0\}]$ is a subspace generated by $A \cup \{x_0\}$, and $[A \cup \{x_0\}] = \{x = a + \alpha x_0 : a \in A, \alpha \in F\}$

Definition(1.1.11)

Let M be a proper subspace of a linear space X on F. We say that M is a Maximal Subspace if the following condition is satisfying

If N is a subspace of X such that $M \subset N \subseteq X$, then $N = X$

It is clear to show that

If M is a proper subspace of a linear space X on a field F, then M is a maximal subspace iff $X = [M \cup \{x_0\}]$ for all $x_0 \notin M$, and hence for all $x \in X$ has a unique representation of the form $x = m + \alpha x_0$, where $\alpha \in F$, $m \in M$.

Definition(1.1.12)

Let M_1 and M_2 be two subspaces of a linear space X over F. We M_1, M_2 are disjoint if $M_1 \cap M_2 = \{0\}$.

Definition(1.1.13)

Let M_1 and M_2 be two subspaces of a linear space X over F. We M_1, M_2 are direct sum (we write $X = M_1 \oplus M_2$), if for all $x \in X$ has a unique representation of the form

$$x = m_1 + m_2 \, , m_1 \in M_1 \, , m_2 \in M_2 \, .$$

We say that M_2 is complement subspace of M_1 in X. It is clear to show that :

(1) $X = M_1 \oplus M_2$ iff $X = M_1 + M_2$ and $M_1 \cap M_2 = \{0\}$

(2) Every subspace of linear space has complement subspace.

Definition(1.1.14)

Let X be a linear space over F. A finite non-empty set $\{x_1,\ldots,x_n\}$ of X is said to be

(1) linear dependent if there exists scalar $\alpha_1, \alpha_2, \cdots \alpha_n \in F$ not all of then zero (some of them may be zero) such that $\alpha_1 x_2 + \alpha_2 x_2 + \cdots \alpha_n x_n = 0$.

(2) linear independent if every relation of the form

$$\alpha_1 x_2 + \alpha_2 x_2 + \cdots \alpha_n x_n = 0 \ , \ \alpha_i \in F, \quad i = 1,2,\cdots,n$$; then $\alpha_i = 0$ for all $i = 1,2,\cdots,n$

An arbitrary subset A of X is said to be linearly independent if every non-empty finite subset of A is linearly independent, otherwise it is linearly dependent.

Remark

Let X be a linear space over F and let $x_0 \in X$, $A \subseteq X$

(1) If $0 \in A$, then A is linearly dependent, hence every subspace is linearly dependent

(2) If $x_0 \neq 0$, then $\{x_0\}$ linearly independent

Theorem(1.1.15)

Let X be a linear space over F and let $A \subseteq B \subseteq X$

(1) If A is linearly dependent, so is B

(2)) If B is linearly independent, so is A

Proof :

(1)Since A is linearly dependent $\Rightarrow$ there exist a subset $\{x_1, x_2, \cdots, x_n\}$ of A is linearly dependent Since $A \subseteq B \quad \Rightarrow \quad \{x_1, x_2, \cdots, x_n\} \subseteq B$ is linearly dependent .

(2)If A is linearly dependent, B is linearly dependent .This contradiction, so B is linearly independent

Definition (1.1.16)

Let be subset of a linear space X on a field F. We say is a basis of X if its linearly independent and generated X (i.e. $X = [A]$).

Remark

- The basis $= \{e_1, e_2, \cdots, e_n\}$ is called the standard ordered basis of F^n, where

$$e_1 = (1,0,\cdots,0), \quad e_2 = (0.1,0,\cdots,0), \cdots, e_n = (0,0,\cdots,0,1)$$

- If $X = \{0\}$, there no basis for X , and any non zero linear space has basis.

Definition(1.1.17)

A linear space X on a field F has dimension n, ($\dim X = n$)if X has a basis $\{x_1, \cdots, x_n\}$.

This means that every $x \in X$ has a unique representation of the form

$$x = \sum_{i=1}^{n} \alpha_i x_i, \qquad \alpha_i \in F$$

- If $X = \{0\}$ **,** then X said to be of dimension zero, i.e. $\dim X = 0$.
- A linear space X is said to be finite dimensional if its dimension is 0 or a positive integer (i.e. $\dim X = 0$ or $\dim X = n$)
- A linear space X is said to be infinite dimensional if the number of elements in its basis is infinite.

Theorem(1.1.18)

The linear space F^n is of dimension n.

Proof :

$$F^n = \{\alpha = (\alpha_1, \cdots, \alpha_n) : \alpha_i \in F, \quad i = 1,2,\cdots,n\}$$

we shall show that the set $= \{e_1, e_2, \cdots, e_n\}$ where

$$e_1 = (1,0,\cdots,0), \quad e_2 = (0.1,0,\cdots,0), \cdots, e_n = (0,0,\cdots,0,1)$$

is a basis for F^n.

First we shall that the set is linearly independent

Let $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$

$\alpha_1(1,0,\cdots,0) + \alpha_2(0,1,0,\cdots,0) + \cdots + \alpha_n(0,0,\cdots,1) = 0$

$(\alpha_1, \alpha_2, \cdots, \alpha_n) = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Therefore the set is linearly independent.

Now we shall show that generates the linear space F^n.

Let $x \in F^n \quad \Rightarrow \quad x = (x_1, \cdots, x_n), \quad x_i \in F, \quad i = 1,2,\cdots,n$

$x = \alpha_1(1,0,\cdots,0) + \alpha_2(0,1,0,\cdots,0) + \cdots + \alpha_n(0,0,\cdots,1) = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$

Therefore the set generates F^n.Hence is a basis of F^n

Since the number of elements in is n , the dimension of F^n is n .

- The linear space $\mathbb{R}^n$ is called the n – dimensional real space, and the linear space $\mathbb{C}^n$ is called the n – dimensional complex space .

Remark

If X is a linear space with dimension n

(1) Any subset of X contain $n+1$elements is linearly dependent .

(2) Let A be a subset of X which contain n elements, then

(a) If A is linearly independent, then its basis of X

(b) If A is generates of X , then its basis of X

Theorem(1.1.19)

Let X be a finite dimensional linear space on a field F

(1)If M is a subspace of X , then $\dim(M) \leq \dim(X)$, and if $\dim(M) = \dim(X)$, then $M = X$

(2) If M_1, M_2 are subspaces of X , then $\dim(M_1 + M_2) = \dim M_1 + \dim M_2 - \dim(M_1 \cap M_2)$

In special case $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$

Definition (1.1.20)

Let X be a real linear space. A partial order relation $\leq$ on X is call linear order if the following axioms are satisfied

(1) $x \leq y \Rightarrow x + z \leq y + z$ for all $x, y, z \in X$

(2) $x \leq y \Rightarrow \alpha x \leq \alpha y$ for all $x, y \in X$ for all $\alpha \geq 0$

A real linear space endowed with a linear order is called an ordered linear space. An element x of an ordered linear space X is said to be positive if $x \geq 0$, and negative if $x \leq 0$. The set of all positive elements of an ordered linear space X with be denoted by X_+, i.e. $X_+ = \{x \in X : x \geq 0\}$, X_+ is called the positive cone of X . It is easy to show that

(1) X_+ is a convex cone of X , i.e. $X_+ + X_+ \subseteq X_+$ and $\alpha X_+ \subseteq X_+$

(2) $X_+ \cap (-X_+) = \{0\}$

1.2 Convexity

Definition(1.2.1)

A subset A of a linear space X over F is called set if $\alpha x + (1-\alpha) y \in A$ whenever $x, y \in A$, $0 \leq \alpha \leq 1$ .Or equivalently if $\alpha A + (1-\alpha) A \subset A$ for all $0 \leq \alpha \leq 1$.

Example(1.2.2)

(1) The empty set and the set consisting of one point are convex.

(2) Every subspace of a linear space is convex, but the converse is not true

Remark

If A is a subset of a linear space X over F , then $(\sim + \quad)A \subset \sim A + \quad A$

Indeed

If $x \in (\sim + \quad)A$, then $x = (\sim + \quad)a \quad , a \in A \Rightarrow x = \sim a + \quad a \in \sim A + \quad A$

In general

$\sim A + \quad A \not\subset (\sim + \quad)A$

Theorem(1.2.3)

If A is a subset of a linear space X over F, then A is convex iff $(\alpha+\beta)A=\alpha A+\beta A$ for $\alpha,\beta\in\mathbb{R}^{+}$

Proof :

Suppose that A is convex

Since $(\alpha+\beta)A\subseteq\alpha A+\beta A$. We must to prove that $\alpha A+\beta A\subseteq(\alpha+\beta)A$

Let $x\in\alpha A+\beta A \Rightarrow x=\alpha a+\beta b$ where $a,b\in A$

$$x=(\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}a+\frac{\beta}{\alpha+\beta}b\right)$$

Put $\lambda=\frac{\alpha}{\alpha+\beta} \Rightarrow 1-\lambda=\frac{\beta}{\alpha+\beta}$

Since $\alpha,\beta\in R^{+} \Rightarrow \lambda\geq 0$

Since $\alpha\leq\alpha+\beta \Rightarrow \lambda\leq 1 \Rightarrow 0\leq\lambda\leq 1$

Since A is convex, then $\lambda a+(1-\lambda)b\in A$, i.e. $\frac{\alpha}{\alpha+\beta}a+\frac{\beta}{\alpha+\beta}b\in A$

$\Rightarrow x\in(\alpha+\beta)A$, hence $\alpha A+\beta A\subseteq(\alpha+\beta)A \Rightarrow (\alpha+\beta)A=\alpha A+\beta A$

The converse, let $(\alpha+\beta)A=\alpha A+\beta A$ for all $\alpha,\beta\in R^{+}$

Let $0\leq\lambda\leq 1 \Rightarrow 1-\lambda\geq 0$, then $\lambda A+(1-\lambda)A=(\lambda+(1-\lambda))A=A$

$\Rightarrow \lambda A+(1-\lambda)A\subseteq A \Rightarrow A$ is convex .

Theorem(1.2.4)

If A and B are convex sets in a linear space X over F, and $\lambda\in F$ then $A\cap B, \alpha A, A+B$ are also convex sets in X .

Proof :

(1) let $x,y\in A\cap B$ and $0\leq\lambda\leq 1 \Rightarrow x,y\in A$ and $x,y\in B$

Since A and B are convex, then $\lambda x+(1-\lambda)y\in A$ and $\lambda x+(1-\lambda)y\in B$

$\Rightarrow \lambda x+(1-\lambda)y\in A\cap B \Rightarrow A\cap B$ is convex set

(2) let $x,y\in\alpha A$ and $0\leq\lambda\leq 1 \Rightarrow x=\alpha z,\ y=\alpha w$ where $z,w\in A$

Since A is convex, then $\lambda z+(1-\lambda)w\in A \Rightarrow \alpha(\lambda z+(1-\lambda)w)\in\alpha A$

Since $\alpha(\lambda z+(1-\lambda)) = \lambda(\alpha z)+(1-\lambda)\alpha w=\lambda x+(1-\lambda)y\in\alpha A \Rightarrow \alpha A$ is convex

(3) Let $x,y\in A+B$ and $0\leq\lambda\leq 1$

$x=a_1+b_1$, $y=a_2+b_2$ and $a_1,a_2\in A,\ b_1,b_2\in B$

Since A and B are convex, then $\lambda a_1+(1-\lambda)a_2\in A$ and $\lambda b_1+(1-\lambda)b_2\in B$

Since $\lambda x+(1-\lambda)y=\lambda(a_1+b_1)+(1-\lambda)(a_2+b_2)=(\lambda a_1+(1-\lambda)a_2)+(\lambda b_1+(1-\lambda)b_2)$

$\Rightarrow \lambda x+(1-\lambda)y\in A+B \Rightarrow A+B$ is convex .

Definition(1.2.5)

Let A be a subset of a linear space X over F. The smallest convex set in X which contains A is called the convex hull (or generated) by A and denoted by $conv\,(A)$.

It is clear to show that

(1) $A \subseteq conv\,(A)$

(2) $conv(A)$ = intersection of all convex sets of X which containing A

(3) A is a convex iff $A = conv\,(A)$

Definition(1.2.6)

Let X be a linear space over F, and let $x_1, x_2, ..., x_n \in X$. A vector $x \in X$ is called a convex combination of $x_1, x_2, ..., x_n$ if $x = \sum_{i=1}^{n} \alpha_i x_i$, where $\alpha_i \geq 0$, and $\sum_{i=1}^{n} \alpha_i = 1$

Theorem(1.2.7)

Let A be a subset of a linear space X over F. Then

$$conv(A) = \{\sum_{k=1}^{n} \alpha_k x_k : \alpha_k \geq 0,\ x_k \in A,\ \sum_{k=1}^{n} \alpha_k = 1\}$$

Proof :

Let $B = \left\{ x = \sum_{i=1}^{n} \alpha_i\, x_i : \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1, x_i \in A \right\}$

and let $x \in B \Rightarrow x = \sum_{i=1}^{n} \alpha_i x, \quad x_i \in A, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1$

Since $A \subset conv(A) \Rightarrow x_i \in conv(A)$

We want to show that $x \in conv(A)$. Now, we shall prove by induction on n.

If $n = 2$, then $x = \alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 x + (1 - \alpha_1) x_2$

Since $x_1, x_2 \in conv(A)$ and $conv(A)$ is convex, then $x \in conv(A)$

We therefore assume the statement to be true for $n-1$ and proceed to n.

If $\alpha_1 + \alpha_2 + ... + \alpha_{n-1} = 0$, then $\alpha_n = 1$, so

$$x = \sum_{i=1}^{n} \alpha_i x_i = x_n \in conv(A)$$

If $\alpha_1 + \alpha_2 + ... + \alpha_{n-1} \neq 0$, put $\beta = \alpha_1 + \alpha_2 + ... + \alpha_{n-1}, \quad \Rightarrow \quad \beta > 0, \quad \beta + \alpha_n = 1$

Let $\beta_i = \frac{\alpha_i}{\beta}$, we have $\sum_{i=1}^{n-1} \beta_i = \sum_{i=1}^{n-1} \frac{\alpha_i}{\beta} = \frac{1}{\beta} \sum_{i=1}^{n-1} \alpha_i = \frac{1}{\beta}(\beta) = 1 \Rightarrow \beta_1 x_1 + ... + \beta_{n-1} x_{n-1} \in conv(A)$

Since $conv(A)$ is convex, then $\beta(\beta_1 x_1 + ... + \beta_{n-1} x_{n-1}) + \alpha_n x_n \in conv(A)$

Since $\alpha_i = \beta \beta_i \Rightarrow \alpha_1 x_1 + ... + \alpha_{n-1} x_{n-1} + \alpha_n x_n \in conv(A)$, so that $B \subset conv(A)$

Since B is convex and $A \subset B$, but $conv(A)$ is the smallest convex set contains A

$\Rightarrow \quad conv(A) \subset B \quad \Rightarrow \quad conv(A) = B$

دراسات عليا - ماجستير

Functional Analysis تحليل دالي

نظري : 3 مناقشة : 1 عدد الوحدات : 3

3.1 Linear Functions

Definition(1.3.1)

Let X and Y be linear spaces over the same field F. A function $f : X \to Y$ is called a linear if $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$ for all $x, y \in X$ and $\alpha, \beta \in F$.

- A function between linear spaces is often referred to as an operator or a transformation, especially if it is linear.
- Kernel (or null space) of a linear function $f : X \to Y$ is denoted by $\ker(f)$ and defined as :

$$\ker(f)=\{x \in X : f(x)=0\}= f^{-1}(\{0\})$$

The of f is denoted by $\operatorname{Im} g(f)$ and defined as: $\operatorname{Im} g(f)=\{f(x): x \in X\}$

- Linear function of a linear space X into its field F is called linear functional on X.
- Let $L(X,Y)$ denote the set of all linear functions from a linear space X into a linear space Y. Then $L(X,Y)$ is a vector space under the following addition and scalar multiplication
  (1) $(f+g)(x)= f(x)+g(x)$ for all $f, g \in L(X,Y)$
  (2) $(\alpha f)(x)=\alpha f(x)$ for all $f \in L(X,Y)$ and for all $\alpha \in F$

If $Y = X$, we write $L(X)$ instead of $L(X,X)$. The space of all linear functionals defined on a linear space X is called the algebraic dual space and denoted by X', i.e. $X' = L(X,F)$

- We say that X,Y are linear isomorphic(we write $(X \cong Y)$), the there is a bijection linear function $f : X \to Y$ such function is called linear isomorphism.

Theorem(1.3.2)

Let $f : X \to Y$ be a linear function

(1) $f(0)=0$ (2) $f(-x)=-f(x)$ for all $x \in X$

(3) $f(x-y)= f(x)-f(y)$ for all $x, y \in X$

(4) $f(\sum_{i=1}^{n}\alpha_i x_i)=\sum_{i=1}^{n}\alpha_i f(x_i)$, for all $x_1, x_2, \cdots, x_n \in X$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$

(5) If A is a subspace (or convex set, or balanced set) in X, the same is true $f(A)$

(6) If B is a subspace (or convex set, or balanced set) in B, the same is true $f^{-1}(B)$

(7) $\ker(f)$ is a subspace of X and $\operatorname{Im} g(f)$ is a subspace of Y

(8) $\ker(f)=\{0\}$ iff f is an injective

Theorem(1.3.3)

Let X be a linear space over a field F

(1) If $x \in X$, and a function $T_x : X' \to F$ defined by $T_x(f)=f(x)$ for all $f \in X'$, then T_x is linear functional, i.e. $T_x \in X''$, and it is called Evaluation Functional Induced by x.

(2) If the function $\cdot : X \to X''$ defined by $\cdot(x)=T_x$ for all $x \in X$, then $\cdot$ injection linear function and $\cdot$ is called Canonical Function.

Proof :

(1) let $f, g \in X'$, $\sim$, $\in F$

$T_x(\sim f + \; g) = (\sim f + \; g)(x) = (\sim f)(x) + (\; g)(x) = \sim f(x) + \; g(x) = \sim T_x(f) + \; T_x(g) \Rightarrow T_x \in X''$

(2) let $x, y \in X$, $\sim$, $\in F$ $\Rightarrow \cdot(\sim x + \; y) = T_{\sim x + \; y}$ for all $f \in X'$

$T_{\sim x + \; y}(f) = f(\sim x + \; y) = \sim f(x) + \; f(y) = \sim T_x(f) + \; T_x(fy) = (\sim T_x + \; T_x)(fy)$

So that $\cdot(\sim x + \; y) = \sim \cdot(x) + \; \cdot(y) \Rightarrow \cdot$ is linear function

Now to prove $\cdot$ is injection : let $x, y \in X$ such that $\cdot(x) = \cdot(y)$

$\Rightarrow T_x = T_y \Rightarrow T_x(f) = T_y(f)$ for all $f \in X'$ $\Rightarrow f(x) = f(y)$ for all $f \in X'$

$\Rightarrow f(x - y) = 0$ for all $f \in X'$ $\Rightarrow x - y = 0$, so that $x = y \Rightarrow \cdot$ is injection.

Definition(1.3.4)

Let X be a linear space over a field F .we say that X is an Algebraically Reflexive if $\cdot$ is an onto , where $\cdot$ is defined in (1.3.3).

Theorem(1.3.5)

Every finite dimensional space is algebraically reflexive.

Proof :

Let X be a finite dimensional space over a field F. $\Rightarrow$ $\dim X' = \dim X$, so that X' finite dimensional $\Rightarrow$ $\dim X'' = \dim X$, so that X'' finite dimensional.

Since $\cdot : X \to X''$ is injection and X', X'' are finite dimensional, and $\dim X'' = \dim X$ then $\Rightarrow \cdot$ is onto.

Theorem(1.3.6)

Every infinite dimensional space is not algebraically reflexive.

Proof :

Let X be an infinite dimensional space over a field F, and let $B = \{x_i : i \in I\}$ be a basis for X .
Since X is infinite dimensional, therefore the index set I is an infinite set and $x_i \neq x_j$ if $i \neq j$.

Define $f_i : X \to F$ by $f_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$, $f_i \in X'$.

We claim that the set $A = \{f_i : i \in I\}$ is a linearly independent subset of X'

Let $\{f_{i_1}, \cdots, f_{i_n}\}$ be finite set in A and $ª_1, \cdots, ª_n \in F$ such that $ª_1 f_{i_1} + \cdots + ª_n f_{i_n} = 0$

$\Rightarrow (ª_1 f_{i_1} + \ldots + ª_n f_{i_n})(x) = 0$ for all $x \in X$ $\Rightarrow ª_1 f_{i_1}(x) + \ldots + ª_n f_{i_n}(x) = 0$ for all $x \in X$

Form definition of f_i, we have $ª_j = 0$ for all $j = 1, \cdots, n$, so that A is linear independent .

Let be a basis of X' such that $A \subset$ .Let $\{ª \in F : i \in I\}$ such that $ª_i \neq 0$ for all $i \in I$.

To show that $\cdot$ is not onto .Define $g : X' \to F$ by $g(f_i) = ª_i$ and $g(f) = 0$, if $f \in X'$ but $f \neq f_i$ for all $i \in I$ $\Rightarrow g \in X''$.

suppose that $\cdot : X \to X''$ is onto, then there is $x \in X$ such that $\cdot(x) = g$
since $g = g_x$, then $g(f_i) = g_x(f_i)$ for all $i \in I \Rightarrow g_x(f_i) = f_i(x) = \sim_i$, where $\sim_i$ is the coefficient of x_i in the representation of x in the terms of the basis . Now $\sim_i = 0$ for all but a finite number of indices i. Therefore $g(f_i) = 0$ for all but a finite number of indices i.
Thus we get a contradiction. $\Rightarrow \cdot$ is not onto .Hence X is not algebraically reflexive.

Corollary (1.3.8)

A linear space is algebraically reflexive iff it is finite dimensional

1.4 Quotient Spaces

Let M be any subspace of a linear space X over F. Let x be any element of X. The set $x + M = \{x + m : m \in M\}$ is called a left coset of M in X generated by x. Similarly the set $M + x = \{m + x : m \in M\}$ is called a right coset of M in X generated by x. Obviously $x + M$ and $M + x$ are both subsets of X .Since addition in X is commutative, therefore we have $x + M = M + x$. Hence we shall call $x + M$ as simply a coset of M in X generated by x.

It is easy to show that

(1) If $x \in M$, then $x + M = M$. In particular $0 + M = M$ (because $0 \in M$)

(2) $x - y \in M$ iff $x + M = y + M$

Let X/M denote the set of all coset of M in X M in X, i.e. $X/M = \{x + M : x \in X\}$

Theorem(1.4.1)

If M is a subspace of a vector space X over F, then X/M is a vector space over F for the addition and scalar multiplication compositions defined as follows :

$(x+M)+(y+M) = (x+y)+M$ for all $x, y \in X$.

$^{a}(x+M) = {}^{a}x + M$ for all $^{a} \in F$ for all $x \in X$

Proof :

Let $x, y \in X \Rightarrow x + y \in X$. Also $x \in X$ and $^{a} \in F \Rightarrow {}^{a}x \in X$

Therefore $(x+y)+M \in X/M$ and also $^{a}x + M \in X/M$. Thus X/M is closed with respect to addition of cosets and scalar multiplication as defined above.

Now first of all we shall these two compositions are will defined

Let $x + M = x' + M$, $x, x' \in X$ and $y + M = y' + M$, $y, y' \in X$

$x + M = x' + M \Rightarrow x - x' \in M$ and $y + M = y' + M \Rightarrow y - y' \in M$

Since M is a subspace of X, we have $(x - x') + (y - y') \in M$

$\Rightarrow (x+y) - (x'+y') \in M \Rightarrow (x+y) + M = (x'+y') + M$

Therefore addition in X/M is well defined.

Again $x - x' \in M$, $^{a} \in F \Rightarrow {}^{a}(x - x') \in M \Rightarrow {}^{a}x - {}^{a}x' \in M \Rightarrow {}^{a}x + M = {}^{a}x' + M \Rightarrow$ scalar multiplication in X/M is also well defined.

It is clear to show that X/M satisfies the conditions of linear space.

Remark

The linear space X/M is called the Quotient space of X relative to M. The coset M is the zero vector of this linear space.

Theorem(1.4.2)

Let M be a subspace of a linear space X over F. Then the function $® : X \to X/M$ defined by $®(x) = x + M$ for all $x \in X$ is an onto linear function and $\ker(®) = M$. (® is called the canonical function or Normal function, or the quotient function).

Proof :

(1) Let $x, y \in X$, $\sim, ª \in F$

$®(\sim x + \ y) = (\sim x + \ y) + M = \sim(x + M) + \ (y + M) = \sim ®(x) + \ ®(y) \Rightarrow ®$ is linear

(2) Let $y \in X/M$, then there is $x \in X$ such that $y = x + M$

$®(x) = x + M = y \quad \Rightarrow \quad ®$ is onto

(3) $\ker(®) = \{x \in X : ®(x) = M\} = \{x \in X : x + M = M\} = \{x \in X : x \in M\} = M$

Remark

In general, the natural function is not one-to-one, because, if $x, y \in X$ such that $x - y \in M$, then $x + M = y + M$, so $®(x) = ®(y)$.

Theorem(1.4.3)

Let M_1 and M_2 be subspaces of a linear space X over F such that $X = M_1 \oplus M_2$. Then $M_1 \cong X/M_2$ and $M_2 \cong X/M_1$.

Proof :

Define $f : M_1 \to X/M_2$ by $f(x) = x + M_2$ for all $x \in M_1$

We shall show that f is an isomorphism of M_1 onto X/M_2

(1) f is linear : let $x_1, x_2 \in M_1$ and $\sim, \ \in F$

$f(\sim x_1 + \ x_2) = (\sim x_1 + \ x_2) + M_2 = \sim(x_1 + M_2) + \ (x_2 + M_2) = \sim f(x_1) + \ f(x_2) \Rightarrow f$ is linear

(2) f is one-to-one : let $x_1, x_2 \in M_1$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 + M_2 = x_2 + M_2 \quad \Rightarrow \quad x_1 - x_2 \in M_2$

Since $x_1 - x_2 \in M_1 \quad \Rightarrow \quad x_1 - x_2 \in M_1 \cap M_2$

But $M_1 \cap M_2 = \{0\} \quad \Rightarrow \quad x_1 - x_2 = 0 \quad \Rightarrow \quad x_1 = x_2 \quad \Rightarrow \quad f$ is one-to-one

(3) f is onto : let $y \in X | M_2$, then there is $x \in X$ such that $y = x + M_2$

Since $X = M_1 \oplus M_2 \quad \Rightarrow \quad x = x_1 + y_1$ where $x_1 \in M_1$ and $y_1 \in M_2$

$\Rightarrow \quad y_1 = x - x_1 \in M_2 \quad x + M_2 = x_1 + M_2 \quad \Rightarrow \quad y = f(x_1) \quad \Rightarrow \quad f$ is onto

Theorem(1.4.4)

Let M be a subspace of a finite dimensional linear space X over F. Then

$$\dim(X/M) = \dim(X) - \dim(M)$$

Proof :

Let $\dim X = n, \quad \dim M = m$

Since M be a subspace of a finite dimensional linear space X, therefore there exists a subspace M_1 of X such that $X = M \oplus M_1$

Also $\dim X = \dim M + \dim M_1 \quad \Rightarrow \quad \dim M_1 = n - m$

Since $M_1 \cong X/M$, then $\dim(X/M) = \dim M_1 = n - m$