2.Tological Linear Spaces

2.1 Topological Spaces

Let ² be a collection of subsets of a set X. We say that ² is a topology on X if the following axioms are satisfied

(1) ¤ $\in$ ² and $X \in$ ²

(2) union of every members in ² is also in ²

(3) Intersection of any finite members in ² is also in ²

The set X together with ² is called a topological space and is denoted by $(X,$²$)$ or simply X. A members of ² are called open sets, i.e. a subset A of X is called an open set in X if $A \in$ ² and we say that A is called a closed set in X if A^c is open set in X. A neighborhood of a point $x \in X$ is any open set that contains x. The interior $\text{int}(A)$ of A is the union of all open sets in X that are subsets of A. The closure $\overline{A}$ of A is the intersection of all closed sets in X that contain A. A topological space is called a Hausdorff space if two different points are always two disjoint neighborhoods. A function f from a topological space X into topological space Y ($f : X \to Y$) is called continuous at a point $x \in X$ if for every neighborhood U of $f(x)$ in Y there is a neighborhood V of x in X such that $f(V) \subseteq U$. If f is continuous at every point, it is called continuous. A function $f : X \to Y$ is continuous iff each open (rsp. closed) set U in Y the set $f^{-1}(U)$ is open (rsp. closed) set in X. Two topological spaces X and Y are called homeomorphic if there exist a bijective function $f : X \to Y$ such that f and f^{-1} are continuous. The function f is called a homeomorphism.

2.2 Linear Topology

Definition(2.2.1)

A topology ² on a linear space X over F is called a linear topology if the functions $+ : X \times X \to X$ and $\cdot : F \times X \to X$ are continuous.

A linear space endowed with a linear topology is called a topological linear space. A topological linear space is called real or complex if it is real or complex as a linear space. The zero element of a topological linear is also called the origin. A subspace of a topological linear space is itself of topological linear space with the relative topology.

A local base at x of a topological linear space X is thus a collection $_x$ of neighborhoods of x such that every neighborhoods of x contains a members of $_x$, i.e. every neighborhoods U of x, there is a $V \in {}_x$ such that $x \in V \subseteq U$. The local base of 0 is denoted by instead of $_0$. A local base is called balanced local base if its members are balanced sets.

Remark

A function $+ : X \times X \to X$ which is defined by $+(x, y) = x + y$ for all $x, y \in X$ is continuous at a point (x, y) if for every neighborhood U_{x+y} in X there is a neighborhoods V_x and V_y in

X such that $V_x + V_y \subset U_{x+y}$. Similarly a function $.: F \times X \to X$ which is defined by $.(\alpha, x) = \alpha x$ for all $\alpha \in F$ and for all $x \in X$ is continuous at a point (α, x) if for every neighborhood $U_{\alpha x}$ in X there is $r > 0$ and there is a neighborhoods V_x in X such that $V_x \subset U_{\alpha x}$ whenever $| \;\; - \alpha| < r$.

Theorem(2.2.2)

Let X be a topological linear space and let $a \in X$ and $\alpha \in F$ such that $\alpha \neq 0$. Defined the function $T_a : X \to X$ by $T_a(x) = a + x$ for all $x \in X$ and the function $«_\alpha : X \to X$ by $«_\alpha(x) = \alpha x$ for all $x \in X$, then the functions T_a and $«_\alpha$ are homeomorphism (i.e. T_a and $«_\alpha$ are bijection and continuous, also T_a^{-1} , $«_\alpha^{-1}$ are continuous such that $T_a^{-1} = T_{-a}$ and $«_\alpha^{-1} = «_{\frac{1}{\alpha}}$) In general the function $f : X \to X$ defined by $f(x) = a + \alpha x$ for all $x \in X$ is homeomorphism

It is clear to show that

(1) If A is a subset in X and $a \in X$, then A is an open set in X iff $a + A$ is an open set in X. Hence if A is an open set in X and B is any subset of X, then $A + B$ is an open set in X (because $A + B = \cup\{b + A : b \in B\}$)

(2) If V is a neighborhood of 0 in X and $x \in X$, then $x + V$ is a neighborhood at x in X

(3) If is a local base at 0 in X and $x \in X$, then $x +$ is a local base at x in X

Theorem (2.2.3)

Let X be a topological linear space and let $\alpha \in F$, $A, B \subseteq X$, then

(1) $\overline{A} = \cap\{A + V\}$, where V runs through all neighborhoods of 0.

(2) $\overline{\alpha A} = \alpha \overline{A}$

(3) $\overline{A} + \overline{B} \subset \overline{A + B}$

(4) If A is a subspace of X, so is $\overline{A}$

(5) If A is a balanced subset of X, so is $\overline{A}$

(6) If A is a balanced subset of X and $0 \in \text{int}(A)$, then $\text{int}(A)$ is balanced

Proof :

(1) Let $x \in \overline{A} \;\Rightarrow$ for every neighborhood V of 0, then $(x + V) \cap A \neq \phi$

There is $y \in (x + V) \cap A \;\Rightarrow\; y \in x + V \wedge y \in A$

Since $y \in x + V \;\Rightarrow\; y = x + b$ such that $b \in V \;\Rightarrow\; x = y - b$ such that $b \in V$, $y \in A$

$\Rightarrow\; x \in A - V$

Hence $x \in A + V$ is a neighborhood of 0

(3) Let $x \in \overline{A} + \overline{B} \;\Rightarrow\; x = a + b$ where $a \in \overline{A}, b \in \overline{B}$ and let W be a neighborhood of $a + b$

Since the function $+ : X \times X \to X$ is continuous , there are neighborhoods V_a, V_b such that $V_a + V_b \subset W$

Since $a \in \overline{A} \;\Rightarrow\; V_a \cap A \neq \phi \;\Rightarrow\; \exists y \in V_a \cap A \;\Rightarrow\; y \in V_a, y \in A$

Since $b \in \overline{B} \Rightarrow V_b \cap B \neq \phi \Rightarrow \exists z \in V_b \cap B \Rightarrow z \in V_b, z \in A$

$y + z \in V_a + V_b$ and $y + z \in A + B$

$y + z \in V_a + V_b \Rightarrow y + z \in W \Rightarrow y + z \in W \cap (A + B)$

$\Rightarrow W \cap (A+B) \neq \phi$ for each a neighborhood W of $a + b \Rightarrow \overline{A} + \overline{B} \subset \overline{A+B}$

(4) Let $\lambda, \mu \in F$, we shall to show that $\lambda\overline{A} + \mu\overline{A} \subset \overline{A}$

Since A is a subspace of X, then $\lambda A + \mu A \subset A \Rightarrow \overline{\lambda A + \mu A} \subset \overline{A}$

If $\lambda = 0 \Rightarrow \lambda A = \{0\}$ and if $\lambda \neq 0 \Rightarrow \lambda\overline{A} = \overline{\lambda A}$

$\Rightarrow \lambda\overline{A} + \mu\overline{A} = \overline{\lambda A} + \overline{\mu A} \subset \overline{\lambda A + \mu A} \subset \overline{A} \Rightarrow \overline{A}$ is a subspace of X.

(5) Let $\alpha \in F$ such that $|\alpha| \leq 1$

Since A is a balanced $\alpha A \subset A \Rightarrow \overline{\alpha A} \subset \overline{A}$

Since $\overline{\alpha A} = \alpha\overline{A} \Rightarrow \alpha\overline{A} \subset \overline{A} \Rightarrow \overline{A}$ is a balanced

(6)

(i) If $0 < |\alpha| \leq 1 \Rightarrow \alpha\,\mathrm{int}(A) = \mathrm{int}(\alpha A) \subset \alpha A \subset A$

Since αA^0 is open set and $\alpha\,\mathrm{int}(A) \subset A$, then $\alpha\,\mathrm{int}(A) \subset \mathrm{int}(A)$

(because $\mathrm{int}(A)$ is greatest open set which contain in A)

(ii) if $\alpha = 0$

Since $0 \in \mathrm{int}(A) \Rightarrow \alpha\,\mathrm{int}(A) = \{0\} \subset \mathrm{int}(A) \Rightarrow \mathrm{int}(A)$ is balanced.

Theorem(2.2.4)

Let A be a convex subset of a topological linear space X. Then

(1) If $x \in \mathrm{int}(A)$ and $y \in \overline{A}$, then $\alpha x + (1-\alpha)y \in \mathrm{int}(A)$, where $0 < \alpha < 1$

(2) $\mathrm{int}(A)$ and $\overline{A}$ are convex sets

(3) If $\mathrm{int}(A) \neq \phi$, then $\overline{A} = \overline{\mathrm{int}(A)}$ and $\mathrm{int}(A) = \mathrm{int}(\overline{A})$

Proof :

(1) Put $Z = \alpha x + (1-\alpha)y$

If $Z = 0 \Rightarrow y = -\dfrac{\alpha}{1-\alpha}x \Rightarrow y = \lambda x$ such that $\lambda < 0$.

Since the function $\varphi_\lambda : X \to X$, $\varphi_\lambda(x) = \lambda x$ for all $x \in X$ is homeomorphism of X, and $x \in \mathrm{int}(A)$, $\mathrm{int}(A)$ is open set in $X \Rightarrow \lambda\,\mathrm{int}(A)$ is a neighborhood of $y = \lambda x$

Since $y \in \overline{A} \Rightarrow A \cap (\lambda\,\mathrm{int}(A)) \neq \phi$, there is at least $W_0 \in A \cap (\lambda\,\mathrm{int}(A))$, i.e. there is $a \in \mathrm{int}(A)$ such that $W_0 = \lambda a \in A$.

Put $\beta = \dfrac{\lambda}{\lambda - 1} \Rightarrow \beta a + (1-\beta)\lambda a = 0, \quad 0 < \beta < 1$

Since the function $f : X \to X$, $f(W) = \beta W + (1-\beta)\lambda a$ for all $W \in X$ is homeomorphism of X, and $f(a) = 0$, then $U = \{\beta W + (1-\beta)\lambda a : W \in \mathrm{int}(A)\}$ is a neighborhood of 0

Since $W \in \text{int}(A) \Rightarrow \sim a \in A \Rightarrow U \subset A \Rightarrow 0 \in \text{int}(A) \Rightarrow Z \in \text{int}(A)$
If $Z \neq 0$, then the prove by translation for A.

(2)

(a) Let $x, y \in \text{int}(A)$ and $0 \leq \alpha \leq 1$

Since $\text{int}(A) \subset A \subset \overline{A} \Rightarrow y \in \overline{A}$

By (1) , we have $\alpha x + (1-\alpha) y \in \text{int}(A) \Rightarrow \text{int}(A)$ is convex

(b) since A is convex $\Rightarrow \alpha A + (1-\alpha) A \subset A$ for all $0 \leq \alpha \leq 1$

$\Rightarrow \overline{\alpha A + (1-\alpha) A} \subset \overline{A}$

Since $\alpha \overline{A} + (1-\alpha)\overline{A} = \overline{\alpha A} + \overline{(1-\alpha)A} \subset \overline{\alpha A + (1-\alpha)A}$

$\Rightarrow \alpha \overline{A} + (1-\alpha)\overline{A} \subset \overline{A} \Rightarrow \overline{A}$ is convex

(3)

(a) since $\text{int}(A) \subset A \Rightarrow \overline{\text{int}(A)} \subset \overline{A}$

Let $y \in \overline{A}$

If $x \in \text{int}(A)$, then $Z_\alpha = \alpha x + (1-\alpha) y \in \text{int}(A)$

From $y = \lim_{\alpha \to 0} Z_\alpha \in \overline{\text{int}(A)} \Rightarrow \overline{A} \subset \overline{\text{int}(A)} \Rightarrow \overline{A} = \overline{\text{int}(A)}$

(b) since $A \subset \overline{A} \Rightarrow \text{int}(A) \subset \text{int}(\overline{A})$

It suffices to show that if B is convex set and $\text{int}(B) \neq \emptyset$, then $0 \in \text{int}(\overline{B}) \Rightarrow 0 \in \text{int}(B)$

Theorem(2.2.5)

Let X be a topological linear space and let V be a neighborhood of 0 in X, then
(1) there a symmetric neighborhood U of 0 in X such that $U + U \subset V$
(2) V contains a balanced neighborhood of 0 in X
(3) X contains a balanced local base
(4) V is absorbing set

Proof :

(1) Since V be a neighborhood of 0 in X and $0 + 0 = 0$

$\Rightarrow V$ be a neighborhood of $0 + 0$ in X

Since $+ : X \times X \to X$ is continuous functions

there are neighborhoods V_1, V_2 of 0 in X such that $V_1 + V_2 \subset V$

Take $U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$

$\Rightarrow U$ is a symmetric neighborhood of 0 in X which satisfies $U + U \subset V$.

(2) Since $. : F \times X \to X$ is continuous functions , then there $r > 0$ and there is a neighborhood U of 0 in X such that $\alpha U \subset V$ whenever $|\alpha| < r$.

Let $W = \cup\{\alpha U : |\alpha| \leq r\} \Rightarrow W$ neighborhood of 0 in X and $W \subset V$

We now to show that W is balanced

Let $\sim \in F$ such that $|\sim| \leq 1$

Since $|ª| < r \Rightarrow |\simª| < r \Rightarrow (\simª)U \subset V$ and $(\simª)U \subset W$ (by definition of W)

$\Rightarrow \sim W \subset W \Rightarrow W$ is balanced

Theorem(2.2.6)

Let X be a topological linear space. Then every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

Proof :

By theorem (2.5), there is a balanced neighborhood W of 0 in X such that $W \subset V$.

Since W is a balanced $\Rightarrow \sim^{-1} W = W$ when $|\sim| = 1$

Since $W \subset V \Rightarrow \sim W \subset \sim V \Rightarrow W \subset \sim V$

Put $A = \cap\{\sim V : \sim \in F, |\sim| = 1\} \Rightarrow W \subset A$

Since $\text{int}(A)$ is greatest open set contain in $A \Rightarrow W \subset \text{int}(A)$

Since $0 \in W \Rightarrow 0 \in \text{int}(A)$

$\Rightarrow \text{int}(A)$ is a convex neighborhood of 0 subset of V.

Chose r and such that $0 \leq r \leq 1$, $| \; | = 1$. Then

$r \; A = \cap\{r \; \sim V : \sim \in F, |\sim| = 1\} = \cap\{r\sim V : \sim \in F, |\sim| = 1\}$

Since $\sim V$ is convex set contain $0 \Rightarrow r\sim V \subset \sim V$

$r \; A \subset A \Rightarrow A$ is balanced and $0 \in \text{int}(A)$

$\Rightarrow \text{int}(A)$ is a balanced convex neighborhood of 0 subset of V.

Definition(2.2.7)

A topological linear space X is called a locally convex if there is a convex local base, i.e. there is a local base at 0 in X such that every members of are convex sets. (every open set in X is a union of convex open sets)

It is clear to show that :

Every locally convex space has a balanced convex local base.

Theorem(2.2.8)

Let X be a topological linear space, then $A = \{0\}$ is closed in X iff for any element $x \neq 0$ there exists a neighborhood V of 0 in X such that $x \notin V$

Proof :

A is closed iff $\overline{A} \subset A$

A is closed iff for all $x \notin A$, then $x \notin \overline{A}$

A is closed iff for all $x \neq 0$, then there exists a neighborhood W of 0 in X such that $(x+W) \cap A = ¤$

A is closed iff for all $x \neq 0$, then there exists a neighborhood W of 0 in X such that $0 \notin x + W$

A is closed iff for all $x \neq 0$, then there exists a neighborhood W of 0 in X such that $-x \notin W$

A is closed iff for all $x \neq 0$, then there exists a neighborhood $V = -W$ of 0 in X such that $x \notin V$

Theorem (2.2.9)

A topological linear space X is a Hausdorff iff for any element $x \neq 0$ there exists a neighborhood V of 0 in X such that $x \notin V$

Proof :

Suppose that X is a Hausdorff and let $x \in X$ such that $x \neq 0$

By definition of Hausdorff space, there exists a neighborhood V of 0 in X and a neighborhood U of x in X such $V \cap U = \phi$

Since $x \in U \Rightarrow x \notin V$

The converse : in order to prove that X is a Hausdorff space it is sufficient to show that the set $A = \{(x,x) : x \in X\}$ is closed in $X \times X$

Since for any element $x \neq 0$ there exists a neighborhood V of 0 in X such that $x \notin V$, then $\{0\}$ is closed in X (by the above theorem)

Since the function $f : X \times X \to X$ which defined by $f(x,y) = x - y$ is continuous , then $f^{-1}(\{0\})$ is closed in $X \times X$

But $f^{-1}(\{0\}) = \{(x,x) : x \in X\} = A \Rightarrow$ a set A is closed set in $X \times X \Rightarrow X$ is Hausdorff space.

Definition(2.2.10)

Suppose now that τ is a linear topology on a linear space X and that M is a closed subspace of X. Let τ_M be the collection of all sets $A \subseteq X/M$ for which $\pi^{-1}(A) \in \tau$ .Then τ_M turns out to be a topology on X/M, called the quotient topology.

$$\tau_M = \{A \subseteq X/M : \pi^{-1}(A) \in \tau\}$$

i.e. $A \in \tau_M \Leftrightarrow \pi^{-1}(A) \in \tau$

Theorem(2.2.11)

Let M be a closed subspace of a topological linear space X

(1) τ_M is a topology on X/M

(2) π is continuous and open

(3) τ_M is a linear topology on X/M

(4) If is a local base for τ , then the collection of all sets $\pi(V)$ with $V \in$ is a local base for τ_M

Proof :

(1) (a) Since $\phi, X \in \tau$ and $\pi^{-1}(\phi) = \phi$, $\pi^{-1}(X/M) = X \Rightarrow \phi, X/M \in \tau_M$

(b) let $A_\alpha \in \tau_M$, $\alpha \in \Lambda \Rightarrow \pi^{-1}(A_\alpha) \in \tau \Rightarrow \bigcup_{\alpha \in \Lambda} \pi^{-1}(A_\alpha) \in \tau$

Since $\pi^{-1}(\bigcup_{\alpha \in \Lambda} A_\alpha) = \bigcup_{\alpha \in \Lambda} \pi^{-1}(A_\alpha) \Rightarrow \pi^{-1}\bigcup_{\alpha \in \Lambda}(A_\alpha) \in \tau \Rightarrow \bigcup_{\alpha \in \Lambda} A_\alpha \in \tau_M$

(c) let $A, B \in \tau_M \Rightarrow \pi^{-1}(A) \in \tau, \pi^{-1}(B) \in \tau \Rightarrow \pi^{-1}(A) \cap \pi^{-1}(B) \in \tau$

Since $\pi^{-1}(A \cap B) = \pi^{-1}(A) \cap \pi^{-1}(B) \Rightarrow \pi^{-1}(A \cap B) \in \tau \Rightarrow A \cap B \in \tau_M$

$\Rightarrow \tau_M$ is a topology on X/M

(2) (a) let A be an open set in X/M, then $A\in\tau_M \Rightarrow \pi^{-1}(A)\in\tau$

$\Rightarrow \pi^{-1}(A)$ is an open set in X, then π is continuous

(b) let V be an open set in X, then $V\in\tau$

Since $\pi^{-1}(\pi(V)=V+M$ and $V+M\in\tau \Rightarrow \pi^{-1}(\pi(V))\in\tau \Rightarrow \pi(V)\in\tau_M \Rightarrow \pi$ is open

(3) Let W be a neighborhood of 0 in X/M

$\Rightarrow W\in\tau_M \Rightarrow \pi^{-1}(W)\in\tau \Rightarrow \pi^{-1}(W)$ is a neighborhood of 0 in X

Since the addition is continuous on $X\times X$ into X, then there is a neighborhood V of 0 in X such that $V+V\subset\pi^{-1}(W)$, so $\pi(V)+\pi(V)\subset W$

Since π is open , then $\pi(V)$ is a neighborhood of 0 in X/M

$\Rightarrow +:(X/M)\times(X/M)\to(X/M)$ is continuous

Similarly, to prove $\cdot:F\times(X/M)\to(X/M)$ is continuous

$\Rightarrow \tau_M$ is a vector topology on X/M

2.3 Linear Metric

Let X be non-empty set. A function $d:X\times X\to\mathbb{R}$ is called a metric function, if

(1) $d(x,y)\geq 0$ for all $x,y\in X$ (2) $d(x,y)=0$ iff $x=y$

(3) $d(x,y)=d(y,x)$ for all $x,y\in X$ (4) $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$

The set X together with d is called a metric space and is denoted by (X,d) or simply X and $d(x,y)$ is called the distance between x and y. The set ${}_r(x_0)=\{x\in X: d(x,x_0)<r\}$ is called an open ball with center at x_0 and radius $r>0$. A topology τ is on a set X is said to be metrizable if there is a metric d on X which is compatible with τ. In that case ,the balls with radius $\frac{1}{n}$ centered at x form a local base at x.

Definition (2.3.1)

Let X be a linear space over F and let d be a metric function on X .We say that d is an invariant metric on X if $d(x+z,y+z)=d(x,y)$ for all $x,y,z\in X$.

Remark

$$d(-x,0)=d(x,0) \quad \text{for all } x\in X,$$

because $d(-x,0)=d(-x+x,0+x)=d(0,x)=d(x,0)$

Theorem(2.3.2)

Let X be a linear space over F and let d be an invariant metric on X

(1) $d(nx,0)\leq nd(x,0)$ for every $x\in X$ and for $n=1,2,\cdots$

(2) If $\{x_n\}$ is a sequence in X and if $x_n\to 0$ as $n\to\infty$, then there are positive scalar λ_n such that $\lambda_n\to\infty$ and $\lambda_n x_n\to 0$ as $n\to\infty$

Proof :

(1) $d(nx,0) \le \sum_{k=1}^{n} d(kx,(k-1)x) = nd(x,0)$

(2) since $x_n \to 0 \Rightarrow$ there is an increasing sequence of positive integer $\{n_k\}$ such that $d(x_n,0) < \frac{1}{k^2}$ if $n \ge n_k$

Put $\lambda_n = \begin{cases} 1 & ,n < n_k \\ k & ,n_k \le n \le n_{k+1} \end{cases}$

For such n, we have $d(\lambda_n x_n,0) = d(kx_n,0) \le kd(x_n,0) < \frac{1}{k}$. Hence $\lambda_n x_n \to 0$ as $n \to \infty$

Definition(2.3.3)

- A topological linear space X with topology τ is called an F-space if its topology τ is induced a complete invariant d .
- A topological linear space X is called Fréchet space if X is a locally convex F-space .

2.4 Boundeness

Definition (2.4.1)

Let A be a subset of a topological linear space X over F . We say that A is a **bounded** if for any neighborhood V of 0 in X ,there exists a real number $\lambda > 0$ such that $A \subset \lambda V$, and we say that X is locally bounded if there is a bounded neighborhood V of 0 in X .

A basis of bounded sets in X , a set F of bounded subsets of X such that every bounded set A in X , there exists $B \in F$ with $A \subset B$.

Theorem(2.4.2)

Let X be a topological linear space X over F and let $A, B \subseteq X$, then

(1) If A is finite, then A is bounded

(2) If B is bounded and $A \subseteq B$, then A is bounded

(3) If A, B are bounded sets, then $A \cap B, A \cup B, A + B$ are bounded sets

(4) If A is bounded, then αA is bounded for all $\alpha \in F$

(5) If A is bounded, then $\overline{A}$ is bounded

Proof :

(1) Since A is finite set, then $A = \{a_1, a_2, \cdots, a_n\}$

Let V be a neighborhood of 0 in X , then there exists a balanced neighborhood W of 0 in X such that $W \subset V$

Since every neighborhood is absorbing set, then W is absorbing set, so that for all $x \in X$, there exists $\lambda > 0$ such that $\lambda x \in W$

Since $A \subset X \Rightarrow a_i \in X$ for all $i = 1, \ldots, n \Rightarrow$ there exists $\lambda_i > 0$ such that $\lambda_i a_i \in W$ for all $i = 1, \cdots, n$

Take $ª = \max\{ª_1, ª_2, \cdots ª_n\}$

Since W is balanced set $\Rightarrow \bigcup_{i=1}^{n} ª_i W = ª W$

Since $A \subseteq \bigcup_{i=1}^{n} ª_i W \Rightarrow A \subset ª W \Rightarrow A \subset ª V \Rightarrow A$ is balanced

(2) Let V be a neighborhood of 0 in X
Since B is bounded set, then there exists $ª > 0$ such that $B \subset ª V$
Since $A \subseteq B \Rightarrow A \subset ª V \Rightarrow A$ is bounded

(3)

(i) since $A \cap B \subseteq A$ and A is bounded $\Rightarrow A \cap B$ is bounded

(ii) Let V be a neighborhood of 0 in X, there is a balanced neighborhood W of 0 in X such that $W \subset V$
Since A, B are bounded , then there exists $ª_1, ª_2 > 0$ such that $A \subset ª_1 W$ and $B \subset ª_2 W$
Take $ª > \max\{ª_1, ª_2\}$
Since W is balanced $\Rightarrow A \cup B \subset ª W$
Since $W \subset V \Rightarrow ª W \subset ª V \Rightarrow A \cup B \subset ª V \Rightarrow A \cup B$ is bounded

(iii) Let V be a neighborhood of 0 in X, there is a symmetric neighborhood W of 0 in X such that $W + W \subset V$
$\Rightarrow$ there is a balanced neighborhood U of 0 in X such that $U \subset W$
Since A, B are bounded , then there exists $ª_1, ª_2 > 0$ such that $A \subset ª_1 U$ and $B \subset ª_2 U$
Take $ª > \max\{ª_1, ª_2\}$
Since U is balanced $\Rightarrow ª_1 U + ª_2 U \subset ª(U + U)$
Since $U \subset W \Rightarrow ª(U + U) \subset ª(W + W) \subset ª V \Rightarrow A + B$ is bounded

(4) If $\sim = 0$, then $\sim A = \{0\} \Rightarrow \sim A$ is bounded

If $\sim \neq 0$
Let V be a neighborhood of 0 in X, there is a balanced neighborhood W of 0 in X such that $W \subset V$
Since A is bounded, there is $ª > 0$ such that $A \subset ª W$
Take $r = ª|\sim| \Rightarrow r > 0$
Since W is balanced and $\sim \leq |\sim| \Rightarrow \sim W \subset |\sim| W \Rightarrow ª \sim W \subset ª|\sim| W$
Since $A \subset ª W \Rightarrow \sim A \subset ª \sim W \subset ª|\sim| W$
Since $W \subset V \Rightarrow rW \subset rV \Rightarrow \sim A \subset rV \Rightarrow \sim A$ is bounded

(5) Let V be a neighborhood of 0 in X, there is a neighborhood W of 0 in X such that $\overline{W} \subset V$
Since A is bounded, there is $ª > 0$ such that $A \subset ª W \Rightarrow \overline{A} \subset \overline{ª W} = ª \overline{W}$
Since $\overline{W} \subset V \Rightarrow ª \overline{W} \subset ª V \Rightarrow \overline{A} \subset ª V \Rightarrow \overline{A}$ is bounded.

Definition (2.4.3)

Let A be a subset of a topological linear space X over F. We say that A is a Totally bounded if for any neighborhood V of 0 in X, there exists a finite subset B of X such that $A \subset B+V$

Theorem(2.4.4)

If A is a totally bounded of a topological linear space X over F, then for any neighborhood V of 0 in X, there exists a finite subset A_0 of A such that $A \subset V + A_0$

Proof :

Let V be a neighborhood of 0 in X, there is a balanced neighborhood W of 0 in X such that $W+W \subset V$

Since A is totally bounded , there exists a finite subset B of X such that $A \subset B+V$

$\Rightarrow \quad B=\{b_1,b_2,...,b_n\}$

Since $W+B=\bigcup_{i=1}^{n}\{W+b_i : b_i \in B\}$, then $A\cap(W+b_i)\neq$ ¤ for all $i=1,...,n$

If $a_i \in A\cap(W+b_i) \Rightarrow a_i = x_i + b_i$, $a_i \in A$ where $b_i \in B$, $x_i \in W$

$\Rightarrow \quad b_i = a_i - x_i \quad \Rightarrow \quad b_i \in A+W$

Setting $A_0=\{a_1,a_2,...,a_n\} \quad \Rightarrow \quad A \subset W + A_0$

Theorem(2.4.5)

Let X be a topological linear space X over F and let $A,B \subseteq X$, then

(1) If A is finite, then A is a totally bounded

(2) If A is totally bounded, then A is a bounded

(3) If B is a totally bounded and $A \subseteq B$, then A is a totally bounded

(4) If A,B are totally bounded sets, then $A\cap B, A\cup B, A+B$ are totally bounded sets

Proof :

(1) Since $A \subset A+V$ for every neighborhood V of 0 in $X \quad \Rightarrow \quad A$ is a totally bounded

(2) Let V be a neighborhood of 0 in X, there is a balanced neighborhood W of 0 in X such that $W \subset V$

Since A is a totally bounded set, there exists a finite subset B of X such that $A \subset B+W$

Since B is finite, then B is bounded, then there is $\sim\, > 0$ such that $B \subset \sim W$

Since W is balanced $\Rightarrow \quad \sim W + W \subset (\sim +1)W$

Take ª $= \sim +1 \Rightarrow \quad A \subset$ ª$W \subset$ ª$V \quad \Rightarrow \quad A$ is bounded set.

(3) Let V be a neighborhood of 0 in X

Since B is a totally bounded, there exists a finite subset D of X such that $B \subset D+V$

Since $A \subset B \quad \Rightarrow \quad A \subset D+V \quad \Rightarrow \quad A$ is a totally bounded

(4)

(i) since $A\cap B \subseteq A$ and A is totally bounded $\quad \Rightarrow \quad A\cap B$ is totally bounded

(ii) Let V be a neighborhood of 0 in X

Since A, B are totally bounded , then there are finite subsets D_1, D_2 such that
$A \subset D_1 + V$ and $B \subset D_2 + V$
Take $D = D_1 \cup D_2 \Rightarrow D$ is finite and $A \cup B \subset D \cup V \Rightarrow A \cup B$ is totally bounded

(iii) Let V be a neighborhood of 0 in X, there is a symmetric neighborhood W of 0 in X such that $W + W \subset V$
Since A, B are totally bounded , then there are finite subsets D_1, D_2 such that $A \subset D_1 + W$ and $B \subset D_2 + W$
Take $D = D_1 \cup D_2 \Rightarrow D$ and $A + B \subset D + W + W \subset D + V \Rightarrow A + B$ is totally bounded

2.5 Convergence

Definition (2.5.1)

Let X be a topological linear space X over F.

(1) A sequence $\{x_n\}$ in X is said to converge to the point $x \in X$ if for every neighborhood V of 0 in X, there exists $k \in \mathbb{Z}^+$ such that $x_n \in x + V$ for all $n \geq k$, and we write

$$x_n \to x \text{ or } \lim_{n \to \infty} x_n = x$$

(2) A sequence $\{x_n\}$ in X is said to Cauchy sequence if for every neighborhood V of 0 in X, there exists $k \in \mathbb{Z}^+$ such that $x_n - x_m \in V$ for all $n, m \geq k$

Theorem(2.5.2)

Let A be a subset of a topological space X .Then the following two statements are equivalent :

(1) A is bounded

(2) If $\{x_n\}$ is a sequence in A and $\{\alpha_n\}$ is a sequence in F such that $\alpha_n \to 0$ as $n \to \infty$, then $\alpha_n x_n \to 0$ as $n \to \infty$

Proof :

(1) $\Rightarrow$ (2)

Let V be a neighborhood of 0 in X, there is a balanced neighborhood W of 0 in X such that $W \subset V$
Since A is bounded set, there is $\alpha > 0$ such that $A \subset \alpha W$
Since $x_n \in A \Rightarrow x_n \in \alpha W$
Since $\alpha_n \to 0$ and $\frac{1}{\alpha} > 0 \Rightarrow$ there is $k \in \mathbb{Z}^+$ such that $|\alpha_n - 0| < \frac{1}{\alpha}$ for all $n \geq k$
$\Rightarrow \alpha |\alpha_n| < 1$ for all $n \geq k$
Since $x_n \in \alpha W \Rightarrow \alpha_n x_n \in \alpha_n \alpha W \Rightarrow \alpha_n x_n \in W$ if $n > k$
Since $W \subset V \Rightarrow \alpha_n x_n \in V$ for all $n > k \Rightarrow \alpha_n x_n \to 0$
(2) $\Rightarrow$ (1)

Assume A is not bounded , there is a neighborhood V of 0 in X such that $A \not\subset nV$ for all $n \in \mathbb{Z}^+$. Hence for any $n \in \mathbb{Z}^+$, there exists $x_n \in A$ such that $x_n \notin nV$ for all $n \in \mathbb{Z}^+$. Thus $n^{-1}x_n \notin V$ for all $n \in \mathbb{Z}^+$ $\Rightarrow$ $n^{-1}x_n \not\to 0$ This contradiction

2.6 Compactness

Definition(2.6.1)

A topological space X is said to be compact if every open cover of X has a finite subcover.

- A subset A of a topological space X is called if it is compact space in the relative .In view definition of the relative topology, this equivalent to saying that a subset A of X is compact if every covering of A by open sets of X has a finite subcovering.
- A topological space X is said to be locally compact if for all $x \in X$,there exist a neighborhood V of x such that $\overline{V}$ is compact in X

Theorem(2.6.2)

(1) Every closed subset of a compact space is itself compact.
(2) A compact subset of a Hausdorff space is closed.
(3) A compact subset of a metric space is closed and bounded.
(4) A continuous image of a compact set is compact.
(5) Every compact space is locally compact, but the converse not true.

Theorem(2.6.3)

Let X be a Hausdorff topological space and let $A, B \subseteq X$ and $\sim, \quad \in F$
(1) If A is compact set , then it is totally bounded
(2) if A, B are compact sets ,then $\sim A + \quad B$ is also compact set
(3) If A is compact set, and B is closed set then $A + B$ is closed set

Proof :

(1) Let V be a neighborhood of 0 in X , $\Rightarrow$ $\cup\{x + V : x \in A\}$ is open cover of A
Since A is compact , then A has finite subcover $\Rightarrow$ A is totally bounded.

(2) Since A and B are compact, then $A \times B$ is also compact
Since the function $f : X \times X \to X$ which defined by $f(x, y) = \sim x + \quad y$ for all $x, y \in X$ and for all $\sim, \quad \in F$ is continuous, $f(A \times B) = \sim A + \quad y$ is compact

(3) Let $x_0 \notin A + B$
Since for any $x \in A$, then $B + x$ is closed set in X
$\Rightarrow$ there is a neighborhood V of 0 in X such that $(V(x) + x_0) \cap (B + x) = ¤$
$\Rightarrow$ there is a symmetric neighborhood $W(x)$ of 0 in X such that $W(x) + W(x) \subset V(x)$
The system $\{W(x) + x : x \in A\}$ forms an open cover of A
Since A is compact, there exists a finite subcovering $\{W(x_i) + x_i : \quad i = 1,2,\ldots,n\}$

Setting $W = \bigcap_{i=1}^{n} W(x_i)$. One has $W + A \subset \bigcup_{x \in A} (V(x) + x)$

Since $x_0 \notin A + B \Rightarrow x_0 \notin V + B + x \Rightarrow x \notin W + A + B$. Hence $(W + x_0) \cap (A + B) = ¤$

$x_0 \notin \overline{A + B} \Rightarrow \overline{A + B} = A + B \Rightarrow A + B$ is closed

Theorem(2.6.4)

Let X be a Hausdorff topological space

(1) If $0 < r_1 < r_2 < \cdots$ and $r_n \to \infty$ as $n \to \infty$ and if V is a neighborhood of 0 in X , then $X = \bigcup_{n=1}^{\infty} r_n V$

(2) Every compact subset A of X is bounded

(3) If $¢_1 > ¢_2 > \cdots$ and $¢_n \to 0$ as $n \to \infty$ and if V is a neighborhood of 0 in X , then the collection $\{¢_n V : n = 1,2,...\}$ is a local base for X .

Proof :

(1) Let $x \in X$ and let A be the set of all $\sim \in F$ with $\sim x \in V$,i.e. $A = \{\sim \in F : \ \sim x \in V\}$

since $\sim \longrightarrow \sim x$ is a continuous function from F into $X \Rightarrow A$ is open set

Since $0 \in V$ and $0 = 0x \Rightarrow 0x \in V \Rightarrow 0 \in A \Rightarrow A$ is open set, contain 0

Since $r_n \longrightarrow \infty$ as $n \to \infty \Rightarrow \frac{1}{r_n} \to 0$ as $n \to \infty \Rightarrow \frac{1}{r_n} \in A$ for all large n

$\Rightarrow \frac{1}{r_n} x \in V$ for all large $n \Rightarrow x \in r_n V$ for all large $n \Rightarrow X \in \bigcup_{n=1}^{\infty} r_n V \Rightarrow X \subset \bigcup_{n=1}^{\infty} r_n V$

Since $\bigcup_{n=1}^{\infty} r_n V \subset X \Rightarrow X = \bigcup_{n=1}^{\infty} r_n V$

(2) Let V be a neighborhood of 0 in X ,there is a balanced neighborhood W of 0 in X such that $W \subset V$. From (1) , we have $X = \bigcup_{n=1}^{\infty} nW$

since $A \subset X \Rightarrow A \subset \bigcup_{n=1}^{\infty} nW$

since A is compact set, there are integers $n_1 < ... < n_r$ such that

$$A \subset n_1 W \cup n_2 W \cup ... \cup n_r W = n_r W$$

If $ª > n_r$, then $A \subset ª W \subset ª V \Rightarrow A$ is bounded

(3) Let U be a neighborhood of 0 in X

Since V is bounded set, there is $ª > 0$ such that $V \subset ª U$

If n is so large that $ª ¢_n < 1$, it follows that $V \subset (\frac{1}{¢_n}) U \Rightarrow ¢_n V \subset U$

Heine–Borel Property

A topological linear X has the Heine – Borel Property if every closed and bounded subset of X is compact.

دراسات عليا - ماجستير

Functional Analysis تحليل دالي

نظري : 3   مناقشة : 1   عدد الوحدات : 3

Exercises (2)

2.1 Suppose A, B are disjoint subsets of a topological linear space X , A is compact , B is closed. Show that 0 has a neighborhood V such that $(A+V)\cap(B+V)=\emptyset$

2.2 In a topological linear space X . Show that

(1) The balanced hull of a bounded set is a bounded set.

(2) $A\subseteq X$ is bounded iff every countable subset of A is bounded.

(3) If A is a totally bounded , then $\overline{A}$ is also totally bounded.

2.3 If M is a subspace of a topological linear space X and $A\subseteq M$. Show that A is bounded in M iff A is bounded in X .

2.4 Suppose M is a subspace of a topological linear space X , and M is locally compact, in the topology inherited from X . Show that M is closed subspace of X .

2.5 Suppose M is a subspace of a complex topological linear space X , and $\dim M = n > 0$. Show that every isomorphism of $\mathbb{C}^n$ onto M is a homeomorphism, and M is closed.

2.6 Show that every locally compact topological linear space X has finite dimension.

2.7 If X is a locally bounded topological linear space with Heine-Borel property. Show that X has finite dimension.

2.8 If X is a topological linear space with a countable local base . Show that there is a metric d on X such that

(1) d is compatible with the topology of X .

(2) The open balls centered at zero balanced

(3) d is invariant metric.

2.9 Suppose M is a subspace of a topological linear space X , and M is an F-space. Show that M is closed subspace of X .

2.10 Prove or disprove :Every bounded set in a Hausdorff topological linear space is compact .