5. Separation Theorems

5.1 The Hahn -Banach Theorem

Let X be a linear space over F. If $F = \mathbb{C}$, then by complex - liner functional on X. According to our first lemma, real-linear functionals can be characterized as the real parts of associated complex-linear functionals.

Lemma (6.1.1)

Let X be a complex linear space.

(1) If f is a complex–linear functional on X and u is the real part of f ,then u is the real linear functional on X and $f(x) = u(x) - iu(ix)$ $\cdots(1)$ for all $x \in X$

(2) If $u : X \to \mathbb{R}$ is real –linear on X and f is defined by the equation (1), then, f is a complex linear functional on X

(3) If $\circ$ is a seminorm on X , f, u u are related as in equation (1), then $|u(x)| \le \circ(x)$ for all $x \in X$ iff $|f(x)| \le \circ(x)$ for $x \in X$

(4) If X is normed space , f, u are related as in equation (1) and either f or u is bounded, then both functionals are bounded and $\|f\| = \|u\|$

(5) If X is a complex topological linear space. A complex–linear functional on X is in X^* iff its real part is continuous, and that every real linear $u : X \to \mathbb{R}$ is the real part of a unique $f \in X^*$

Proof :

(1) Let $v = \mathrm{Im}(f)$

Since $u = \mathrm{Re}(f) \Rightarrow f(x) = u(x) + iv(x)$ for all $x \in X$

$if(x) = iu(x) + i^2 v(x) = -v(x) + iu(x)$ and $f(ix) = u(ix) + iv(ix) \Rightarrow if(x) = u(ix) + iv(ix)$

$\Rightarrow -v(x) + iu(x) = u(ix) + iv(ix) \Rightarrow u(ix) = -v(x), \; v(ix) = u(x)$

$\Rightarrow f(x) = u(x) + iv(x) = u(x) - iu(ix)$

Let $x, y \in X$ and $\sim, \; \in \mathbb{R}$

Since f is linear functionals

$f(\sim x + \; y) = \sim f(x) + \; f(y) = \sim(u(x) - iu(ix)) + \; (u(y) - iu(iy)) = \sim u(x) + \; u(y) - i(\sim u(ix) + \; u(iy))$

Also $f(\sim x + \; y) = u(\sim x + \; y) - iu(i(\sim x + \; y))) \Rightarrow u(\sim x + \; y) = \sim u(x) + \; u(y) \Rightarrow u$ is linear

(2) Let $x, y \in X$ and $\sim, \; \in \mathbb{C}$

$\Rightarrow f(\sim x + \; y) = u(\sim x + \; y) - iu(i(\sim x + \; y)) = \sim u(x) + \; u(y) - i(i \sim u(x) + i \; u(y))$

$= \sim(u(x) - iu(ix)) + \; (u(y) - iu(iy)) = \sim f(x) + \; f(y)$

$\Rightarrow f$ is a complex–linear functional on X

(3) Suppose that $|f(x)| \le \circ(x)$ for $x \in X$

Since $u(x) = \mathrm{Re}(f(x)) \Rightarrow u(x) \le |f(x)|$ for $x \in X \Rightarrow u(x) \le \circ(x)$

Since

$-u(x) = u(-x) = \mathrm{Re}(f(-x)) \le |f(-x)| = |f(x)| \Rightarrow -u(x) \le \circ(x) \Rightarrow u(x) \ge -\circ(x)$

$\Rightarrow -\circ(x) \le u(x) \le \circ(x) \Rightarrow |u(x) \le \circ(x)|$

Conversely , If $|u(x)| \le p(x)$ for all $x \in X$

Let $f(x) = re^{i\theta}, \quad r \ge 0 \quad \Rightarrow \quad |f(x)| = |re^{i\theta}| = |r||e^{i\theta}| = r \times 1 = r$

Since $f(x) = re^{i\theta} \quad \Rightarrow \quad r = e^{-i\theta} f(x) = f(e^{-i\theta} x) \quad \Rightarrow \quad |f(x)| = f(e^{-i\theta} x)$

Since $f(x) = u(x) - iu(ix) \quad \Rightarrow \quad r = f(e^{-i\theta} x) = u(e^{-i\theta} x) - iu(ie^{-i\theta} x)$

Since r is real $\Rightarrow \quad u(ie^{-i\theta} x) = 0 \quad \Rightarrow \quad r = u(e^{-i\theta} x) \quad \Rightarrow \quad |f(x)| = r = u(e^{-i\theta} x)$

Since $u(x) \le p(x) \quad \Rightarrow \quad u(e^{-i\theta} x) \le p(e^{-i\theta} x)$

$\Rightarrow \quad |f(x)| \le p(e^{-i\theta} x) = p(x)$ for $x \in X$

(4) If f is bounded, then as $|u(x)| \le |f(x)|$ for all $x \in X$, u is bounded and $\|u\| \le \|f\|$.
For each
$x \in X$ there exists $\alpha \in \mathbb{C}$ such that $\|\alpha\| = 1$ and $f(\alpha x) = \alpha f(x) = |f(x)|$, then
$f(\alpha x) \in \mathbb{R}$, so
$|f(x)| = f(\alpha x) = \mathrm{Re}(f(\alpha x)) = u(\alpha x) \le \|u\|\|\alpha x\| = \|u\|\|x\|$. Hence $\|f\| \le \|u\|$, and therefore
$\|f\| = \|u\|$

Finally, if u is bounded, then for all $x \in X$ with $\|x\| \le 1$ we have
$|f(x)| \le |u(x)| + |u(ix)| \le \|u\|(\|x\| + \|ix\|) \le 2\|u\|$.So f is bounded. By the forgoing, $\|f\| = \|u\|$.

Theorem (5.1.2)

Let M be a proper subspace of a linear space X over F , and let $x_0 \in X$, $x_0 \notin M$.
Define $M_0 = [M \cup \{x_0\}] = \{m + \alpha x_0 : m \in M, \alpha \in F\}$, then

(1) M_0 is a subspace of X

(2) If $g \in M'$, then there exists $f_0 \in (M_0)'$ such that $f_0(x) = g(x)$ for all $x \in M$.
Moreover if X is a normed space then $\|f_0\| = \|g\|$

Proof :

Case (1) : X is real linear space , i.e. $F = \mathbb{R}$

(1) It is obvious

(2) Define $f_0 : M_0 \to R$, by $f_0(x) = f_0(m + \alpha x_0) = g(m) + \alpha r_0, \quad r_0 \in \mathbb{R}$. We must to prove

(i) f_0 is linear :

Let $x, y \in M_0$ and $\lambda, \mu \in R$, then $x = m_1 + \alpha_1 x_0, \quad y = m_2 + \alpha_2 x_0$

$\lambda x + \mu y = \lambda(m_1 + \alpha_1 x_0) + \mu(m_2 + \alpha_2 x_0) = (\lambda m_1 + \mu m_2) + (\lambda \alpha_1 + \mu \alpha_2) x_0$

$f_0(\lambda x + \mu y) = g(\lambda m_1 + \mu m_2) + (\lambda \alpha_1 + \mu \alpha_2) r_0 = \lambda g(m_1) + \mu g(m_2) + \lambda(\alpha_1 r_0) + \mu(\alpha_2 r_0)$

$= \lambda(g(m_1) + \alpha_1 r_0) + \mu(g(m_2) + \alpha_2 r_0) = \lambda f_0(x) + \mu f_0(y)$

$\Rightarrow \quad f_0$ is linear $\Rightarrow \quad f_0 \in (M_0)'$

(ii) $f_0(x) = g(x)$ for all $x \in M$

Let $x \in M \quad \Rightarrow \quad x = x + (0)x_0 \quad \Rightarrow \quad f_0(x) = f_0(x + (0)x_0) = g(x) + (0)r_0 = g(x)$

Now if X is a normed space, we now prove that $\|f_0\| = \|g\|$

Now $\|f_0\| = \sup\{|f_0(x)| : x \in M_0, \|x\| \le 1\}$

since $M \subseteq M_0 \quad \Rightarrow \quad \sup\{|f_0(x)| : x \in M_0, \|x\| \le 1\} \ge \sup\{|f_0(x)| : x \in M, \|x\| \le 1\}$

$\Rightarrow \quad \|f_0\| \ge \sup\{|f_0(x)| : x \in M, \|x\| \le 1\}$

Since $f_0(x) = g(x)$ for all $x \in M$

$\Rightarrow \quad \sup\{|f_0(x)| : x \in M, \|x\| \le 1\} = \sup\{|g(x)| : x \in M_0, \|x\| \le 1\} = \|g\|$

$\Rightarrow \quad \|f_0\| \ge \|g\|$

Let $x_1, x_2 \in M$, then

$g(x_2) - g(x_1) = g(x_2 - x_1) \le |g(x_2 - x_1)| \le \|g\|\|x_2 - x_1\| = \|g\|\|(x_2 + x_0) - (x_1 + x_0)\|$

$\Rightarrow \quad g(x_2) - g(x_1) \le \|g\|\|x_2 + x_0\| + \|g\|\|-(x_1 + x_0)\| = \|g\|\|x_2 + x_0\| + \|g\|\|x_1 + x_0\|$

Thus $-g(x_1) - \|g\|\|x_1 + x_0\| \le -g(x_2) + \|g\|\|x_2 + x_0\|$ .Since this inequality holds for arbitrary $x_1, x_2 \in M$, we see that $\sup_{y \in M}\{-g(y) - \|g\|\|y + x_0\|\} \le \inf_{y \in M}\{-g(y) + \|g\|\|y + x_0\|\}$

Choose r_0 to be real number such that

$\sup_{y \in M}\{-g(y) - \|g\|\|y + x_0\|\} \le r_0 \le \inf_{y \in M}\{-g(y) + \|g\|\|y + x_0\|\}$

It follows that $-g(y) - \|g\|\|y + x_0\| \le r_0 \le -g(y) + \|g\|\|y + x_0\|$ for all $y \in M$

Putting $y = \frac{x}{\sim}$, we have $-g(\frac{x}{\sim}) - \|g\|\left\|\frac{x}{\sim} + x_0\right\| \le r_0 \le -g(\frac{x}{\sim}) + \|g\|\left\|\frac{x}{\sim} + x_0\right\|$ (1)

If $\sim > 0$, then right hand inequality in (1) gives $r_0 \le -g(\frac{x}{\sim}) + \|g\|\left\|\frac{x}{\sim} + x_0\right\|$

$\Rightarrow \quad r_0 \le -\frac{1}{\sim} g(x) + \frac{1}{\sim}\|g\|\|x + \sim x_0\| \quad \Rightarrow \quad g(x) + \sim r_0 \le \|g\|\|x + \sim x_0\| \quad \Rightarrow \quad f_0(x + \sim x_0) \le \|g\|\|x + \sim x_0\|$

$\Rightarrow \quad f_0(z) \le \|g\|\|z\|$, where $z = x + \sim x_0$

If $\sim < 0$, then left hand inequality in (1) gives $-g(\frac{x}{\sim}) - \|g\|\left\|\frac{x}{\sim} + x_0\right\| \le r_0$

$\Rightarrow \quad -\frac{1}{\sim} g(x) - \left|\frac{1}{\sim}\right|\|g\|\|x + \sim x_0\| \ge r_0 \quad \Rightarrow \quad -\frac{1}{\sim} g(x) + \frac{1}{\sim}\|g\|\|x + \sim x_0\| \ge r_0 \quad \Rightarrow \quad \sim r_0 \le -g(x) + \|g\|\|x + \sim x_0\|$

$\Rightarrow \quad g(x) + \sim r_0 \le \|g\|\|x + \sim x_0\| \quad \Rightarrow \quad f_0(z) \le \|g\|\|z\|$, where $z = x + \sim x_0$

Thus we have show that when $\sim \ne 0$, then $f_0(z) \le \|g\|\|z\|$ for all $z \in M_0$

Since g is bounded , then f_0 is bounded linear functional

Since $\|f_0\| = \sup\{|f_0(x)| : x \in M_0, \|x\| \le 1\}$,then $\|f_0\| \le \|g\|$. It follows that $\|f_0\| = \|g\|$

Case (2) : X is complex linear space , i.e. $F = \mathbb{C}$

Let u be real part of g , by lemma(5.1.1), we have $u \in M'$ and $g(x) = u(x) - iu(ix)$ for all $x \in M$. Moreover $\|g\| = \|u\|$ .By case(1), there exists $u_0 \in X'$ such that $u_0(x) = u(x)$ for all $x \in M$ and $\|u_0\| = \|u\|$, so $\|g\| = \|u_0\|$, put $f_0(x) = u_0(x) - iu_0(ix)$ for $x \in X$, by lemma (6.2), we have $f_0 \in X'$ and $\|f_0\| = \|u_0\|$. Since $\|g\| = \|u_0\| \Rightarrow \quad \|f_0\| = \|g\|$

Theorem (5.1.3)

Let M be a subspace of linear space X and let $g \in M'$, then there exists $f \in X'$ such that $f(x) = g(x)$ for all $x \in M$. Moreover if X is a normed space then $\|f\| = \|g\|$

Proof :

Let $\mathcal{G}$ be the collection of all ordered pairs $(f_{¥}, M_{¥})$ such that

(i) $M_{¥}$ is a subspace of X and $M \subset M_{¥}$ (ii) $f_{¥} \in (M_{¥})'$ such that $f_{¥}(x) = g(x)$ for $x \in M$

$\Rightarrow$ G is non-empty and partially ordered by

$(f_{¥}, M_{¥}) \le (f_{˜}, M_{˜}) \Leftrightarrow M_{¥} \subset M_{˜}$ & $f_{¥}(x) = f_{˜}(x) \quad \forall x \in M_{¥}$

Let $\Phi = \{(f_{¥}, M_{¥})\}$ be a totally ordered set in G . then it is easy to see that Φ has an upper bound $(\Psi, \cup M_{˜})$ where $\Psi(x) = f_{˜}(x)$ for all $x \in M_{˜}$. By using Zorn's Lemma, there exists a maximal element (f, H) in G. To complete the proof, we must show that $H = X$.

Suppose that $H \neq X$, then there exists $a \in X$ such that $a \notin H$

Put $H_0 = [H \cup \{a\}]$ by using first part in this proof, we have $h \in H_0'$ such that $h(x) = f(x)$ for all $x \in H_0$. But contradicts the maximally of (f, H) (f, H).

Consequently, we must have $H = X$ and the proof is complete.

Theorem(5.1.4)

Let M be a proper subspace of a real linear space X ,and let $x_0 \notin M$,then there exists $f \in X'$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in M$.

Proof :

Let $M_0 = [M \cup \{x_0\}] = \{m + ª x_0 : m \in M, ª \in \mathbb{R}\}$, then M_0 is a subspace of X.

Define $g : M_0 \to \mathbb{R}$, by $g(x) = g(m + ª x_0) = ª$ for all $x \in M_0$ .We must to prove

(1) g is linear : let $x, y \in M_0$ and $\sim, \ \in \mathbb{R}$, $\quad x = m_1 + ª_1 x_0$ and $y = m_2 + ª_2 x_0$

$\sim x + \ y = \sim(m_1 + ª_1 x_0) + \ (m_2 + ª_2 x_0) = (\sim m_1 + \ m_2) + (\sim ª_1 + \ ª_2) x_0$

$g(\sim x + \ y) = (\sim ª_1 + \ ª_2) = \sim g(x) + \ g(y) \quad \Rightarrow \quad g \in M_0'$

(2) $g(x_0) = 1$ and $g(x) = 0$ for all $x \in M$

since $x_0 = 0 + (1)x_0 \quad \Rightarrow \quad g(x_0) = 1$

Let $x \in M \quad \Rightarrow \quad x = x + (0)x_0 \quad \Rightarrow \quad g(x) = 0$

If $M_0 = X$, then we finish ; either if $M_0 \neq X$, then M_0 is a proper subspace of X and $g \in M_0'$, by using theorem (5.1.3), there exists $f \in X'$ such that $f(x) = g(x)$ for all $x \in M$. Hence $f(x_0) = 1$ and $f(x) = 0$ for all $x \in M$.

Corollary (5.1.5)

Let X be a real linear space. If $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$

Proof :

Let $x_0 \neq 0$

Put $M = \{0\} \quad \Rightarrow \quad M$ is a subspace of X and $x_0 \notin M$ .By using theorem (5.1.**4**), there exists $f \in X'$ such that $f(x_0) = 1$.This contradiction $\Rightarrow \quad x_0 = 0$

Definition(5.1.6)
Let X be a linear space over F. The function $p: X \to \mathbb{R}$ is called sub-linear functional on X if
(1) $p(x+y) \le p(x)+p(y)$ for all $x, y \in X$ (Sub-additivity)
(2) $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and for all $\alpha \ge 0$ (Positive homogeneity)
If in addition, P satisfies the condition
(3) $P(x) \ge 0$ for all $x \in X$, then P is called a convex functional
A convex functional P is said to be symmetric if we have $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and $\alpha \in \mathbb{R}$.

Example (5.1.7)
Let $X = \mathbb{R}^n$. Define $P: X \to \mathbb{R}$ by $P(x) = \sum_{i=1}^{n} |x_i|$ for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then P is a sub-linear functional on X and Convex Functional .

Theorem(5.1.8) Generalized Hahn-Banach Theorem
Suppose
(1) M is a subspace of a real linear space X (2) P is a sublinear functional on X
(3) $g \in M'$ such that $g(x) \le p(x)$ for all $x \in M$. Then there exists $f \in X'$ such that
(i) $f(x) = g(x)$ for all $x \in M$ (ii) $f(x) \le p(x)$ for all $x \in X$

Proof :
Let $x_0 \in M$ and $M_0 = [M \cup \{x_0\}] = \{m + \alpha x_0 : m \in M, \alpha \in \mathbb{R}\}$, then M_0 is a subspace of X.
Define $f_0 : M_0 \to \mathbb{R}$, by $f_0(x) = f_0(m + \alpha x_0) = g(m) + \alpha r_0,\quad r_0 \in \mathbb{R}$
It is easy to see that f_0 is linear and $f_0(x) = g(x)$ for all $x \in M$
Now to prove : $f_0(x) \le p(x)$ for all $x \in M_0$
Let $m_1, m_2 \in M$
$g(m_2) - g(m_1) = g(m_2 - m_1) \le p(m_2 - m_1) = p((m_2 + x_0) + (-m_1 - x_0)) \le p(m_2 + x_0) + p(-m_1 - x_0)$
$-g(m_1) - p(-m_1 - x_0) \le -g(m_2) + p(m_2 + x_0)$ for all $m_1, m_2 \in M$
so that $\sup_{y \in M}\{-g(y) - p(-y - x_0)\} \le \inf_{y \in M}\{-g(y) + p(y + x_0)\}$
Choose r_0 such that $\sup_{y \in M}\{-g(y) - p(-y - x_0)\} \le r_0 \le \inf_{y \in M}\{-g(y) + p(y + x_0)\}$
It follows that $-g(y) - p(-y - x_0) \le r_0 \le -g(y) + p(y + x_0)$ $(*)$ for all $y \in M$
Let $x \in M_0 \Rightarrow x = m + \alpha x_0$
If $\alpha = 0 \Rightarrow x = m$, then $f_0(x) = g(m) \le p(m) = p(x)$
If $\alpha \ne 0$, put $y = \frac{m}{\alpha} \Rightarrow y \in M$ in $(*)$ to obtain

$$-g\left(\frac{m}{\alpha}\right) - p\left(-\frac{m}{\alpha} - x_0\right) \le r_0 \le -g\left(\frac{m}{\alpha}\right) + p\left(\frac{m}{\alpha} + x_0\right) \qquad (**) \text{ for all } m \in M$$

if $\alpha > 0$, then the right hand inequality in $(**)$ gives $r_0 \le -\frac{1}{\alpha} g(m) + \frac{1}{\alpha} p(m + \alpha x_0) \Rightarrow$
$\alpha r_0 \le -g(m) + p(m + \alpha x_0) \Rightarrow g(m) + \alpha r_0 \le p(m + \alpha x_0) \Rightarrow f_0(x) \le p(x)$

and if $ª > 0$, then the right hand inequality in $(**)$ gives $-\frac{1}{ª} g(m) + \frac{1}{ª} p(m + ªx_0) \le r_0$

since $ª < 0$, then $-g(m) + p(m + ªx_0) \ge ª\, r_0 \Rightarrow g(m) + ªr_0 \le p(m + ªx_0)$

thus when $ª \ne 0$, obtain $f_0(x) \le p(x)$ for all $x \in M$. Thus $f_0 \in M_0'$ and $f_0(x) = g(x)$ for all $x \in M$. Hence $f_0(x) \le p(x)$ for all $x \in M_0$. If $M_0 = X$ complete proof, either if $M_0 \ne X$

Let G be the collection of all ordered pairs $(f_¥, M_¥)$ such that

(i) $M_¥$ is a subspace of X and $M \subset M_¥$ (ii) $f_¥ \in (M_¥)'$ such that $f_¥(x) = g(x)$ for $x \in M$

(iii) $f_¥(x) \le p(x)$ for all $x \in M_¥$.

$\Rightarrow$ G is non-empty and partially ordered by

$(f_¥, M_¥) \le (f_~, M_~) \Leftrightarrow M_¥ \subset M_~$ & $f_¥(x) = f_~(x) \quad \forall x \in M_¥$

Let $\Phi = \{(f_¥, M_¥)\}$ be a totally ordered set in G . then it is easy to see that Φ has an upper bound $(\Psi, \cup M_~)$ where $\Psi(x) = f_~(x)$ for all $x \in M_~$ .By using Zorn's Lemma, there exists a maximal element (f, H) in G. To complete the proof, we must show that $H = X$.

Suppose that $H \ne X$, then there exists $a \in X$ such that $a \notin H$

Put $H_0 = [H \cup \{a\}]$ by using first part in this proof, we have $h \in H_0'$ such that $h(x) = f(x)$ for all $x \in H_0$. But contradicts the maximally of (f, H) (f, H).

Consequently, we must have $H = X$ and the proof is complete.

Remark

Let M be a subspace of a complex linear space X, such that

(1) The function $p : X \to \mathbb{R}$ satisfies the conditions

(i) $p(x + y) \le p(x) + p(y)$ for all $x, y \in X$ (ii) $p(ªx) = |ª|\, p(x)$ for all $x \in X$ and for all $ª \in \mathbb{C}$

(2) $g \in M'$ such that $|g(x)| \le p(x)$ for all $x \in M$,

Then there exists $f \in X'$ such that

(i) $f(x) = g(x)$ for all $x \in M$ (ii) $|f(x)| \le p(x)$ for all $x \in X$

5.2 Minkowski' Functional

Definition(5.2.1)

Let A be an absorbing subset of a linear space X over F. The functional $«_A : X \to \mathbb{R}$, $«_A(x) = \inf\{ª > 0 : x \in ªA\}$ for all $x \in X$ is called the Minkowski's functional of A.

It clear to show that

(1) $«_A(x) < \infty$ for all $x \in X$, because that A is an absorbing

(2) If $x \in ªA$, then $«_A(x) \le ª$. In special case if $y \in «_A(x)$, then $«_A(y) \le «_A(x)$

(3) If $x \notin ªA$ for some $ª > 0$, then $«_A(x) \ge ª$

(4) If A is open in topological linear space X, then $ªA = \{x \in X : «_A(x) < ª\}$

Theorem(5.2.2)

Suppose P is a seminorm on a linear space X over F. If $A = \{x \in X : P(x) < 1\}$, then $P = «_A$

Proof :

Since A is convex , absorbing, balanced set $x \in X$
Since A is absorbing, there exists $ª > 0$ such that $x \in ªA \Rightarrow «_A(x) \le ª$
and $ª^{-1}x \in A \Rightarrow p(ª^{-1}x) < 1 \Rightarrow p(x) < ª$, so that $«_A \le P$
since P semi-norm, then $P(x) \ge 0$, there exist ~ such that $0 < \sim \le P(x)$
$\Rightarrow P(\sim^{-1}x) \ge 1 \Rightarrow \sim^{-1}x \notin A$, so that $P(x) \le «_A(x) \Rightarrow P \le «_A$. Hence $P = «_A$

Theorem(5.2.3)

Suppose A is a convex absorbing set in a linear space X over F. Define $H_A(x) = \{ª > 0 : x \in ªA\}$ for all $x \in X$ .If $\sim \in H_A(x)$, then $\in H_A(x)$ for all $> \sim$.

Proof :

Since $\sim \in H_A(x) \Rightarrow x \in \sim A \Rightarrow \sim^{-1}x \in A$
Since A is a convex and $0, \sim^{-1}x \in A$, then $^{-1}x = {}^{-1}(- \sim)(0) + {}^{-1}\sim(\sim^{-1}x) \in A$
$\Rightarrow x \in A \Rightarrow \in H_A(x)$

Theorem(5.2.4)

Suppose A is a convex absorbing set in a linear space X over F. Then
(1) $«_A$ is a sublinear functional.
(2) If $B = \{x \in X : «_A(x) < 1\}$ and $C = \{x \in X : «_A(x) \le 1\}$, then $B \subset A \subset C$ and $«_B = «_A = «_C$
(3) If A is balanced, then $«_A$ is a seminorm.

Proof :

(1) Let $x, y \in X$. For all $£ > 0$, there exists $ª_1 \in H_A(x)$ and $ª_2 \in H_A(y)$ such that $ª_1 < «_A(x) + £$ and $ª_2 < «_A(x) + £$, then
$(«_A(x) + £) \in H_A(x)$ and $(«_A(y) + £) \in H_A(y)$, $x \in («_A(x) + £)A$ and $y \in («_A(y) + £)A$
$$(«_A(x) + £)^{-1}x \in A \text{ and } («_A(y) + £)^{-1}y \in A$$
Put $ª = («_A(x) + £)(«_A(x) + «_A(y) + 2£)^{-1} \Rightarrow 0 < ª < 1$
since A is convex
$ª(«_A(x) + £)^{-1}x + (1 - ª)(«_A(y) + £)^{-1}y \in A \Rightarrow («_A(x) + «_A(y) + 2£)^{-1}(x + y) \in A$
It is clear to show that $«_A(0) = 0$. Let $x \in X$ and $\sim > 0$, then
$«_A(\sim x) = \inf\{ª > 0 : \sim x \in ªA\} = \inf\{ª > 0 : x \in \sim^{-1}ªA\} = \sim \inf\{\sim^{-1}ª : x \in \sim^{-1}ªA, ª > 0\} = \sim«_A(x)$

(3) since A is a balanced set , then $^{-1}A = A$ for all $\in F$ such that $| \, | = 1$
so $\{ª > 0 : \sim x \in ªA\} = \{ª > 0 : |\sim| x \in ªA\} \Rightarrow «_A(\sim x) = |\sim| «_A(x) \Rightarrow «_A$ is a semi-norm on X .

5.3 Separation Theorems For Normed Spaces

theorem(5.3.1)

Let M be a subspace of a normed space X . if $g \in M^*$, then there exists $f \in X^*$ such that $f(x) = g(x)$ for all $x \in M$ and $\|f\| = \|g\|$

Proof :

Case (1): consider g is a real –linear functional on M
Define $p : X \to \mathbb{R}$ by $p(x) = \|g\|\|x\|$ for all $x \in X$. Then p is a sub linear .We also observe $|g(x)| \le \|g\|\|x\| = p(x)$ for all $x \in M$. By theorem (5.1.8) , there exists $f \in X'$ such that (i) $f(x) = g(x)$ for all $x \in M$ (ii) $f(x) \le p(x)$ for all $x \in X \Rightarrow f(x) \le \|g\|\|x\|$ for all $x \in X$

$\Rightarrow \quad |f(x)| = \max\{f(x), -f(x)\} \le \|g\|\|x\|$ for all $x \in X \quad \Rightarrow \quad f$ is bounded and $\|f\| \le \|g\| \quad \Rightarrow \quad f \in X^*$
Since f extends g ,so $\|f\| \ge \|g\|$ and therefore $\|f\| = \|g\|$.

Case (2) : when g is a complex –linear functional on M

Let u be real part of g , by lemma(5.1.1), we have $u \in M^*$ and $g(x) = u(x) - iu(ix)$ for all $x \in M$. Moreover $\|g\| = \|u\|$.By case(1), there exists $u_0 \in X^*$ such that $u_0(x) = u(x)$ for all $x \in M$ and $\|u_0\| = \|u\|$, so $\|g\| = \|u_0\|$, put $f_0(x) = u_0(x) - iu_0(ix)$ for $x \in X$, by lemma (5.1.2), we have $f_0 \in X^*$ and $\|f_0\| = \|u_0\|$. Since $\|g\| = \|u_0\| \Rightarrow \quad \|f_0\| = \|g\|$

Theorem (5.3.2)

If x_0 is a non zero element of a normed space X over F , then there exists $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$. In particular X^* separated points on X , i.e. if $x, y \in X$ such that $x \ne y$, then there exists an $f \in X^*$ such that $f(x) \ne f(y)$.

Proof :

Let $M = [x_0] = \{ª x_0 : ª \in F\}$, then M_0 is a subspace of X .

Define $g : M \to F$, by $g(x) = g(ª x_0) = ª \|x_0\|$ for all $x \in M$

(1) g is linear : let $x_1, x_2 \in M$ and $\sim,\ \ \in F \Rightarrow \quad x_1 = ª_1 x_0, \quad x_2 = ª_2 x_0$

$g(\sim x_1 + \ x_2) = g(\sim ª_1 x_0 + \ ª_2 x_0) = g((\sim ª_1 + \ ª_2) x_0) = (\sim ª_1 + \ ª_2)\|x_0\| = \sim ª_1 \|x_0\| + \ ª_2 \|x_0\|$

$= \sim g(ª_1 x_0) + \ g(ª_2 x_0) = \sim g(x_1) + \ g(x_2)$

$\Rightarrow \quad g$ is linear

(2) g is bounded : let $x \in M \quad \Rightarrow \quad x = ª x_0 \quad \Rightarrow \quad \|x\| = \|ª x_0\| = |ª|\|x_0\|$

$|g(x)| = |g(ª x_0)| = |ª \|x_0\|| = |ª|\|x_0\| = \|x\| < 2\|x\| \Rightarrow \quad g$ is bounded

(3) $\|g\| = 1$: $\|g\| = \sup\{|g(x)| : x \in M, \|x\| \le 1\}$

Since $|g(x)| = \|x\| \quad \Rightarrow \quad \|g\| = \sup\{\|x\| : x \in M, \|x\| \le 1\} = 1$

By corollary (5.3.1) exists $f \in X^*$ such that $f(x) = g(x)$ for all $x \in M$ and $\|f\| = \|g\|$

Since $g(ª x_0) = ª \|x_0\|$ for all $ª \in F$. Put $ª = 1 \quad \Rightarrow \quad g(x_0) = \|x_0\|$

Since $\|g\| = 1 \quad \Rightarrow \quad \|f\| = 1$

Now let $x, y \in X$ such that $x \ne y \quad \Rightarrow \quad x_0 = x - y \ne 0$, then by above theorem, there exists $f \in X^*$ such that $f(x_0) = \|x_0\|$

$f(x - y) = \|x - y\| \ne 0 \quad \Rightarrow \quad f(x) - f(y) \ne 0 \quad \Rightarrow \quad f(x) \ne f(y)$.

Corollary (5.3.3)

Let X be a normed space and suppose $f(x) = 0$ for all $f \in X^*$, then $x = 0$

Proof :

Suppose $x \ne 0$. Then by theorem(6.11), there exists $f \in X^*$ such that $f(x) = \|x\| > 0$ which contradicts the hypothesis that $f(x) = 0$ for all $f \in X^*$**.** Hence we must have $x = 0$.

Corollary (5.3.4)

Let X be a normed space and suppose , $\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}$ for all $x \in X$.

Proof :

If $x = 0$, the conclusion is trivial. If $x \neq 0$, then for all $f \in X^*$ with $\|f\| = 1$ we have

$$|f(x)| \leq \|f\| \|x\| = \|x\|$$

Since , by theorem(5.3.2), there exists $f \in X^*$ such that $\|f\| = 1$and $f(x) = \|x\|$, the result follows.

Theorem (5.3.5)

Let M be a closed subspace of a normed space X and $x_0 \in X$ **,** but $x_0 \notin M$ .Then there exists $f \in X^*$ such that $f(x_0) \neq 0$ and $f(x) = 0$ for all $x \in M$.

Proof :

Consider the natural function $® : X \to X / M$ by $®(x) = x + M$, then $®$ is continuous linear function.

Let $x \in M \quad \Rightarrow \quad ®(x) = x + M = M = 0$ (0 denote the zero vector M of X / M)

Also since $x_0 \notin M$, we have $®(x_0) = x_0 + M \neq 0$

Hence by theorem(5.3.2), there exists $g \in (X / N)^*$ such that $g(x_0 + M) = \|x_0 + M\| \neq 0$

We now define f by $f(x) = g(®(x))$ for all $x \in X$

(1) f is linear : Let $x, y \in X$ and $\sim, \quad \in F$

$f(\sim x + \quad y) = g(®(\sim x + \quad y)) = g((\sim x + \quad y) + M) = g(\sim(x + M) + \quad (y + M))$

$f(\sim x + \quad y) = \sim g(x + M) + \quad g(y + M) = \sim g(®(x)) + \quad g(®(y)) = \sim f(x) + \quad f(y)$

$\Rightarrow \quad f$ is linear

(2) f is bounded

$$|f(x)| = |g(®(x))| \leq \|g\| \|®(x)\| \leq \|g\| \|®\| \|x\|$$

Since $\|®\| \leq 1 \quad \Rightarrow \quad |f(x)| \leq \|g\| \|x\| \quad \Rightarrow \quad f$ is bounded $\Rightarrow \quad f \in X^*$

Also $f(x_0) = g(®(x_0)) = g(x_0 + M) = \|x_0 + M\| \neq 0$ and

$f(x) = g(®(x)) = g(x + M) = g(0) = 0$ for all $x \in M$.

Theorem (5.3.6)

Let A be a nonempty open convex subset of a normed space X , and $x_0 \in X$, but $x_0 \notin A$. Then there exists an $f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in A$.

Proof :

By translation, we may assume that $0 \in A$

Define $P : X \to \mathbb{R}$ by $P(x) = \inf\{ª > 0 : x \in ªA\}$ for all $x \in X$.

It is clear to show that P is sub-linear and $P(x) < 1$ iff $x \in A$.

Let $M_0 = [x_0]$, i.e. $M_0 = \{ªx_0 : ª \in \mathbb{R}\} \quad \Rightarrow \quad M_0$ is subspace of X .

Define $g : M_0 \to \mathbb{R}$ by $f(ªx_0) = ª$ for all $ª \in \mathbb{R} \quad \Rightarrow \quad g \in M_0'$ and $g(x) \leq P(x)$ for all $x \in M_0$ by using theorem (5.1.8), there exist $f \in X'$ such that $f(x) = g(x)$ for all $x \in M_0$ and and $f(x) \leq P(x)$ for all $x \in X$.

Since $A \subseteq X \quad \Rightarrow \quad f(x) \leq P(x)$ for all $x \in A$

Since $P(x) < 1$ iff $x \in A \Rightarrow \quad f(x) \leq 1$ for all $x \in A$

Since $x_0 = 1.x_0 \in M \Rightarrow f(x_0) = g(x_0) = g(1.x_0) = 1 \Rightarrow f(x) \le f(x_0)$ for all $x \in A$

It is clear to show that $\ker(f)$ is closed , then f is continuous $\Rightarrow f \in X^*$

Theorem (5.3.7)

Let A be a nonempty closed convex subset of a normed space X , and $x_0 \in X$, but $x_0 \notin A$. Then there exists an $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha < f(x_0)$ for all $x \in A$.

Proof :

Choose $r > 0$ such that $_r(x_0) \cap A = \emptyset$

Let $D = A + {}_r(0)$

Since $_r(0)$ is open, then D is open

Since A and $_r(0)$ are convex , then D is convex

Since $x_0 \in {}_r(x_0) \Rightarrow x_0 \notin A$

By theorem(5.3.6), there exists an $f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in D$.

Since f is not identically 0, $f(b) > 0$ for some $b \in {}_r(0)$

Taking $\alpha = f(x_0) - f(b)$, we see that for all $x \in A$, $f(x) = f(x+b) - f(b) < \alpha < f(x_0)$.

Theorem (5.3.8)

Let A and B be convex sets of real normed X . If A is compact in X and B is closed, then there is $f \in X^*$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $f(x) \le \alpha_1 < \alpha_2 \le f(y)$ for all $x \in A$ and for all $y \in B$.

Proof :

Theorem (5.3.9)

Let A and B be disjoint, nonempty, convex sets of normed X such that A is compact in X and B is closed, then there is $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) > \alpha$ for all $x \in A$ and $f(y) < \alpha$ for all $y \in B$.

5.4 Separation Theorems For Topological Linear Spaces

Theorem(5.4.1)

Let X be a topological linear space, $x_0 \in X$. If V is a neighborhood of 0 in X such that $x_0 \notin V$, then there is $f \in X^*$ such that $f(x_0) = 1$ and $f(x) < 1$ for all $x \in V$.

Proof :

Since every neighborhood of 0 is absorbing set, then V is absorbing convex set.

Define $P : X \to \mathbb{R}$ by $P(x) = \inf\{\alpha > 0 : x \in \alpha V\}$ for all $x \in X$.

It is clear to show that P is sub-linear and $P(x) \ge 0$ for all $x \in X$.

Let $M_0 = [x_0]$, i.e. $M_0 = \{\alpha x_0 : \alpha \in \mathbb{R}\} \Rightarrow M_0$ is subspace of X .

Define $g : M_0 \to \mathbb{R}$ by $f(\alpha x_0) = \alpha$ for all $\alpha \in \mathbb{R} \Rightarrow g \in M_0'$ and $g(x) \le P(x)$ for all $x \in M_0$ by using theorem (5.1.8), there exist $f \in X'$ such that $f(x) = g(x)$ for all $x \in M_0$ and $-P(-x) \le f(x) \le P(x)$ for all $x \in X$.

Since $x_0 = 1.x_0 \in M \Rightarrow f(x_0) = g(x_0) = g(1.x_0) = 1$

Since V is open set $\Rightarrow \alpha V = \{x \in X : P(x) < \alpha\}$

If $\alpha = 1 \Rightarrow V = \{x \in X : P(x) < 1\}$, so $f(x) \le P(x) < 1$ for all $x \in V$.

Let $y \in -V \;\Rightarrow\; -y \in V \;\Rightarrow\; f(-y) < 1 \;\Rightarrow\; f(y) > -1$

For all $y \in -V \;\Rightarrow\; |f(x)| < 1$ for all $x \in W = V \cap (-V)$

Since V is a neighborhood of 0 in $X \;\Rightarrow\; W$ is a neighborhood of 0 in X, then f is bounded function for some neighborhood W of 0 in X, so that by using theorem (), we have f is bounded function $\Rightarrow\; f \in X^*$.

Theorem(5.4.2)

Let A and B be disjoint, nonempty, convex sets in a topological linear space X. If A is open in X, then there is $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha \le f(y)$ for all $x \in A$ and for all $y \in B$.

Proof :

Let $x_0 = b_0 - a_0$ where $a_0 \in A,\; b_0 \in B$

Let $V = A - B + x_0 \;\Rightarrow\; V = (A - a_0) - (B - b_0)$

Since A and B are convex sets, then V is convex set

Since A is open set , then V is open set

Since $a_0 \in A \;\Rightarrow\; 0 = a_0 - a_0 \in A - a_0$, also $b_0 \in B \;\Rightarrow\; 0 = b_0 - b_0 \in B - b_0$

$\Rightarrow\; 0 \in V \;\Rightarrow\; V$ is a convex neighborhood of 0 in X

To prove $x_0 \notin V$: let $x_0 \in V$

$\Rightarrow\; x_0 \in A - B + x_0 \;\Rightarrow\; 0 = x_0 - x_0 \in A - B$

since $0 = 0 - 0 \;\Rightarrow\; 0 \in A, 0 \in B \;\Rightarrow\; 0 \in A \cap B \;\Rightarrow\; A \cap B \ne \emptyset$

This contradiction $\Rightarrow\; x_0 \notin V$

By using theorem (5.4.1), there exist $f \in X^*$ such that $f(x_0) = 1$ and $f(z) < 1$ for all $z \in V$

Now

For all $x \in A$ and for all $y \in B$

$\Rightarrow\; x - y + x_0 \in V \;\Rightarrow\; f(x - y + x_0) < 1 \;\Rightarrow\; f(x) - f(y) + f(x_0) < 1 \;\Rightarrow\; f(x) < f(y)$ For all $x \in A$ and for all $y \in B$

Since A and B are non-empty disjoint convex sets, then $f(A), f(B)$ are disjoint convex sets in R such that $f(A) \subset f(B)$

Since every non constant convex functional on X is open and A is open in X , then $f(A)$ is open in $\mathbb{R}$

Let α be a right limit of $f(A)$, i.e. $f(x) < \alpha$ for all $x \in A$

$\Rightarrow\; f(x) < \alpha \le f(y)$ for all $x \in A$ and for all $y \in B$

Corollary (5.4.3)

Let A and B be disjoint, nonempty, convex sets in a locally convex X. If A is compact in X and B is closed, then there is $f \in X^*$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $f(x) < \alpha_1 \le \alpha_2 \le f(y)$ for all $x \in A$ and for all $y \in B$.

Exercises(5)

5.1 Let M be a closed subspace of a normed space X and $x_0 \in X$, but $x_0 \notin M$ .Then there exists $f \in X^*$ such that $f(x_0) = d$, $\|f\| = 1$ and $f(x) = 0$ for all $x \in M$,where $d = d(x_0, M)$, i.e. d is the distance from x_0 to M .

5.2 Let M be a closed subspace of a normed space X and $x_0 \in X$, but $x_0 \notin M$ .Then there exists $f \in X^*$ such that $f(x_0) = 1$, $\|f\| = \frac{1}{d}$ and $f(x) = 0$ for all $x \in M$,where $d = d(x_0, M)$,

5.3 Let M be a subspace of a locally convex space X and $x_0 \in X$. If $x_0 \notin \overline{M}$ then there exists $f \in X^*$ such that $f(x_0) = 1$, but $f(x) = 0$ for all $x \in M$.

5,4 If X is a locally convex space then X^* separated points on X .

5,5 Let M be a subspace of a locally convex space X and $x_0 \in X$. If $x_0 \in \overline{M}$ then $f(x_0) = 0$ for every continuous linear functional f on X that vanishes on M .

5.6 Let M be a subspace of a locally convex space X . If $g \in M^*$, then there exists $f \in X^*$ Such that $f(x) = g(x)$ for all $x \in M$.

5.7 Suppose A is a convex, balanced, closed set in a locally convex space X , $x_0 \in X$, but $x_0 \notin A$. Then there exists $f \in X^*$ such that $|f(x)| \le 1$ for all $x \in A$,but $f(x_0) > 1$

5.8 Suppose is a convex balanced local base in a topological linear space X . Associate to every $V \in$ its Minkowski functional $«_V$. Show that $\{«_V : V \in \}$ is a separating family of continuous seminorms on X .

5.9 Suppose $\mathcal{G}$ is a separating family of seminorms on a linear space X . Associate to each $P \in \mathcal{G}$ and each positive integer n the set $V(P,n) = \{x \in X : P(x) < \frac{1}{n}\}$. Let be the collection of all finite intersections of all the sets $V(P,n)$. Show that is a convex balanced local base for topology ² on X , which turns into a locally convex space such that (1) Every $P \in \mathcal{G}$ is continuous, and (2) set $A \subseteq X$ is bounded iff every $P \in \mathcal{G}$ is bounded on A .

5.10 Show that a topological linear space X is normable iff its has origin has a convex bounded neighborhood.