6. Bipolar Theorem

6.1 Dual linear Spaces

Definition(6.1.1)

Let X, Y and Z be linear spaces over F and a function $G : X \times Y \to Z$ associate to each $a \in X$ and to each $b \in Y$ the functions $G_b : X \to Z$ and $G_a : Y \to Z$ by defining $G_a(b) = G(a,b) = G_b(a)$. We say that G is a bilinear function if every G_a and every G_b are linear.

Theorem(6.1.2)

Let X and Y be linear spaces over F and a function $G : X \times Y \to F$ is a bilinear functional. Put $N_X = \{x \in X : G(x,y) = 0, \quad \forall y \in Y\}$ and $N_Y = \{y \in Y : G(x,y) = 0, \quad \forall x \in X\}$

Then N_X is a subspace of X and N_Y is a subspace of Y. N_X and N_Y are called null spaces.

Proof :

Since $G(0,0) = G_{\circ}(0) = 0 \quad \Rightarrow \quad 0 \in N_X \quad \Rightarrow \quad N_X \neq ¤$

Let $x_1, x_2 \in N_X$ and $\sim, \ \in F$. For all $y \in Y$

$G(\sim x_1 + \ x_2, y) = G_y(\sim x_1 + \ x_2) = \sim G_y(x_1) + \ G_y(x_2) = \sim G(x_1, y) + \ G(x_2, y) = \sim(0) + \ (0) = 0$

$\sim x_1 + \ x_2 \in N_X \quad \Rightarrow \quad N_X$ is a subspace of X. Similarly to prove N_Y is a subspace of Y.

Definition(6.1.3)

Let X and Y be linear spaces over F. A bilinear functional $G : X \times Y \to F$ is called a non-degenerate if $N_X = \{0\}$ and $N_Y = \{0\}$.

Remarks

(1) $N_X = \{0\}$, ,means that : $\forall\ x \neq 0, x \in X$, $\exists y \in Y$ such that $G(x,y) \neq 0$,

i.e. if $G(x,y) = 0$ for all $y \in Y$, then $x = 0$

(2) A non-degenerate bilinear functional $G : X \times Y \to F$ will be denoted by $\langle\, , \rangle$,

i.e. $G(x,y) = \langle x, y \rangle$

Definition(6.1.4)

Tow linear spaces X, Y over F are said to be dual spaces, if there is a non-degenerate bilinear functional $G : X \times Y \to F$.

Example(6.1.5)

Let X be a linear space over F. Show that X, X' are dual spaces.

Ans:

$X' = \{\cdot \ : X \to F \ , \quad \cdot \ $ is linear functional $\}$. Define $G : X' \times X \to F$ by

$G(\cdot\ , x) = \langle \cdot\ , x \rangle = \cdot\ (x)$ for all $\cdot\ \in X'$ and for all $x \in X$.

It is clear to show that G is a bilinear functional

$N_X = \{x \in X : G(\cdot\ , x) = 0, \quad \forall \cdot\ \in X'\} = \{x \in X : \cdot\ (x) = 0, \quad \forall \cdot\ \in X'\}$

It is clear to show that N_X is a subspace of X and $N_X = \{0\}$

$N_{X'} = \{f \in X' : f(x) = 0, \ \forall x \in X\} \Rightarrow N_{X'}$ and $N_{X'} = \{0\} \Rightarrow G$ is a non -degenerate bilinear functional , then X, X' are dual spaces.

Theorem(6.1.6)

Let X and Y be dual linear spaces over F. For any element $y \in Y$, define the functional $\phi_y : X \to F$ by $\phi_y(x) = \langle x, y\rangle$ for all $x \in X$.

(1) ϕ_y is linear functional

(2) $F_Y = \{\phi_y : y \in Y\}$ is a subspace of X'

(3) $Y \approx F_Y$

Proof :

(1) Let $x_1, x_2 \in X$ and $\alpha, \beta \in F$

$\phi_y(\alpha x_1 + \beta x_2) = \langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle = \alpha\phi_y(x_1) + \beta\phi_y(x_2) \Rightarrow \phi_y$ is linear

(2) since $0 \in Y \Rightarrow \phi_y = 0 \Rightarrow 0 \in F_Y \Rightarrow F_Y \neq \phi$

Let $\phi_{y_1}, \phi_{y_2} \in F_Y$ and $\alpha, \beta \in F$

$$(\alpha\phi_{y_1} + \beta\phi_{y_2})(\alpha_1 x_1 + \alpha_2 x_2) = (\alpha\phi_{y_1})(\alpha_1 x_1 + \alpha_2 x_2) + (\beta\phi_{y_2})(\alpha_1 x_1 + \alpha_2 x_2)$$
$$= \alpha[\alpha_1\phi_{y_1}(x_1) + \alpha_2\phi_{y_1}(x_2)] + \beta[\alpha_1\phi_{y_2}(x_1) + \alpha_2\phi_{y_2}(x_2)]$$
$$= \alpha_1[\alpha\phi_{y_1}(x_1) + \beta\phi_{y_2}(x_1)] + \alpha_2[\alpha\phi_{y_1}(x_2) + \beta\phi_{y_2}(x_2)]$$
$$= \alpha_1(\alpha\phi_{y_1} + \beta\phi_{y_2})(x_1) + \alpha_2(\alpha\phi_{y_1} + \beta\phi_{y_2})(x_2)$$

$\Rightarrow \alpha\phi_{y_1} + \beta\phi_{y_2} \in F_Y \Rightarrow F_Y$ is a subspace of X'

(3) Define $H : Y \to F_Y$ by $H(y) = \phi_y$ for all $y \in Y$

(i) let $y_1, y_2 \in Y$ and $\alpha, \beta \in F$. For all $x \in X$

$$H(\alpha y_1 + \beta y_2)(x) = \phi_{\alpha y_1 + \beta y_2}(x) = \langle x, \alpha y_1 + \beta y_2\rangle = \alpha\langle x, y_1\rangle + \beta\langle x, y_2\rangle$$
$$= \alpha\phi_{y_1}(x) + \beta\phi_{y_2}(x) = (\alpha\phi_{y_1} + \beta\phi_{y_2})(x) = (\alpha H(y_1) + \beta H(y_2))(x)$$

$\Rightarrow H(\alpha y_1 + \beta y_2) = \alpha H(y_1) + \beta H(y_2) \Rightarrow H$ is linear

(ii) let $y_1, y_2 \in Y$ such that $H(y_1) = H(y_2)$

$\Rightarrow \phi_{y_1} = \phi_{y_2} \Rightarrow \phi_{y_1}(x) = \phi_{y_2}(x)$ for $x \in X \Rightarrow \langle x, y_1\rangle = \langle x, y_2\rangle$ for $x \in X$

$\Rightarrow G(x, y_1) = G(x, y_2)$ for $x \in X \Rightarrow G(x, y_1 - y_2) = 0$ for $x \in X$

$\Rightarrow y_1 - y_2 \in N_Y$

Since $N_Y = \{0\} \Rightarrow y_1 - y_2 = 0 \Rightarrow y_1 = y_2 \Rightarrow H$ is one to one

(iii) Let $\phi_y \in F_Y \Rightarrow y \in Y \Rightarrow G(y) = \phi_y \Rightarrow H$ is onto.

So that $Y \approx F_Y$

6.2 Weak Topology

Definition(6.2.1)

Let X and Y be dual topological linear spaces over F.

- The weakest topology on X, for which all functional ¤$_y$ are continuous ,is called the weak topology on X and it is denoted by $\pm(X,Y)$, the member of this topology is weakly open sets. Similarly: one may define the weak topology $\pm(Y,X)$ on Y.
- $\pm(X,Y)$ is a locally convex topology as it is defined by the family $\{P_y\}_{y\in Y}$ of all seminorms $P_y(x)=|\langle x,y\rangle|$
- A weakly continuous ($\pm$ - continuous) functional on X is by definition, a continuous functional in the weak topology. We also use the notions of weakly closed (or $\pm$ - closed) ser, weakly compact set, etc.

It is clear to show that : A subset of X is weakly open if for every $x_0\in A$, there is an $£>0$ and there are $y_1,y_2,\cdots,y_n\in Y$ such that

$$\bigcap_{i=1}^{n}\{x\in X:\mathrm{Re}<x-x_0,y_i>\le £\}\subseteq A$$

Theorem(6.2.2)

Let X and Y be dual topological vector spaces over F. Any weakly continuous linear functional f on X has a unique representation of the form $f(x)=\langle x,y\rangle$ for all $x\in X$

Proof :

There exists $y_i\in Y$, $i=1,2,\cdots,n$ such that

$$|f(x)|\le \max\{|\langle x,y_i\rangle|:i=1,2,\cdots,n\}$$

Denoting

$$f_i(x)=\langle x,y_i\rangle,\quad i=1,2,\cdots,n$$

We have $f(x)=0$ whenever $f_i(x)=0 \quad i=1,2,\cdots,n$

Hence f is a linear combination of $f_i \quad i=1,2,\cdots,n$

6.3 Bipolar Sets

Definition(6.3.1)

Let X and Y be dual topological linear spaces over F, and let $A\subseteq X$. The polar set of A is denoted by A° and defined as : $A^\circ=\{y\in Y:\mathrm{Re}<x,y>\le 1\,,\forall x\in A\}$.The set $A^{\circ\circ}$, i.e. the polar set of A°, is called the bipolar set of A

$$A^{\circ\circ}=(A^\circ)^\circ=\{x\in X:\mathrm{Re}\langle x,y\rangle\le 1\,,\forall y\in A^\circ\}$$

It is clear to show that $A\subset A^{\circ\circ}$. In particular, if X is a Hausdorff locally convex space and $Y=X^*$, then

$A^{\circ}=\{f\in X^{*}:|f(x)|\le 1,\ \forall x\in A\}$ $\quad A^{\circ\circ}=\{x\in x:|f(x)|\le 1,\ \forall f\in A^{\circ}\}$

Theorem(6.3.2)

Let X and Y be dual topological linear spaces over F, and let $A,B\subseteq X$

(1) If $A\subset B$, then $B^{\circ}\subset A^{\circ}$

(2) If $\lambda\in F$ and $\lambda\neq 0$, then $(\lambda A)^{\circ}=\lambda^{-1}A^{\circ}$

(3) If A absorbs B, then B° absorbs A°

(4) If A is balanced, then so is A° and $A^{\circ}=\{y\in Y:|\langle x,y\rangle\le 1 \quad \forall x\in A\}$

(5) If A is a subspace of X, then A° is a subspace of Y and
$A^{\circ}=\{y\in Y:\langle x,y\rangle=0 \quad \forall x\in A\}$

(6) $\emptyset^{\circ}=Y$ and $X^{\circ}=\{0\}$

(7) A° is convex

(8) $A^{\circ}=\bigcap_{x\in A}\{y\in Y:\operatorname{Re}\langle x,y\rangle\le 1\}$

(9) A° is weakly closed

Proof :

(1) Let $y\in B^{\circ} \Rightarrow y\in Y$ and $\operatorname{Re}\langle x,y\rangle\le 1$ for all $x\in B$

Since $A\subseteq B \Rightarrow \operatorname{Re}\langle x,y\rangle\le 1$ for all $x\in A \Rightarrow y\in A^{\circ} \Rightarrow B^{\circ}\subseteq A^{\circ}$

(2) Since $\langle \lambda x,y\rangle=\langle x,\lambda y\rangle=\langle x,\lambda y\rangle$

$\Rightarrow \operatorname{Re}\langle x,\lambda y\rangle\le 1$ for all $x\in A \Rightarrow \lambda y\in A^{\circ}$

Since $\lambda\neq 0 \Rightarrow \lambda^{-1}(\lambda y)\in \lambda^{-1}A^{\circ} \Rightarrow y\in \lambda^{-1}A^{\circ} \Rightarrow (\lambda A)^{\circ}\subseteq \lambda^{-1}A^{\circ}$

Similarly $\lambda^{-1}A^{\circ}\subseteq(\lambda A)^{\circ}$ so that $(\lambda A)^{\circ}=\lambda^{-1}A^{\circ}$

(3) Since A is absorbs $B \Rightarrow \lambda_0\in F$ such that $B\subset \lambda A_0$ whenever $|\lambda|\ge|\lambda_0|$

Since $B\subset \lambda A \Rightarrow (\lambda A)^{\circ}\subset B^{\circ} \Rightarrow \lambda^{-1}A^{\circ}\subset B^{\circ} \Rightarrow A^{\circ}\subset \lambda B^{\circ}$

$\Rightarrow B^{\circ}$ absorbs A°

(4) Since A is balanced $\Rightarrow \lambda A\subset A$ for all $\lambda\in F$ with $|\lambda|\le 1$

$A^{\circ}\subset(\lambda A)^{\circ}=\lambda^{-1}A^{\circ} \Rightarrow \lambda A^{\circ}\subset A^{\circ}$ for all $\lambda\in F$ with $|\lambda|\le 1$

$\Rightarrow A^{\circ}$ is balanced.

Put $D=\{y\in Y:|\langle x,y\rangle|\le 1,\ \forall x\in A\}$

Let $y\in D \Rightarrow |\langle x,y\rangle|\le 1$ for all $x\in A$

Since $\operatorname{Re}\langle x,y\rangle\le|\langle x,y\rangle| \Rightarrow \operatorname{Re}\langle x,y\rangle\le 1$ for all $x\in A$

$y\in A^{\circ} \Rightarrow D\subseteq A^{\circ}$

Now let $z\in A^{\circ} \Rightarrow \operatorname{Re}\langle x,z\rangle\le 1$ for all $x\in A$

Take $\lambda\in F$ such that $|\lambda|=1$

Since A is balanced $\Rightarrow \lambda A=A$

$\Rightarrow \quad \text{Re}\langle ªx, z\rangle \le 1$ for all $x \in A \Rightarrow \quad |\langle x, y\rangle| \le 1$ for all $x \in A$

$\Rightarrow \quad z \in D \quad \Rightarrow \quad A^{\circ} \subseteq D \quad \Rightarrow \quad A^{\circ} = D$

(5) Since A is a subspace of $X \;\Rightarrow\;$ A is a balanced set

$\Rightarrow \quad A^{\circ} = \{y \in Y : |\langle x, y\rangle| \le 1 \,,\, \forall x \in A\}$

Put $\quad M = \{y \in Y : |\langle x, y\rangle| = 0 \,,\, \forall x \in A\}$

Let $y \in M \quad \Rightarrow \quad \langle x, y\rangle = 0$ for all $x \in A$

$\Rightarrow \quad |\langle x, y\rangle| \le 1$ for all $x \in A \;\Rightarrow\; y \in A^{\circ} \quad \Rightarrow \quad M \subseteq A^{\circ}$

Now

Let $z \in A^{\circ} \quad \Rightarrow \quad |\langle m, z\rangle| \le 1$ for all $m \in A$

Since A is a subspace $\quad \Rightarrow \quad ªx \in A$ for all $x \in A$ and for all $ª \in F$

$\Rightarrow \quad |\langle x, y\rangle| |< 1$ for all $x \in A$ and for all $ª \in F$

$\langle x, y\rangle = 0$ for all $x \in A \quad \Rightarrow \quad y \in M \quad \Rightarrow \quad A^{\circ} \subseteq M \quad \Rightarrow \quad A^{\circ} = M$

$\Rightarrow \quad A^{\circ} = \{y \in Y : \langle x, y\rangle = 0 \,, \quad \forall x \in A\}$

Let $y_1, y_2 \in A^{\circ}$ and $\sim, \quad \in F$

For all $x \in A$

$\langle x, \sim y_1 + \quad y_2\rangle = \sim\langle x, y_1\rangle + \quad \langle x, y_2\rangle = 0 \;\; \sim y_1 + \quad y_2 \in A^{\circ} \quad \Rightarrow \quad A^{\circ}$ is a subspace.

(6)

(a) $¤^{\circ} = \{y \in Y : \text{Re}\langle x, y\rangle \le 1 \quad \forall\, x \in ¤\}$

Since ¤ is empty set $\quad \Rightarrow \quad ¤^{\circ} = \{y \in Y\} = Y$

(b) Since X is a vector space $\Rightarrow \quad X^{\circ} = \{y \in Y : \langle x, y\rangle = 0 \,,\, \forall x \in X\}$

Since $\langle x, y\rangle = 0$ for all $x \in X \quad \Rightarrow \quad y = 0$, i.e. $N_Y = \{0\} \Rightarrow \quad X^{\circ} = \{0\}$

(7) Let $y_1, y_2 \in A^{\circ}$ and $0 \le ª \le 1$

$$\text{Re} < x, ªy_1 + (1-ª)y_2 >= \text{Re}(ª < x, y_1 > +(1-ª) < x, y_2 >)$$

$$= ª\,\text{Re} < x, y_1 > +(1-ª)\,\text{Re} < x, y_2 > \le ª + (1-ª) = 1$$

$ªy_1 + (1-ª)y_2 \in A^{\circ} \quad \Rightarrow \quad A^{\circ}$ is convex set.

(8) Take $\; D = \bigcap_{x \in A} \{y \in Y : \text{Re}\langle x, y\rangle \le 1\}$

Let $y \in A^{\circ} \quad \Rightarrow \quad y \in Y$ and $\text{Re}\langle x, y\rangle \le 1$ for all $x \in A$

$\Rightarrow \quad y \in D \quad \Rightarrow \quad A^{\circ} \subseteq D$. Similarly to prove $D \subseteq A^{\circ} \quad \Rightarrow \quad D = A^{\circ}$

Theorem (6.3.3)

Let X and Y be dual topological linear spaces over F and let $A \subseteq X$, then $A^{\circ\circ} = \overline{co}(A \cup \{0\})$ where the closure "$-$" taken in the weak topology.

Proof :

Let $B = co(A \cup \{0\}) \Rightarrow \overline{B}$ is smallest closed convex set contained A
Since A° is closed convex set in $Y \Rightarrow A^{\circ\circ}$ is closed convex set in X contained $A \Rightarrow \overline{B} \subset A^{\circ\circ}$. We have to show that $A^{\circ\circ} \subset \overline{B}$
Since $A^\circ = B^\circ \Rightarrow A^{\circ\circ} = B^{\circ\circ}$. To show that $B^{\circ\circ} \subset \overline{B}$
Let us assume that there exists an $x_0 \in B^{\circ\circ}$ such that $x_0 \notin \overline{B}$. By second separation theorem, there exists $y_0 \in Y$ such that $\text{Re}\langle x_0, y_o\rangle > 1$ and $\text{Re}\langle x, y_o\rangle < 1$ for all $x \in B$
Since $\text{Re}\langle x, y_o\rangle < 1$ for all $x \in B$, then $y_0 \in B^\circ$, but $\text{Re}\langle x_0, y_o\rangle > 1$, then $x_\circ \notin B^{\circ\circ}$. This contradiction $B^{\circ\circ} \subset \overline{B} \Rightarrow A^{\circ\circ} \subset \overline{B} \Rightarrow A^{\circ\circ} = \overline{B}$

Corollary (6.3.4)

Let X and Y be dual topological linear spaces over F
(1) If M is a subspace of X, then $M^{\circ\circ} = \overline{M}$
(2) If $A \subseteq X$, then $A^{\circ\circ\circ} = A^\circ$

Proof :

(1) Since $M \subseteq X \Rightarrow M^{\circ\circ} = \overline{co}(M \cup \{0\})$
Since M is a subspace , then $0 \in M \Rightarrow M \cup \{0\} = M \Rightarrow M^{\circ\circ} = \overline{co}(M)$
Since M is a subspace , then M is convex set $\Rightarrow co(M) = M \Rightarrow M^{\circ\circ} = \overline{M}$

Exercises (6)

6.1
6.2