دراسات عليا - ماجستير

Functional Analysis تحليل دالي

نظري : 3   مناقشة : 1   عدد الوحدات : 3

7. Fundamental Theorems For Normed Spaces

7.1 Riesz Representation

Theorem(7.1.1) Riesz lemma

Let M be closed proper subspace of a normed space X , and let α be a real number such that $0<\alpha<1$. Then there exists a vector $x_\alpha \in X$ such that $\|x_\alpha\|=1$ and $\|x-x_\alpha\|\ge\alpha$ for all $x\in M$.

Proof :

Since M be closed proper subspace of $X \Rightarrow M \neq X \Rightarrow$ there exists $x_0\in X$ such that $x_0\notin M$

Let $d=d(x_0,M)$, i.e. $d=\inf\{\|x-x_0\|:x\in M\}$

Since $x_0\notin M \Rightarrow d>0$ (because if $d=0$, then $x_0\in M$), Since $0<\alpha<1 \Leftarrow \frac{d}{\alpha}>d$

By the definition of infimum, there exists $x_1\in M$ such that $d<\|x-x_1\|\le\frac{d}{\alpha}$ …(1)

Let $x_\alpha=k(x_0-x_1)$ where $k=\|x_0-x_1\|^{-1}>0$ (note that $x_0\neq x_1$)

Then $\|x_\alpha\|=\|k(x_0-x_1)\|=k\,\|(x_0-x_1)\|=k\times k^{-1}=1$

Now let $x\in M \Rightarrow k^{-1}x+x_1\in M$

$\Rightarrow \|x-x_\alpha\|=\|x-k(x_0-x_1)\|=k\,\|(k^{-1}x+x_1)-x_0\|\ge k\,d$ …(2)

By (1), we have $\frac{1}{k}\le\frac{d}{\alpha}$, so $kd\ge\alpha$, hence $\|x-x_\alpha\|\ge\alpha$ for all $x\in M$.

Theorem(7.1.2)

Let X be a normed space, and suppose the $A=\{x\in X:\|x\|=1\}$ is compact .Then X is finite dimensional.

Proof :

We know that in metric space, a subset is compact iff it is sequentially compact, i.e. iff every sequence has a convergent subsequence.

Suppose that X is not finite dimensional.

Choose $x_1\in A$,and let M_1 be the subspace spanned by x_1, i.e. $M_1=[x_1]=\{\alpha x:\alpha\in F\}$

Then M_1 is a proper subspace of X

Since M_1 is finite dimensional $\Rightarrow M_1$ is complete , so that M_1 is closed.

Hence by Reisz-lemma there exists a vector $x_2\in A$ such that $\|x_2-x_1\|\ge\frac{1}{2}$

Let M_2 be the closed proper subspace of X generated by $\{x_1, x_2\}$, then as before, there must exist $x_3 \in A$ such that $\|x_3 - x\| \geq \frac{1}{2}$ for all $x \in M_2$.

This sequence can therefore have no convergent subsequence. But this contradicts the hypothesis A compact. Hence X must be finite dimensional.

Theorem(7.1.3)

Let x_0 be a fixed vector in a Hilbert space X and let $f_{x_0} : X \to F$ be a function defined by $f_{x_0}(x) = \langle x, x_0 \rangle$ for all $x \in X$, then $f_{x_0} \in X^*$ and $\|f_{x_0}\| = \|x_0\|$

Proof :

Let $x, y \in X$ and $\sim, \ \in F$

$f_{x_0}(\sim x + \ y) = \langle \sim x + \ y, x_0 \rangle = \sim \langle x, x_0 \rangle + \ \langle y, x_0 \rangle = \sim f_{x_0}(x) + \ f_{x_0}(y) \Rightarrow f_{x_0} \in X'$

To prove f_{x_0} is continuous . For every $x \in X$, we have $f_{x_0}(x) = \langle x, x_0 \rangle$

$\Rightarrow \ |f_{x_0}(x)| = |\langle x, x_0 \rangle| \leq \|x\| \|x_0\|$

Let $\|x_0\| = k \Rightarrow k > 0$. Therefore we have $|f_{x_0}(x)| \leq k\|x\|$ for all $x \in X$

Therefore the function f_{x_0} is bounded and every bounded function is continuous.

Hence f_{x_0} is functional on X and so $f_{x_0} \in X^*$

Now we shall show that $\|f_{x_0}\| = \|x_0\|$. As shown above, for every $x \in X$ we have

$|f_{x_0}(x)| \leq \|x\| \|x_0\| \quad \ldots(1)$

Now by definition, $\|f_{x_0}\| = \sup\{|f_{x_0}(x)| : \ \|x\| \leq 1\}$

If $\|x\| \leq 1 \Rightarrow \|x\| \|x_0\| \leq \|x_0\|$ and therefore (1) gives $|f_{x_0}(x)| \leq \|x_0\|$ for all $x \in X$ such that $\|x\| \leq 1$

$\Rightarrow \ \sup\{|f_{x_0}(x)| : \ \|x\| \leq 1\} \leq \|x_0\| \Rightarrow \|f_{x_0}\| \leq \|x_0\| \quad \ldots(2)$

If $x_0 = 0 \Rightarrow \|x_0\| = 0$. Also $f_{x_0}(x) = \langle x, x_0 \rangle = \langle x, 0 \rangle = 0 \Rightarrow f_{x_0}(x) = 0$ for all $x \in X$

Let us now take $x_0 \neq 0 \Rightarrow X \neq \{0\}$, we have $\|f_{x_0}\| = \sup\{|f_{x_0}(x)| : \ \|x\| = 1\}$

$\|f_{x_0}\| = \sup\{|f_{x_0}(x)| : \ \|x\| = 1\}$

Put $z = \dfrac{x_0}{\|x_0\|} \Rightarrow \|z\| = 1$

$$f_{x_0}(z) = \langle z, x_0 \rangle = \left\langle \frac{x_0}{\|x_0\|}, x_0 \right\rangle = \frac{1}{\|x_0\|} \langle x_0, x_0 \rangle = \frac{1}{\|x_0\|} \|x_0\|^2 = \|x_0\|$$

But $\|f_{x_0}\| \geq \|f_{x_0}(z)\| \Rightarrow \|f_{x_0}\| \geq \|x_0\| \quad \ldots(3)$

From (2),(3), we have $\|f_{x_0}\| = \|x_0\|$

Remark

From this theorem we conclude that the function $\cdot : X \to X^*$ such that $\cdot(x_0) = f_{x_0}$ is a norm preserving function.

Theorem(7.1.4)

Let X be a Hilbert space, and let $f \in X^*$. Then there exists a unique vector x_0 in X such that $f = f_{x_0}$, i.e. $f(x) = \langle x, x_0 \rangle$

Proof :

If f is a zero functional , then $f(x) = 0$ for $x \in X$, then $x_0 = 0$ is such that $f(x) = \langle x, x_0 \rangle$ for all $x \in X$. Now suppose that f is not zero functional, i.e. $f(x) \neq 0$ for some $x \in X$.

Let $M = \ker(f) \Rightarrow M = \{x \in X : f(x) = 0\}$, then M is a proper subspace of X.

Since f is continuous , then M is closed, hence M is a proper closed subspace of X.

Therefore there exists a non-zero vector $y_0 \in X$ such that $y_0 \perp M$

$\Rightarrow y_0 \in M^\perp \Rightarrow y_0 \notin M$ (if $y_0 \in M$, then $y_0 = 0$ this contradiction) $\Rightarrow f(y_0) = 0$

Put $x_0 = \sim y_0$ such that $\sim = \dfrac{\overline{f(y_0)}}{\|y_0\|^2} \Rightarrow \bar{\sim} = \dfrac{f(y_0)}{\|y_0\|^2}$, then

$f(y_0) = \bar{\sim}\|y_0\|^2 = \bar{\sim}\langle y_0, y_0 \rangle = \langle y_0, \sim y_0 \rangle = \langle y_0, x_0 \rangle$

If $m \in M \Rightarrow f(m) = 0$

Since $y_0 \perp M \Rightarrow \langle m, y_0 \rangle = 0 \Rightarrow \bar{\sim}\langle m, y_0 \rangle = 0 \Rightarrow \langle m, \sim y_0 \rangle = 0$

$\Rightarrow \langle m, x_0 \rangle = 0 \Rightarrow f(m) = \langle m, x_0 \rangle$

Now . If $x \in X$, then $f(x) = \dfrac{f(x)}{f(y_0)} f(y_0) = \quad f(y_0), \qquad = \dfrac{f(x)}{f(y_0)}$

$f(x) - \quad f(y_0) = 0 \Rightarrow f(x - \quad y_0) = 0 \Rightarrow x - \quad y_0 \in M$

Put $m = x - \quad y_0 \Rightarrow x = m + \quad y_0$

$f(x) = f(m + \quad y_0) = f(m) + \quad f(y_0) = \langle m, x_0 \rangle + \quad \langle y_0, x_0 \rangle = \langle m + \quad y_0, x_0 \rangle = \langle x, x_0 \rangle$

To prove unique .

Suppose $x_1, x_2 \in X$ such that $f(x) = \langle x, x_1 \rangle$ for all $x \in X$ and $f(x) = \langle x, x_2 \rangle$ for all $x \in X$

$\Rightarrow \langle x, x_1 \rangle = \langle x, x_2 \rangle$ for all $x \in X \Rightarrow \langle x, x_1 - x_2 \rangle = 0$ for all $x \in X$

Since $x_1 - x_2 \in X \Rightarrow \langle x_1 - x_2, x_1 - x_2 \rangle = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$.

Theorem(7.1.5)

Let x_0 be a fixed vector in a Hilbert space X , and let f_{x_0} be a functional on X defined by $f_{x_0} = \langle x, x_0 \rangle$ for all $x \in X$, then f_{x_0} is continuous linear functional on X , and $\|x_0\| = \|f_{x_0}\|$

Proof : H.w

Theorem(7.1.6)

Let X be a Hilbert space, and let $\cdot : X \to X^*$ defined by $\cdot(f) = f_y$ such that $f_y = \langle x, y \rangle$ for all $x \in X$. Then $\cdot$ is one-to-one, onto, additive but not linear, and an Isometry.

Proof :

(1) $\cdot$ is one-one : Let $y_1, y_2 \in X$ such that $\cdot(y_1) = \cdot(y_2) \Rightarrow f_{y_1} = f_{y_2}$

$\Rightarrow f_{y_1}(x) = f_{y_2}(x)$ for all $x \in X \Rightarrow \langle x, y_1 \rangle = \langle x, y_2 \rangle$ for all $x \in X \Rightarrow \langle x, y - y_2 \rangle = 0$ for all $x \in X$

Since $y_1 - y_2 \in X \Rightarrow \langle y_1 - y_2, y_1 - y_2 \rangle = 0 \Rightarrow y_1 - y_2 = 0 \Rightarrow y_1 = y_1 \Rightarrow \cdot$ is one-one

(2) $\cdot$ is onto : Let $f \in X^*$ by theorem(7.1.4), there exists a unique vector x_0 in X such that $f(x) = \langle x, x_0 \rangle$ for all $x \in X$, i.e. $f = f_{x_0}$ this mean $\cdot(x_0) = f_{x_0} = f \Rightarrow \cdot$ is onto, so that $\cdot$ is bijective

(3) $\cdot$ is additive : Let $y_1, y_2 \in X \Rightarrow \cdot(y_1 + y_2) = f_{y_1 + y_2}$

Now for every $x \in X$,we have

$f_{y_1+y_2}(x) = \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle = f_{y_1}(x) + f_{y_2}(x) = (f_{y_1} + f_{y_2})(x)$

$\Rightarrow f_{y_1+y_2} = f_{y_1} + f_{y_2} \Rightarrow \cdot(y_1 + y_2) = \cdot(y_1) + \cdot(y_2) \Rightarrow \cdot$ is additive.

(4) $\cdot$ is not linear : Let $y \in X$, $\sim \in F$

$\cdot(\sim y) = f_{\sim y}, \quad f_{\sim y}(x) = \langle x, y \rangle = \bar{\sim} \langle x, y \rangle = \bar{\sim} f_y(x) \Rightarrow f_{\sim y} = \bar{\sim} f_y \Rightarrow \cdot(\sim y) = \bar{\sim} \cdot(y)$

$\Rightarrow \cdot$ is not linear

(5) $\cdot$ is an Isometry : Let $y_1, y_2 \in X$

$\|\cdot(y_1) - \cdot(y_2)\| = \|f_{y_1} - f_{y_2}\| = \|f_{y_1} + f_{-y_2}\| = \|f_{y_1+(-y_2)}\| = \|f_{y_1-y_2}\| = \|y_1 - y_2\|$

By theorem(8.14), we have $\|y\| = \|f_y\| \Rightarrow \cdot$ is an Isometry.

Remark

In our further discussion we shall represent the functionals in X^* by $f_x, f_y, f_z, \cdots$ where $x, y, z, \cdots$ are their corresponding vectors in X .

Theorem(7.1.7)

If X is a Hilbert space, then X^* is also a Hilbert space with respect to the inner product defined by $\langle f_x, f_y \rangle = \langle y, x \rangle$.

Proof :

(1) To prove X^* is a pre-Hilbert space

(i) let $f_x \in X^*$, then $\langle f_x, f_x \rangle = \langle x, x \rangle = \|x\|^2 \geq 0$

(ii) $\langle f_x, f_y \rangle = 0 \;\Leftrightarrow\; \|x\|^2 = 0 \;\Leftrightarrow\; \|f_x\|^2 = 0 \;\Leftrightarrow\; f_x = 0$

(iii) let $f_x, f_y \in X^*$, then $\overline{\langle f_x, f_y \rangle} = \overline{\langle y, x \rangle} = \langle x, y \rangle = \langle f_y, f_x \rangle$

(iv) let $f_x, f_y, f_z \in X^*$ and let $\sim, \;\; \in F$, then

$\langle \sim f_x + \;\; f_y, f_z \rangle = \langle f_{\bar{\sim} x} + f_{\bar{\;} y}, f_z \rangle == \langle f_{\bar{\sim} x + \bar{\;} y}, f_z \rangle = \langle z, \bar{\sim} x + \bar{\;} y \rangle = \sim \langle z, x \rangle + \;\; \langle z, y \rangle = \sim \langle f_x, f_z \rangle + \;\; \langle f_y, f_z \rangle$

$\Rightarrow$ X^* is a pre-Hilbert space

(2) To prove X^* is complete

Since X is a Hilbert space, then X is a normed space , so X^* is complete.

$\Rightarrow$ X^* is a Hilbert space.

Corollary(7.1.8)

If we denote the elements of X^{**} by $F_f, G_g, \cdots$ where f, g are their corresponding elements in X^*, then by theorem(), we conclude that X^{**} is also a Hilbert space with respect to the inner product defined by $\langle F_f, G_g \rangle = \langle g, f \rangle$.

7.2 Strong and Weak Convergence

Convergence of sequence of elements in a normed space was defined in section 3, from now on, will be called strong convergence, to distinguish it from " weak convergence" to be introduced shortly. Hence we first state

Definition(7.2.1)

A sequence $\{x_n\}$ in a normed space X is said to be **Strongly convergent**(or convergent in the norm) if there is an $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. This is written $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

The element x is called the strong limit of $\{x_n\}$, and we say that $\{x_n\}$ converges strongly to x .

Weak convergence is defined in terms of bounded linear functionals on X as follows.

Definition(7.2.2)

A sequence $\{x_n\}$ in a normed space X is said to be Weakley Convergent if there is an $x \in X$ such that for every $f \in X^*$, we have $f(x_n) \to f(x)$. This is written $x_n \xrightarrow{w} x$. The element x is called the weak limit of $\{x_n\}$, and we say that $\{x_n\}$ converges weakly to x .

Theorem(7.2.3)

Let $\{x_n\}$ be a weakly convergent sequence in a normed space X , say $x_n \xrightarrow{w} x$

(1) The weak limit x of $\{x_n\}$ is unique.

(2) Every subsequence of $\{x_n\}$ converges weakly to x .

(3) The sequence $\{\|x_n\|\}$ is bounded.

Proof :

(1) Suppose that $x_n \xrightarrow{w} x$, $y_n \xrightarrow{w} y$. To prove that $x = y$

Let $f \in X^*$ $\Rightarrow$ $f(x_n) \to f(x)$, $f(x_n) \to f(y)$

Since the limit is unique , we have $f(x) = f(y)$ for all $f \in X^*$

$\Rightarrow$ $f(x - y) = 0$ for all $f \in X^*$ $x - y = 0$ $\Rightarrow$ $x = y$.

(2) Since $\{f(x_n)\}$ is convergent sequence in F for all $f \in X^*$,so that every subsequence of $\{f(x_n)\}$ converges and has the same limit as the sequence.

(3) Since $\{f(x_n)\}$ is convergent sequence in F for all $f \in X^*$ $\Rightarrow$ $\{f(x_n)\}$ is bounded $\Rightarrow$ there exists $M_f > 0$ such that $|f(x_n)| \le M_f$ for all n, where M_f is a constant depending of f but not on n. Using the canonical function $\cdot : X \to X^{**}$, we can define $g_n \in X^{**}$ by $g_n(f) = f(x_n)$ for all $f \in X^*$. Then for all n , $|g_n(f)| = |f(x_n)| \le M_g$ that is, the sequence $\{|g_n(f)|\}$ is bounded for every $f \in X^*$.

Since X^* is complete (X^* is Banach space) $\Rightarrow$ $\{\|g_n\|\}$ is bounded.

Now since $\|g_n\| = \|x_n\|$ $\Rightarrow$ $\{\|x_n\|\}$ is bounded.

Theorem (7.2.4)

If $\{x_n\}$ and $\{y_n\}$ are sequence in a normed space X such that $x_n \xrightarrow{w} x$, $y_n \xrightarrow{w} y$, then

(1) $x_n + y_n \xrightarrow{w} x + y$ (2) $\alpha x_n \xrightarrow{w} x + y$ for all $\alpha \in F$.

Proof : obvious

Theorem(7.2.5)

Let $\{x_n\}$ be a sequence in a normed space X such that $x_n \to x$, then $x_n \xrightarrow{w} x$ and the converse not true.

Proof :

Since $x_n \to x$ $\Rightarrow$ $\|x_n - x\| \to 0$ as $n \to \infty$

Let $f \in X^*$ $\Rightarrow$ $|f(x_n) - f(x)| = |f(x_n - x)| \le \|f\| \|x_n - x\|$

$\Rightarrow$ $|f(x_n) - f(x)| \to 0$ as $n \to \infty$ $\Rightarrow$ $x_n \xrightarrow{w} x$.

Example for the converse : Let X be a Hilbert space over F and let $f \in X^*$. By using Riesz representation, there exists $x_0 \in X$ such that $f(x) = \langle x, x_0 \rangle$ for all $x \in X$.

Let $\{x_n\}$ be an orthonormal sequence in X $\Rightarrow$ $f(x_n) = \langle x_n, x_0 \rangle$.

Now the Bessel inequality is $\sum_{i=1}^{\infty} |\langle x_n, x_0 \rangle| \le \|x_0\|^2$

Hence the series on the left converges, so that its terms must approach zero as $n \to \infty$. i.e. $|\langle x_n, x_0 \rangle| \to 0$ as $n \to \infty$. This implies $f(x_n) = \langle x_n, x_0 \rangle \to 0$ since $f \in X^*$ $\Rightarrow$ $x_n \xrightarrow{w} 0$, but $\{x_n\}$ does not converge to zero. because

$\|x_n - x_m\|^2 = \langle x_n - x_m, x_n - x_m \rangle = \|x_n\|^2 + \|x_m\|^2 = 1+1=2, \quad n \neq m$.

Theorem(7.2.6)

Let $\{x_n\}$ be a sequence in a finite dimensional normed space X such that $x_n \xrightarrow{w} x$, then $x_n \to x$.

Proof :

Let $\dim X = m$ and let $\{x_1, \cdots, x_m\}$ be any basis for X

Since $x \in X \quad \Rightarrow \quad x$ has unique representation $x = \sum_{i=1}^{m} \alpha_i \, x_i, \quad \alpha_i \in F$

Also $x_n \in X \quad \Rightarrow \quad x_n$ has unique representation $x_n = \sum_{i=1}^{m} \alpha_{in} \, x_i, \quad \alpha_{in} \in F$

Define $f_i : X \to F$ by $f_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$. It is clear to show that $f_i \in X^*$ for all $i = 1, \cdots, n$

since $x_n \xrightarrow{w} x \quad \Rightarrow \quad f_i(x_n) \to f_i(x)$ as $n \to \infty$

since $f_i(x) = \alpha_i \quad , \quad f_i(x_n) = \alpha_{in} \quad \Rightarrow \quad \alpha_{in} \to \alpha_i$ as $n \to \infty \quad \Rightarrow \quad |\alpha_{in} - \alpha_i| \to 0$ as $n \to \infty$

$$\|x_n - x\| = \left\| \sum_{i1}^{m} \alpha_{in} x_i - \sum_{i1}^{m} \alpha_i x_i \right\| = \left\| \sum_{i1}^{m} (\alpha_{in} - \alpha_i) x_i \right\| \leq \sum_{i=1}^{m} |\alpha_{im} - \alpha_i| \|x_i\|$$

$\Rightarrow \quad \|x_n - x\| \to 0$ as $n \to \infty \quad \Rightarrow \quad x_n \to x$ as $n \to \infty$.

Definition(7.27)

Let X and Y be normed spaces over F , and let $\{f_n\}$ be a sequence in $B(X,Y)$. A sequence $\{f_n\}$ is said to be

(1) Uniformly Convergent if $\{f_n\}$ converges in the norm on $B(X,Y)$. i.e.

If there exists $f \in L(X,Y)$ such that $\|f_n - f\| \to 0$ as $n \to \infty$

(2) **Strongly Convergent** if $\{f_n(x)\}$ converges strongly in Y for every $x \in X$, i.e.

If there exists $f \in L(X,Y)$ such that $\|f_n(x) - f(x)\| \to 0$ as $n \to \infty$ for every $x \in X$.

(3) **Weakly Convergent** if $\{f_n(x)\}$ converges weakly in Y for every $x \in X$, i.e.

If there exists $f \in L(X,Y)$ such that $\|g(f_n(x)) - g(f(x))\| \to 0$ as $n \to \infty$ for every $x \in X$, and for every $g \in X^*$.

It is not difficult to show that $(1) \Rightarrow (2) \Rightarrow (3)$, but the converse is not generally true , as can be seen from the following examples.

Example (7.2.8)

(1) In the space ℓ^2 we consider a sequence $\{f_n\}$, where $f_n = \ell^2 \to \ell^2$ is defined by

$$f_n(x) = (\underbrace{0, 0, \cdots, 0}_{n-0}, x_{n+1}, x_{n+2}, \cdots)$$

where $x = (x_1, x_2, \cdots) \in \ell^2 \Rightarrow f_n \in B(\ell^2)$ for all n.

$\{f_n\}$ is strongly convergent to 0. (because $f_n(x) \to 0$ as $n \to \infty$), but $\{f_n\}$is not uniformly convergent (because $\|f_n - 0\| = \|f_n\| = 1$)

(2)) In the space ℓ^2 we consider a sequence $\{f_n\}$, where $f_n = \ell^2 \to \ell^2$ is defined by

$$f_n(x) = (\underbrace{0,0,\cdots,0}_{n-0's}, x_1, x_2, \cdots)$$

where $x = (x_1, x_2, \cdots) \in \ell^2 \Rightarrow f_n \in B(\ell^2)$ for all n.

We show that $\{f_n\}$ is weakly convergent to 0, but not strongly convergent

Let $g \in (\ell^2)^* \Rightarrow g$ is bounded linear functional on ℓ^2. By Riesz representation there is $y \in \ell^2$ such that $g(x) = \langle x, z \rangle$ where $x \in \ell^2$.

$\Rightarrow g(x) = \sum_{i=1}^{\infty} x_i \overline{y_i}$ where $x = (x_1, x_2, \cdots, 1, y_1, y_2, 0) \in \ell^2$

$g(f_n(x)) = \sum_{i=n+1}^{\infty} x_{i-n} \overline{y_i} = \sum_{k=1}^{\infty} x_k y_{n+k}$. by the Cauchy-Schwarz inequality , we have

$$|g(f_n(x)|^2 = \left|\sum_{k=1} x_n y_{n+k}\right|^2 \le \left(\sum_{k=1} |x_k|^2\right)\left(\sum_{m=n+1} |y_m|^2\right)$$

The last series is the remainder of a convergent series. Hence the right-hand side approaches 0 as $n \to \infty$. Thus $g(f_n(x)) \to 0 \Rightarrow \{f_n\}$ is weakly convergent to 0.

However $\{f_n\}$ is not strongly convergent because for $x = (1,0,0,\cdots)$ we have

$\|f_m(x) - f_n(x)\| = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad n \ne m$.

Definition(7.2.9)

Let X be normed space over F, and let$\{f_n\}$ be a sequence in X^*. A sequence $\{f_n\}$is said to be

(1) **Strong Convergent**, if there is an $f \in X^*$ such that $\|f_n - f\| \to 0$ as $n \to \infty$. This written $f_n \to f$. The function f is called the strong limit of $\{f_n\}$

(2) **Weak* Convergence,** if there is an $f \in X^*$ such that $f_n(x) \to f(x)$ for all $x \in X$. This written : $f_n \xrightarrow{w^*} f$. The function f is called the Weak* limit of $\{f_n\}$

Example (7.2.10)

The space X of all sequences $x = (x_n)$ in ℓ^2 with only finitely many nonzero terms, taken with metric on ℓ^2 is not complete . A function $f_n : X \to X$ is defined by

$f_n(x) = (x_1, 2x_2, 3x_3, \cdots, nx_n, x_{n+1}, x_{n+2}, \cdots)$

So that $f_n(x)$ has terms mx_m if $m \le n$ and x_m if $m > n$.

This sequence $\{f_n\}$ converges strongly to the unbounded linear function f defined by $f(x)=(y_i)$ where $y_i = ix_i$.

Theorem(7.2.11)

Let X and Y be normed spaces over F, and let $\{f_n\}$ be a sequence in $B(X,Y)$. If X is a Banach and the X is bounded in Y for all $x \in X$, then the sequence $\{\|f_n\|\}$ is bounded.

Proof :

Let k be natural number. Define A_k by $A_k = \{x \in X : \|f_n(x)\| < k\}$

First : To prove A_k is closed

Let $x \in \overline{A}_k \Rightarrow$ there exists a sequence $\{x_n\}$ in A_k such that $x_m \to x$ as $m \to \infty$

Since $x_m \in A_k \Rightarrow$ for all n, we have $\|f_n(x_m)\| < k$

Since f_n is continuous , then for all n, we have $\|f_n(x)\| < k \Rightarrow x \in A_k \Rightarrow \overline{A}_k \subset A$

But $A_k \subseteq \overline{A}_k \Rightarrow \subset \overline{A}_k = A_k \Rightarrow A_k$ is closed

Since $\{f_n(x)\}$ is bounded in Y for all $x \in X$., then for all n, there exists k_x such that $\|f_n(x)\| \le k_x$

For all $n \Rightarrow x \in A_k$ for some k , so that $X = \bigcup_{k=1}^{\infty} A_k$

Since X is complete, by Baires theorem, A_k contain open ball, say $B_r(x_0) \subset A_{k_0}$

Let x be non zero element in X .

Put $\alpha = \dfrac{r}{2\|x\|}$, $y = x_0 + \alpha x$ $\|y - x_0\| < r \Rightarrow y \in {}_r(x_0) \Rightarrow y \in A_k \Rightarrow \|f_n(y)\| \le k$

Since $x_0 \in {}_r(x_0) \Rightarrow \|f_n(x_0)\| \le k$

Since $x = \dfrac{1}{\alpha}(y - x_0)$. for all n ,

$$f_n(x) = \frac{1}{\alpha} f(y - x_0) = \frac{1}{\alpha}\left[\left(f(y) - f(x_0)\right)\right]$$

$$\|f_n(x)\| = \left\|\frac{1}{\alpha}\left(f(y) - f(x_0)\right)\right\| \le \frac{1}{\alpha}\left(\|f((y)\| - \|f(x_0)\|\right) \le \frac{2}{\alpha} k_0 = \frac{4}{r}\|x\| k_0$$

So that for all n , $\|f_n\| = \sup\{|f_n(x)| : x \in X , \|x\| = 1\}$, then $\|f_n\| \le \dfrac{4}{r} k_0$. Hence $\{\|f_n\|\}$ is bounded.

7.3 Adjoint Operator

Recall that a function $T : X \to Y$ is called an operator form X into Y if X and Y are linear space over the same field F. A linear operator T is an operator such that $T(\sim x + \ y) = \sim T(x) + \ T(y)$ for all $x, y \in X$ and for all $\sim, \ \in F$. Let X and Y be normed spaces over F, $B(X, Y)$ is the space of bounded linear operator from X into Y, $B(X, Y)$ is a normed space with respect to the norm defined by $\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| \le 1\}$ for all $T \in B(X, Y)$, (see section 5.)

Definition(7.3.1)

Let X and Y be normed spaces over F, and let $T \in B(X, Y)$. An operator $T^* : Y^* \to X^*$ which is defined by $(T^*(g))(x) = g(T(x))$ for all $g \in Y^*$ is called an adjoint (or conjugate) of T.

It is clear to show T^* is unique .

Theorem(7.3.2)

Let X and Y be normed spaces over F, and let $T \in B(X, Y)$. Then T^* is bounded linear operator and $\|T^*\| = \|T\|$

Proof :

(1) let $f, g \in Y^*$ and let $\sim, \ \in F$

$$T^*(\sim f + \ g)(x) = (\sim f + \ g)(T(x)) = (\sim f)(T(x)) + (\ g)(T(x)) = \sim(f(T(x)) + \ (g(T(x)))$$

$$= \sim f(T(x)) + \ g(T(x)) = \sim T^*(f)(x) + \ T^*(g)(x) = (\sim T^*(f) + \ T^*(g))(x)$$

$T^*(\sim f + \ g) = \sim T^*(f) + \ T^*(g) \Rightarrow T^*$ is linear

(2)

$$\|T^*\| = \sup\{\|T^*(f)\| : f \in Y^*, \|f\| \le 1\} = \sup\{|T^*(f)(x)| : f \in Y^*, \|f\| \le 1, \|x\| \le 1\}$$

$$= \sup\{|f(T(x))| : f \in Y^*, \|f\| \le 1, \|x\| \le 1\} \le \sup\{\|f\|\|T\|\|x\| : f \in Y^*, \|f\| \le 1, \|x\| \le 1\} \le \|T\|$$

$\Rightarrow \|T^*\| \le \|T\|$ Since T is bounded, then T^* is bounded.

(3) We must prove $\|T^*\| \ge \|T\|$

Since , for each nonzero vector $x \in X$, there exists $f \in Y^*$ such that $\|f\| = 1$ and $f(T(x)) = \|T(x)\|$

$$\|T\| = \sup\{\frac{\|T(x)\|}{\|x\|} : x \ne 0\} = \sup\{\frac{|f(T(x))|}{\|x\|} : f \in Y^*, \|f\| = 1, x \ne 0\} = \sup\{\frac{|T^*(f)(x)|}{\|x\|} : f \in Y^*, \|f\| = 1, x \ne 0\}$$

$$\le \sup\{\frac{\|T^*(f)\|}{\|x\|} : f \in Y^*, \|f\| = 1, x \ne 0\} \le \sup\{\|T^*(f)\|\|T\|\|x\| : f \in Y^*, \|f\| = 1\} = \|T^*\|$$

$\Rightarrow \|T\| \le \|T^*\| \Rightarrow \|T^*\| = \|T\|$

Remark

Let X and Y be normed spaces over F ,and let $B(Y^*,X^*)$ denote the set of all adjoint operator of T , where $T \in B(X,Y)$, i.e. $T^* \in B(Y^*,X^*)$, iff T^* is an adjoint operator of T . It is clear to show that $B(Y^*,X^*)$ is normed space.

Theorem(7.3.3)

Let X,Y,Z be normed spaces over F. Then

(1) $(\sim S + \ T)^* = \sim S^* + \ T^*$ for all $S,T \in B(X,Y)$ and for all $\sim, \ \in F$.

(2) If $T \in B(X,Y)$, $S \in B(Y,Z)$. Then $(S \circ T)^* = T^* \circ S^*$

(3) If $I \in B(X)$, then $I^* = I$, where I is identity operator

(4) Let $T \in B(X,Y)$. If T^{-1} exists and $T^{-1} \in B(Y,X)$. Then $(T^*)^{-1}$ also exists ,
$(T^*)^{-1} \in B(X^*,Y^*)$,
$(T^*)^{-1} = (T^{-1})^*$ and

Proof :

(1) Let $S,T \in B(X,Y)$ and let $\sim, \ \in F$

$$((\sim S + \ T)^*(f))(x) = f((\sim S + \ T))(x) = f(\sim S(x) + \ T(x)) = \sim f(S(x)) + \ f(T(x))$$
$$= \sim (S^*(f))(x) + \ (T^*(f))(x)$$
$$(\sim S + \ T)^*(f) = \sim (S^*(f)) + \ (T^*(f)) = (\sim S^* + \ T^*)(f) \Rightarrow (\sim S + \ T)^* = \sim S^* + \ T^*$$

(2) Let $T \in B(X,Y)$, $S \in B(Y,Z)$

$$((S \circ T)^*(f))(x) = f((S \circ T)(x)) = f(S(T(x))) = (S^*(f)(T(x)) = (T^*(S^*(f)))(x) = ((T^* \circ S^*)(f))(x)$$

Hence $(S \circ T)^* = T^* \circ S^*$

(3) $(I^*(f))(x) = f(I(x)) = f(x) = I(f(x)) = (I(f))(x) \Rightarrow I^* = I$

Theorem(7.3.4)

Let X and Y be normed spaces over F. Define $\ddot{} : B(X,Y) \to B(Y^*,X^*)$ by $\ddot{}(T) = T^*$ for all $T \in B(X,Y)$. Then $\ddot{}$ is an is an isometric isomorphism

Proof :

(1) $\ddot{}$ is one-one : let $\ddot{}(S) = \ddot{}(T)$

$$\Rightarrow S^* = T^* \Rightarrow \|S^* - T^*\| = 0 \Rightarrow \|(S-T)^*\| = 0 \Rightarrow \|S-T\| = 0 \Rightarrow S = T$$

$\Rightarrow$ $\ddot{}$ is one-one

(2) $\ddot{}$ is linear : let $S,T \in B(X,Y)$ and $\sim, \ \in F$

$\ddot{}(\sim S + \ T) = (\sim S + \ T)^* = \sim S^* + \ T^* = \sim \ddot{}(S) + \ \ddot{}(T) \Rightarrow$ $\ddot{}$ is linear

(3) $\ddot{}$ is preserves norm : Let $T \in B(X,T)$

$\|\ddot{}(T)\| = \|T^*\| = \|T\| \Rightarrow$ $\ddot{}$ is preserves norm

Definition(7.3.5)

Let X and Y be linear spaces over F. A function $h: X \times Y \to F$ is called a sesquilinear form (or sesquilinear functional) if

(1) $h(\sim x_1 + \ x_2, y) = \sim h(x_1, y) + \ h(x_2, y)$ for all $x_1, x_2 \in X, y \in Y$ and $\sim, \ \in F$

(2) $h(x, \sim y_1 + \ y_2) = \overline{\sim} h(x, y_1) + \overline{\ } h(x, y_2)$ for all $x \in X, y_1, y_2 \in Y$ and $\sim, \ \in F$

Hence h is linear in the first argument and conjugate linear in the second one.

Let X and Y be normed spaces over F. A sesquilinear form $h: X \times Y \to F$ is called bounded, if there is a real number k such that for all $x \in X, y \in Y$ such that $|h(x, y)| \le k \|x\| \|y\|$.

and the number

$$\|h\| = \sup\{\frac{|h(x, y)|}{\|x\| \|y\|} : x \in X, y \in Y, x \neq 0, y \neq 0\} = \sup\{|h(x, y)| : x \in X, y \in Y, \|x\| = 1, \|y\| = 1\}$$ is

called the norm of G.

Theorem(7.3.6) Riesz representation

Let X and Y be Hilbert spaces over F, and let $h: X \times Y \to F$ be a bounded sesquilinear form. Then h has a representation $h(x, y) = \langle S(x), y \rangle$ where $S: X \to Y$ is bounded linear operator. S is uniquely determined by h and has norm $\|S\| = \|h\|$.

Proof :

Definition(7.3.7)

Let X and Y be Hilbert spaces over F, and let $T \in B(X, Y)$. The Hilbert adjoint operator T^* of T is the operator $T^*: Y \to X$ such that for all $x \in X$ and $y \in Y$, $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$.

Theorem(7.3.8)

Let X and Y be Hilbert spaces over F, and let $T \in B(X, Y)$. The Hilbert adjoint operator T^* of T is unique and is bounded linear operator with norm $\|T^*\| = \|T\|$

Proof :

Define $h: Y \times X \to F$ by $h(y, x) = \langle y, T(x) \rangle$ for all $x \in X$ and $y \in Y$

(1) G is conjugate linear : let $x_1, x_2 \in X$ and $\sim, \ \in F$

$$h(y, \sim x_1 + \ x_2) = \langle y, T(\sim x_1 + \ x_2) \rangle = \langle y, \sim T(x_1) + \ T(x_2) \rangle = \overline{\sim} \langle y, T(x_1) \rangle + \overline{\ } \langle y, T(x_2) \rangle = \overline{\sim} h(y, x_1) + \overline{\ } h(y, x_2)$$

(2) h is bounded

By the Schwarz inequality, we have $|h(y, x)| = |\langle y, T(x) \rangle| \le \|y\| \|T(x)\| \le \|T\| \|x\| \|y\|$

This also implies $\|h\| \le \|T\|$. Moreover we have $\|h\| \ge \|T\|$ from

$$\|h\| = \sup\{\frac{|\langle y, T(x) \rangle|}{\|y\| \|x\|} : x \neq 0, y \neq 0\} \ge \sup\{\frac{|\langle y, T(x) \rangle|}{\|T(x)\| \|x\|} : x \neq 0, y \neq 0\} = \|T\|$$

Together, $\|h\| = \|T\|$

By using theorem (7.3.6) for h; writing T^* for S, we have $h(y,x)=\langle T^*(y),x\rangle$ and we know from that theorem that $T^*:Y \to X$ is a uniquely determined bounded linear operator with norm $\|T^*\|=\|h\|=\|T\|$.

Theorem(7,3.9) Properties of Hilbert adjoint operator

Let X and Y be Hilbert spaces over F, and let $S,T \in B(X,Y)$.

(1) $\langle T^*(y),x\rangle=\langle y,T(x)\rangle$ for all $x\in X$, $y\in Y$ (2) $(\alpha S+\beta T)^*=\bar{\alpha}S^*+\bar{\beta}T^*$ for all $\alpha,\beta\in F$

(3) $(T^*)^*=T$ (4) $\|T^*\circ T\|=\|T\circ T^*\|=\|T\|^2$ (5) $T^*\circ T=0$ iff $T=0$ (6) $(S\circ T)^*=T^*\circ S^*$ (assuming $X=Y$)

(7) Let $T\in B(X,Y)$. If T is bijective, then T^* is also bijective and $(T^*)^{-1}=(T^{-1})^*$

Proof :

Definition(7.3.10)

Let X be a Hilbert space over F, and let $T\in B(X)$. T is said to be Self-adjoint or Hermitian if $T^*=T$.

The Hilbert –adjoint operator T^* of T is defined by $\langle T(x),y\rangle=\langle x,T^*(y)\rangle$. If T is self –adjoint, we have $\langle T(x),y\rangle=\langle x,T(y)\rangle$.

Theorem(7.3.11) (Self-adjointness of product)

The product of two bounded self-adjoint linear operators S and T on a Hilbert space X is self-adjoint iff the operators commute (i.e., $S\circ T=T\circ S$)

Proof :

Since S and T are self –adjoint, then $S^*=S$, $T^*=T$

Since $(S\circ T)^*=T^*\circ S^*$, then $(S\circ T)^*=T\circ S$.

Hence $(S\circ T)^*=S\circ T$ iff $S\circ T=T\circ S$.

Theorem(7.3.12)

Let X be a Hilbert space over F, and let $T,T_n\in B(X)$ such that $T_n\to T$. If T_n is self-adjoint for all n, then T is self-adjoint.

Proof :

Since $T_n\to T \Rightarrow \|T_n-T\|\to 0$ as $n\to\infty$

Since T_n is self-adjoint for all n, $\Rightarrow T_n^*=T_n$ for all n

$T-T^*=(T-T_n)+(T_n-T_n^*)+(T_n^*-T^*)=(T-T_n)+0+(T-T_n)^* \Rightarrow T-T^*=(T-T_n)+(T-T_n)^*$

$\Rightarrow \|T-T^*\|=\|(T-T_n)+(T-T_n)^*\|\le\|T-T_n\|+\|(T-T_n)^*\|=\|T-T_n\|+\|T-T_n\|=2\|T-T_n\|$

Since $\|T_n-T\|\to 0$ as $n\to\infty$, then $\|T-T^*\|\to 0$ as $n\to\infty$.

Hence $\|T-T^*\|=0 \to T^*=T \Rightarrow T$ is self-adjoint.

Theorem(7.3.13)

Let X be a Hilbert space over F. If $S(X)$ denotes the set of all self-adjoint in $B(X)$,then $S(X)$ is a closed subspace of $B(X)$, and therefore a real Banach space which contains the identity linear operator.

Proof :

$S(X)=\{T\in B(X):T \text{ is self-adjoint}\}$

Since $0^*=0 \Rightarrow 0\in S(X) \Rightarrow S(X)\neq$ ¤

Let $\Rightarrow S,T\in S(X) \Rightarrow S^*=S,\ T^*=T$

Let $\sim,\ \ \in\mathbb{R}$, then $(\sim S+\ \ T)^*=(\sim S)^*+(\ \ T)^*=\overline{\sim}S^*+\overline{\ \ }T^*=\sim S^*+\ \ T^*=\sim S+\ \ T$

$\Rightarrow\ \sim S+\ \ T\in S(X)$, so that $S(X)$ is a real subspace of $B(X)$.

Now to show that $S(X)$ is closed subset of $B(X)$

Let $T\in\overline{S(X)} \Rightarrow$ there exists a sequence $\{T_n\}$ in $S(X)$ such that $T_n\to T$

$\|T-T^*\|=\|(T-T_n)+(T_n-T^*)\|\le\|T-T_n\|+\|(T_n-T_n^*)+(T_n^*-T^*)\|$

$\le\|T-T_n\|+\|T_n-T_n^*\|+\|T_n^*-T^*\|=\|T-T_n\|+\|0\|+\|(T_n-T)^*\|=\|T_n-T\|+\|T_n-T\|=2\|T_n-T\|$

Since $T_n\to T \Rightarrow \|T_n-T\|\to 0$ as $n\to\infty$

$\|T-T^*\|=0 \Rightarrow T-T^*=0 \Rightarrow T=T^*$ .so T is self-adjoint

$T\in S(X) \Rightarrow \overline{S(X)}=S(X)$

$\Rightarrow S(X)$ is closed subset of $B(X)\Rightarrow S(X)$ is a real closed subspace of $B(X)$

Since $B(X)$ is complete $\Rightarrow S(X)$ is a real Banach space .

Since $I^*=I \Rightarrow I\in S(X)$.

Theorem(7.3.14)

Let X be a Hilbert space over F, and $T\in B(X)$.Then $T=0$ iff $\langle T(x),y\rangle=0$ foe all $x,y\in X$.

Proof :

Suppose $T=0 \Rightarrow T(x)=0$ for all $x\in X$, we have $\langle T(x),y\rangle=\langle 0,y\rangle=0$

Conversely : suppose that $\langle T(x),y\rangle=0$ foe all $x,y\in X$.

Since $T(x)\in X$. Taking $y=T(x)\Rightarrow \langle T(x),T(x)\rangle=0$ for all $x\in X$

$\Rightarrow T(x)=0$ for all $x\in X \Rightarrow T=0$.

Theorem(7.3.15)

Let X be a Hilbert space over F, and $T\in B(X)$.Then $T=0$ iff $\langle T(x),x\rangle=0$ foe all $x\in X$.

Proof :

Suppose $T=0 \Rightarrow T(x)=0$ for all $x\in X$, we have $\langle T(x),x\rangle=\langle 0,x\rangle=0$

Conversely : suppose that $\langle T(x),x\rangle = 0$ foe all $x \in X$. Then to prove $T = 0$

If $x, y \in X$ and $\alpha, \beta \in F$, then we have

$$\langle T(\alpha x+\beta y),\alpha x+\beta y\rangle = \langle \alpha T(x)+\beta T(y),\alpha x+\beta y\rangle = \alpha\langle T(x),\alpha x+\beta y\rangle + \beta\langle T(y),\alpha x+\beta y\rangle$$

$$\langle T(\alpha x+\beta y),\alpha x+\beta y\rangle = \alpha\bar{\alpha}\langle T(x),x\rangle + \alpha\bar{\beta}\langle T(x),y\rangle + \bar{\alpha}\beta\langle T(y),x\rangle + \beta\bar{\beta}\langle T(y),y\rangle$$

$$\langle T(\alpha x+\beta y),\alpha x+\beta y\rangle = |\alpha|^2\langle T(x),x\rangle + \alpha\bar{\beta}\langle T(x),y\rangle + \bar{\alpha}\beta\langle T(y),x\rangle + |\beta|^2\langle T(y),y\rangle$$

$$\langle T(\alpha x+\beta y),\alpha x+\beta y\rangle - |\alpha|^2\langle T(x),x\rangle - |\beta|^2\langle T(y),y\rangle = \alpha\bar{\beta}\langle T(x),y\rangle + \bar{\alpha}\beta\langle T(y),x\rangle \quad (1)$$

But by hypothesis $\langle T(x),x\rangle = 0$ for all $x \in X$. Therefore the left hand side of (1)is also equal to zero. Thus we have $\alpha\bar{\beta}\langle T(x),y\rangle + \bar{\alpha}\beta\langle T(y),x\rangle = 0 \quad (2)$ for all $x, y \in X$ and $\alpha, \beta \in F$

Put $\alpha = 1, \beta = 1$ in (2), we give $\langle T(x),y\rangle + \langle T(y),x\rangle = 0 \quad (3)$

Again putting $\alpha = i, \beta = 1$ in (2), we get $i\langle T(x),y\rangle - i\langle T(y),x\rangle = 0 \quad (4)$

Multiplying (3) by i and adding to (4), we get $2i\langle T(x),y\rangle = 0$ for all $x, y \in X$

$\Rightarrow \langle T(x),y\rangle = 0$ for all $x, y \in X$. Taking $y = T(x) \Rightarrow \langle T(x),T(x)\rangle = 0$ for all $x \in X$

$\Rightarrow T(x) = 0$ for all $x \in X \Rightarrow T = 0$.

Theorem(7.3.16)

Let X be a Hilbert space over F , and let $T \in B(X)$ .T is self-adjoint iff $\langle T(x),x\rangle$ is real for all $x \in X$.

Proof :

Suppose that T is self-adjoint

Let $x \in X \Rightarrow \langle T(x),x\rangle = \langle x,T^*(x)\rangle = \langle x,T(x)\rangle = \overline{\langle T(x),x\rangle}$

Thus $\langle T(x),x\rangle$ is equal to its own conjugate and is therefore real

Conversely : suppose that $\langle T(x),x\rangle$ is real for all $x \in X$

$\Rightarrow \langle T(x),x\rangle = \overline{\langle T(x),x\rangle} = \overline{\langle x,T^*(x)\rangle} = \langle T^*(x),x\rangle$

From this, we get $\langle T(x),x\rangle - \langle T^*(x),x\rangle = 0$ for all $x \in X$

$\Rightarrow \langle T(x) - T^*(x),x\rangle = 0$ for all $x \in X$

$\Rightarrow \langle (T - T^*)(x),x\rangle = 0$ for all $x \in X \Rightarrow T - T^* = 0 \Rightarrow T^* = T \Rightarrow T$ is self-adjoint.

Definition(7.3.17)

Let X be a Hilbert space over F . We define a relation $\leq$ on $S(X)$ as follows :

If $S,T \in S(X)$, then we write $S \leq T$ if $\langle S(x),x\rangle \leq \langle T(x),x\rangle$ for all $x \in X$

In the following theorem we shall prove that the relation $\leq$ defined on the set of all self-adjoint operators is a partial order relation.

Theorem(7.3.18)

Let X be a Hilbert space over F. Then $S(X)$ is a partially ordered.

Proof :

Let $S,T \in S(X)$, if $S \le T$ then $\langle S(x),x\rangle \le \langle T(x),x\rangle$ for all $x \in X$

(1) reflexive : let $T \in S(X)$

Since $\langle T(x),x\rangle = \langle T(x),x\rangle$ for all $x \in X$ $\Rightarrow$ $\langle T(x),x\rangle \le \langle T(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $T \le T$ therefore the relation $\le$ on $S(X)$ is reflexive.

(2) transitive : let $R,S,T \in S(X)$ such that $R \le S \wedge S \le T$

$\Rightarrow$ $\langle R(x),x\rangle \le \langle S(x),x\rangle$ for all $x \in X$ and $\langle S(x),x\rangle \le \langle T(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $\langle R(x),x\rangle \le \langle T(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $R \le T$ therefore the relation $\le$ on $S(X)$ is transitive.

(3) Anti-symmetric : let $S,T \in S(X)$ such that $S \le T \wedge T \le S$

$\Rightarrow$ $\langle S(x),x\rangle \le \langle T(x),x\rangle$ for all $x \in X$ and $\langle T(x),x\rangle \le \langle S(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $\langle S(x),x\rangle = \langle T(x),x\rangle$ for all $x \in X$ $\Rightarrow$ $\langle S(x - T(x),x\rangle = 0$ for all $x \in X$

$\Rightarrow$ $\langle (S-T)(x),x\rangle = 0$ for all $x \in X$

$\Rightarrow$ $S - T = 0$ $\Rightarrow$ $S = T$ therefore the relation $\le$ on $S(X)$ is Anti-symmetric.

Hence $\le$ is a partial order relation on $S(X)$.

Remark

Let X be a Hilbert space over F, and let $R,S,T \in S(X)$, $\alpha \ge 0$.

(1) If $S \le T$, then $S + R \le T + R$ (2) If $S \le T$, then $\alpha S \le \alpha T$

Proof :

(1) Since $S \le T$ $\Rightarrow$ $\langle S(x),x\rangle \le \langle T(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $\langle S(x),x\rangle + \langle R(x),x\rangle \le \langle T(x),x\rangle + \langle R(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $\langle (S+R)(x),x\rangle \le \langle (T+R)(x),x\rangle$ for all $x \in X$ $\Rightarrow$ $S + R \le T + R$.

(2) Since $S \le T$ $\Rightarrow$ $\langle S(x),x\rangle \le \langle T(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $\alpha\langle S(x),x\rangle \le \alpha\langle T(x),x\rangle$ for all $x \in X$

$\Rightarrow$ $\langle (\alpha S)(x),x\rangle \le \langle (\alpha T)(x),x\rangle$ for all $x \in X$ $\Rightarrow$ $\alpha S \le \alpha T$.

Definition(7.3.19)

Let X be a Hilbert space over F, and let $T \in S(X)$. We say that T is positive if $T \ge 0$, i.e. $\langle T(x),x\rangle \ge 0$ for all $x \in X$.

Example(7.3.20)

(1) Identity and zero operators are both positive operators.

(2) Let X be a Hilbert space over F, and let $T \in B(X)$. Show that $T \circ T^*, T^* \circ T$ are positive.

Ans :

(1) $\langle I(x),x\rangle=\langle x,x\rangle\ge 0$ for all $x\in X$ and $\langle 0(x),x\rangle=\langle 0,x\rangle=0$ for all $x\in X$.

(2) $(T\circ T^*)^*=(T^*)^*\circ T^*=T\circ T^*\;\Rightarrow\; T\circ T^*\in S(X)$

$\langle (T\circ T^*)(x),x\rangle=\langle T(T^*(x),x\rangle=\langle T^*(x),T^*(x)\rangle=\|T^*(x)\|^2\ge 0\Rightarrow\; T\circ T^*$ is positive

Also $(T^*\circ T)^*=T^*\circ(T^*)^*=T^*\circ T\;\Rightarrow\; T^*\circ T\in S(X)$

$\langle (T^*\circ T)(x),x\rangle=\langle T^*(T(x),x\rangle=\langle T(x),T^{**}(x)\rangle=\langle T(x),T(x)\rangle=\|T(x)\|^2\ge 0\;\Rightarrow\; T^*\circ T$ is positive.

Theorem(7.3.21)

Let X be a Hilbert space over F , and let $T\in S(X)$. If T is positive, then $I+T$ is s non singular.

Proof :

In order to show that $I+T$ is s non singular is one-one and onto function from X onto itself.

(1) $I+T$ is one –one : To prove $\ker(I+T)=\{0\}$

Let $x\in\ker(I+T)\;\Rightarrow\;(I+T)(x)=0$

$I(x)+T(x)=0\;\Rightarrow\; x+T(x)=0\;\Rightarrow\; T(x)=-x$

$\langle T(x),x\rangle=\langle -x,x\rangle=-\langle x,x\rangle=-\|x\|^2$

Since $\langle T(x),x\rangle\ge 0\;\Rightarrow\; -\|x\|^2\ge 0\;\Rightarrow\;\|x\|^2\le 0$, but $\|x\|^2\ge 0\;\Rightarrow\;\|x\|^2=0\;\Rightarrow\;\|x\|=0$

$\Rightarrow\; x=0\;\Rightarrow\; I+T$ is one –one.

(2) we shall show that $I+T$ is onto. Let M be the range of $I+T$. Then $I+T$ will be onto if

we show that $M=X$.

First we shall show that M is closed. Let $x\in X$, we have

$\|(I+T)(x)\|^2=\|x+T(x)\|^2=\langle x+T(x),x+T(x)\rangle=\langle x,x\rangle+\langle x,T(x)\rangle+\langle T(x),x\rangle+\langle T(x),T(x)\rangle$

$\|(I+T)(x)\|^2=\|x\|^2+\|T(x)\|^2+\overline{\langle T(x),x\rangle}+\langle T(x),x\rangle$

Since T is positive, then T is self-adjoint $\Rightarrow\;\langle T(x),x\rangle$ is real for all $x\in X$

$\Rightarrow\;\overline{\langle T(x),x\rangle}+\langle T(x),x\rangle$ for all $x\in X\;\Rightarrow\;\|(I+T)(x)\|^2=\|x\|^2+\|T(x)\|^2+2\langle T(x),x\rangle$

Since T is positive, then $\langle T(x),x\rangle\ge 0\;\Rightarrow\;\|(I+T)(x)\|^2\ge\|x\|^2$

Thus $\|x\|\le\|(I+T)(x)\|$ for all $x\in X$

Now let $\{(I+T)(x_n)\}$ be a Cauchy sequence in M . For any two positive integers n,m , we have $\|x_n-x_m\|\le\|(I+T)(x_n-x_m)\|=\|(I+T)(x_n)-(I+T)(x_m)\|$

Since $\{(I+T)(x_n)\}$ be a Cauchy sequence in M , then $\|(I+T)(x_n)-(I+T)(x_m)\|\to 0$

$\|x_n-x_m\|\to 0$. This mean that $\{x_n\}$ is a Cauchy sequence in X . But X is complete .

Therefore the Cauchy sequence $\{x_n\}$ in X is converges to $x\in X$. Now

Since T is continuous $\Rightarrow$ $I+T$ is continuous
Since $x_n \to x$ and $I+T$ is continuous, then $(I+T)(x_n) \to (I+T)(x)$
Thus the Cauchy sequence $\{(I+T)(x_n)\}$ in M converges to $(I+T)(x)$ in M .Therefore M is complete . But every complete subspace of a complete space is closed. Hence M is closed.
Now we show that $M = X$. Suppose $M \neq X$. Then M is a proper closed subspace of X . Therefore there exists a non zero $x_0 \in X$ such that $x_0 \perp M$.
Since

$(I+T)(x_0) \in M \Rightarrow \langle (I+T)(x_0), x_0 \rangle = 0 \Rightarrow \langle x_0 + T(x_0), x_0 \rangle = 0 \Rightarrow \langle x_0, x_0 \rangle + \langle T(x_0), x_0 \rangle = 0$

$\Rightarrow \|x_0\|^2 + \langle T(x_0), x_0 \rangle = 0 \Rightarrow -\|x_0\|^2 = \langle T(x_0), x_0 \rangle$

Since T is positive $\Rightarrow \langle T(x_0), x_0 \rangle \geq 0 \Rightarrow -\|x_0\|^2 \geq 0 \Rightarrow \|x_0\|^2 \leq 0$

Since $\|x_0\|^2 \geq 0 \Rightarrow \|x_0\|^2 = 0 \Rightarrow x_0 = 0$

But this contradicts the fact that $x_0 \neq 0$.Hence we must have $M = X$ and so $I+T$ is onto.

Corollary(7.3.22)

Let X be a Hilbert space over F , and let $T \in S(B)$ .then the operators $I + T \circ T^*$ and $I + T^* \circ T$ are non singular.

Proof :

Since $T \circ T^*, T^* \circ T$ are positive. **(**see example 8.50**)**, then by theorem (8.51), we have $I + T \circ T^*$ and $I + T^* \circ T$ are non singular.

Normal and Unitary operators

Definition(7.3.23)

Let X be a Hilbert space over F ,and let $T \in B(X)$. T is said to be Normal if $T \circ T^* = T \circ T^*$

Example(7.3.24)

Every self-adjoint operator is normal, but the converse is not true

Ans :

Let X be a Hilbert space over F ,and let $T \in S(X)$ .i.e.T is self-adjoint
$\Rightarrow T^* = T \Rightarrow T \circ T^* = T \circ T^* \Rightarrow T$ is normal
The converse, for example
Let X be a Hilbert space over F , if $\mathrm{I}: X \to X$ is the identity operator, then $T = 2i\mathrm{I}$ is normal
Because $T^* = -2i\mathrm{I}$ and $T \circ T^* = T \circ T^* = 4\mathrm{I}$ but $T^* \neq T$ as well as $T^* \neq T^{-1} = -\frac{1}{2} i\mathrm{I}$.

Remark

Let X be a Hilbert space over F , we denotes the set of all normal in $B(X)$ by $N(X)$**.**
From above example we have $S(X) \subset N(X)$, but not $S(X) \neq N(X)$ in general.

Theorem(7.3.25)

Let X be a Hilbert space over F .
(1) $N(X)$ is closed subset of $B(X)$
(2) If $T \in N(X)$ and $\alpha \in F$, then $\alpha T \in N(X)$, i.e. $S(X)$ is a closed under scalar multiplication.

Proof :

(1) $T \in \overline{N(X)}$ $\Rightarrow$ there exist a sequence $\{T_n\}$ in $N(X)$ such that $T_n \to T$. We have

$$\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \to 0 \quad \Rightarrow \quad \|T_n^* - T^*\| \to 0 \quad \Rightarrow \quad T_n^* \to T^*$$

Now

$$\|T \circ T^* - T^* \circ T\| = \|(T \circ T^* - T_n \circ T_n^*) + (T_n \circ T_n^* - T^* \circ T)\| \le \|T^* \circ T - T_n \circ T_n^*\| + \|T_n \circ T_n^* - T^* \circ T\|$$
$$= \|T \circ T^* - T_n \circ T_n^*\| + \|(T_n \circ T_n^* - T^* \circ T) + (T_n^* \circ T_n - T^* \circ T)\| \|T \circ T^* - T_n \circ T_n^*\| + \|T_n \circ T_n^* - T_n^* \circ T_n\| + \|T_n^* \circ T_n - T^* \circ T\|$$
$$= \|T \circ T^* - T_n \circ T_n^*\| + \|T_n^* \circ T_n - T^* \circ T\|$$

Since $T_n \to T, T_n^* \to T^*$, then $\|T \circ T^* - T^* \circ T\| \to 0 \Rightarrow T \circ T^* = T^* \circ T \quad \Rightarrow \quad T$ is a normal
$\Rightarrow \quad T \in N(X) \Rightarrow \quad \overline{N(X)} = N(X) \Rightarrow \quad N(X)$ is closed.
(2) $(\alpha T) \circ (\alpha T)^* = (\alpha T) \circ (\overline{\alpha} T^*) = \alpha \overline{\alpha} (T \circ T^*) = \overline{\alpha} \alpha (T \circ T^*) = (\overline{\alpha} T^*) \circ (\alpha T) = (\alpha T)^* \circ (\alpha T)$
$\Rightarrow \quad \alpha T$ Is normal $\Rightarrow \quad \alpha T \in N(X)$

Theorem(7.3.26)

Let X be a Hilbert space over F ,and let $S, T \in N(X)$ such that $S \circ T^* = T^* \circ S$ or $T \circ S^* = S^* \circ T$. Then $S + T, \quad S \circ T \in N(x)$

Proof :

Since $S, T \in N(X) \quad \Rightarrow \quad S \circ S^* = S^* \circ S$ and $T \circ T^* = T^* \circ T$

$$(S+T)\circ(S+T)^* = (S+T)\circ(S^*+T^*) = S\circ S^* + S\circ T^* + T\circ S^* + T\circ T^* = S^*\circ S + T^*\circ S + S^*\circ T + T^*\circ T$$
$$= S^*\circ(S+T) + T^*\circ(S+T) = (S^*+T^*)\circ(S+T) = (S+T)^*\circ(S+T)$$

$\Rightarrow \quad S + T$ is normal $\Rightarrow \quad S + T \in \mathrm{N}(X)$

$$(S \circ T) \circ (S \circ T)^* = (S \circ T) \circ (T^* \circ S^*) = S \circ (T \circ T^*) \circ S^* = S \circ (T^* \circ T) \circ S^* = (S \circ T^*) \circ (T \circ s^*)$$
$$= (T^* \circ S) \circ (S^* \circ T) = T^* \circ (S \circ S^*) \circ T = T^* \circ (S^* \circ S) \circ T = (T^* \circ S^*) \circ (S \circ T) = (S \circ T)^* \circ (S \circ T)$$

$\Rightarrow \quad S \circ T$ is normal $\Rightarrow \quad S \circ T \in \mathrm{N}(X)$

Definition(7.3.27)

Let X be a Hilbert space over F ,and let $T \in B(X)$. T is said to be Unitary if $T^* = T^{-1}$
(i.e. $T \circ T^* = T^* \circ T = I$)
It is clear to show that
(1) every unitary operator is normal, but the converse is not true

(2) Let X be a Hilbert space over F ,and let $T \in B(X)$. Then T is unitary iff it is bijective.

Theorem(7.3.28)

Let X be a Hilbert space over F ,and let $T \in B(X)$. The following statements are equivalents**.**
(1) $T^* \circ T = 1$ (2) $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in X$ (3) $\|T(x)\| = \|x\|$ for all $x \in X$

Proof :

(1) $\Rightarrow$ (2)

Let $x, y \in X$, we have $\langle T(x), T(y) \rangle = \langle x, T^*(T(y)) \rangle = \langle x, I(y) \rangle = \langle x, y \rangle$

(2) $\Leftarrow$ (3)

Let $x \in X$, by (2), we have $\langle T(x), T(x) \rangle = \langle x, x \rangle \Rightarrow \|T(x)\|^2 = \|x\|^2 \Rightarrow \|T(x)\| = \|x\|$

(3) $\Leftarrow$ (1)

Let $x \in X$, by (3), we have

$\|T(x)\|^2 = \|x\|^2 \Rightarrow \langle T(x), T(x) \rangle = \langle x, x \rangle \Rightarrow \langle (T^* \circ T)(x), x \rangle = \langle x, x \rangle$

$\Rightarrow \langle (T^* \circ T - I)(x), x \rangle = 0 \Rightarrow T^* \circ T - I = 0 \Rightarrow T^* \circ T = 1$

7.4 Projections

Definition(7.4.1)

Let X be a linear space over F . A linear operator $P : X \to X$ is called projection(مسقط) on X if $P^2 = P$, i.e. p is an idempotent (متساوي القوى).

Theorem (7.4.2)

Let M_1 and M_2 be two subspaces of a vector space over F such that $X = M_1 \oplus M_2$ (then every $x \in X$ can be uniquely written as $x = x_1 + x_2$ where , $x_1 \in M_1$ and $x_2 \in M_2$). Define $P : X \to X$ by $P(x) = x_1$, then P is a projection on X .

Proof :

(1) Let $x, y \in X$ and ~, $\in F$

$x = x_1 + x_2$, $x_1 \in M_1$, $x_2 \in M_2$, $y = y_1 + y_2$, $y_1 \in M_1$, $y_2 \in M_2$

$\sim x + \ y = \sim(x_1 + x_2) + \ (y_1 + y_2) = (\sim x_1 + \ y_1) + (\sim x_2 + \ y_2)$

$P(\sim x + \ y) = \sim x_1 + \ y_1 = \sim P(x) + \ P(y) \Rightarrow P$ is linear function

(2) Let $x \in X \Rightarrow x = x_1 + x_2$ where , $x_1 \in M_1$ and $x_2 \in M_2$

$p^2(x) = P(P(x)) = P(x_1) = P(x_1 + 0) = x_1 = P(x) \quad (x_1 \in M_1 , 0 \in M_2) \Rightarrow P^2 = P$

So that P is a projection on X .

Theorem(7.4.3)

A linear operator P on a linear space X is a projection on some subspace iff it is idempotent , i.e. $P^2 = P$.

Proof :

Let $X = M_1 \oplus M_2$ and let P be the projection on M_1 along M_2. To prove $P^2 = P$

Let $x \in X \Rightarrow x = x_1 + x_2, \ x_1 \in M_1, \ x_2 \in M_2 \Rightarrow p(x) = x_1$

$P^2(x) = P(P(x)) = P(x_1) = P(x_1 + 0) = x_1 = P(x)$ for all $x \in X \Rightarrow p^2 = p$

Conversely, let $P^2 = P$. To prove P is projection

Let $M_1 = \{x \in X : P(x) = x\}$, and $M_2 = \{x \in X : P(x) = 0\}$

$\Rightarrow M_1, M_2$ are subspace of X . To prove $X = M_1 \oplus M_2$

Let $x \in X \Rightarrow x = p(x) - [x - p(x)]$

Put $x_1 = P(x), \ x_2 = x - P(x)$

$P(x_1) = P(P(x)) = P^2(x) = P(x) = x_1 \Rightarrow x_1 \in M_1$

$P(x_2) = P(x - P(x)) = P(x) - P(P(x)) = P(x) - P(x) = 0 \Rightarrow x_2 \in M_2$

$x = x_1 + x_2$, where $x_1 \in M_1, x_2 \in M_2 \Rightarrow X = M_1 + M_2$

Let $x \in M_1 \cap M_2 \Rightarrow x \in M_1, x \in M_2$

$P(x) = 0, x = P(x) \Rightarrow x = 0 \Rightarrow M_1 \cap M_2 = \{0\} \Rightarrow X = M_1 \oplus M_2$

Let $x \in X \Rightarrow x = x_1 + x_2, \ x_1 \in M_1, x_2 \in M_2$

$P(x) = P(x_1 + x_2) = P(x_1) + P(x_2) = x_1 + 0 = x_1$

Theorem(7.4.4)

Let P be a projection on a linear space X over F .Then the range of P is the set of all vectors which are fixed under P , i.e. $R_p = \{x \in X : P(x) = x\}$

Proof :

Take $A = \{x \in X : P(x) = x\}$. To prove $R_p = A$

Let $x \in R_p \Rightarrow$ there exists $y \in X$ such that $P(y) = x$

$P(P(y)) = P(x) \Rightarrow P^2(y) = P(x) \Rightarrow P(y) = P(x)$ (because $P^2 = P$)

But $P(y) = x \Rightarrow P(x) = x \Rightarrow x \in A \Rightarrow R_p \subseteq A$

Now let $x \in A \Rightarrow P(x) = x$

Since $x \in X \Rightarrow P(x) \in R_p$

But $x = P(x) \Rightarrow x \in R_p \Rightarrow A \subseteq R_p \Rightarrow R_p = A$.

Theorem(7.4.5)

Let X be a linear space over F , and let $P : X \to X$ be a linear operator. Then P is a projection on X iff $I - P$ is a projection on X

Proof :

Suppose P is a projection on X

First : To prove $I - P$ is linear function

Let $x, y \in X$, ~, $\in F$

$(I-P)(\sim x+\ y)=I(\sim x+\ y)-P(\sim x+\ y)=\sim I(x)+\ I(y)-\sim P(x)-\ P(y)$

$=\sim(I(x)-P(x))+\ (I(y)-P(y))=\sim(I-P)(x)+\ (I-P)(y)$

$\Rightarrow$ $I-P$ is linear operator.

Second : To prove $(I-P)^2=I-P$

$(I-P)^2=(I-P)(I-P)=I-P-P+P^2=I-P \Rightarrow I-P$ is a projection on X

Conversely, let $I-P$ is a projection on X , then $(I-P)^2=I-P$

$\Rightarrow (I-P)(I-P)=I-P \Rightarrow I-P-P+P^2=I-P \Rightarrow P^2=P \Rightarrow P$ is a projection on X .

Remark

From theorems (7.4.3) and (7.4.4), we have

(1)The Projection P on a linear space X , determines a pair of subspaces N,M such that $X=M\oplus N$ where M is the range of P, i.e. $M=\{P(x):x\in X\}$, and N is the kernel of P, i.e. $N=\{x\in X:P(x)=0\}$

(2)The pair of subspace N,M of a linear space X such that $X=M\oplus N$, determines a Projection p on X whose range and kernel space are M and N (p defined by $p(z)=x$, if $z=x+y$ is the unique representation of vector $z\in X$ as a sum of vectors $x\in M, y\in N$

The above remark shows that the study of Projections on a linear space X is equivalent to the study of pair of disjoint subspaces of X generated X .

Recall that a projection P on a linear space X , is a linear operator $P:X\to X$ such that $P^2=P$. In the following definition

Definition(7.4.6)

Let X a normed space , and let $P\in B(X)$. We say that P is a projection on X , if $P^2=P$, i.e. a projection on a normed space X is continuous, linear and idempotent operator on X . Hence a projection on a normed space X is a projection on a linear space X with the additional property that it is continuous.

Theorem(7.4.7)

Let P be a projection on a normed space X and let M and N be its range and null space respectively. Then M and N are closed subspaces of X such that $X=M\oplus N$.

Proof :

Since P is linear function , then N,M are subspaces of X .

Since $P^2=P \Rightarrow X=M\oplus N$

Since P is continuous function, then N is closed,

Since $M=\{x\in X:P(x)=x\} \Rightarrow M=\{x\in X:(I-P)(x)=0\} \Rightarrow M$ is the kernel of $I-P$

Since $I-P$ is continuous function, then M is closed.

Theorem(7.4.8)
Let X be a normed space and suppose that M and N are closed subspaces of X such that $X = M \oplus N$. If $z = x + y$ is the unique representation of a vector in X as a sum of vectors in M and N , then the function P defined by $P(z) = x$ is a projection on X whose range and null spaces are M and N .

Proof :

Since $X = M \oplus N$. Thus P defined by $P(z) = x$

$z \in X$ has a unique representation as $z = x + y$ with $x \in M$ and $y \in N$.

Also the function defined by $P(z) = x$ is an idempotent function whose range and null space respectively in theorem(8.65).

Thus to prove that P is a Projection on a normed space X

If X' denotes the linear space X equipped with the new norm $\| \ \|'$ defined by

$$\|z\|' = \|x\| + \|y\|$$

is normed space . Further

$\|P(z)\| = \|P(x+y)\| = \|x\| \le \|x\| + \|y\| = \|z\|' \Rightarrow \|P(z)\| \le \|z\|' \Rightarrow P$ is bounded and hence continuous from X' into X . It suffices to prove that X' and X have the same topology.

Let T denote the identity function of X' onto X , then

$\|T(z)\| = \|z\| = \|x + y\| \le \|x\| + \|y\| = \|z\|'$

$\Rightarrow$ T is continuous from X' into X . Moreover T is one-one

$\Rightarrow$ T is homeomorphism and so X' and X have the same topology.

Since P is continuous from X' into X $\Rightarrow P$ is continuous from X' into itself $\Rightarrow P$ is a projection.

Definition(7.4.9)
Let X be a Hilbert space over F , and let $P \in B(X)$. We say that P is a Perpendicular projection on X , if $P^2 = P$ and $P^* = P$

Example (7.4.10)
Every zero and identity function are Perpendicular projection

Theorem(7.4.11)
Let X be a Hilbert space over F , and let P is a projection on X , then P is a Perpendicular projection on X iff the range and kernel of P are orthogonal

Proof :

Let M is the range of P ,and N is the kernel of P , i.e.

$$M = \{P(x) : x \in X\} \text{ and } N = \{x \in X : P(x) = 0\}$$

$\Rightarrow$ $X = M \oplus N$

First : suppose P is a Perpendicular projection on $X \Rightarrow P^* = P$

Let $x \in M, y \in N \quad \Rightarrow \quad P(x) = x, \quad P(y) = 0$

$\langle x, y\rangle = \langle P(x), y\rangle = \langle x, P^*(y)\rangle = \langle x, P(y)\rangle = \langle x, 0\rangle = 0 \quad \Rightarrow \quad x \perp y \quad \Rightarrow \quad M \perp N$.

Second : suppose that $M \perp N$

Let $z \in X$, then z can be uniquely written as $z = x + y$ where

$x \in M, y \in N \quad \Rightarrow \quad P(z) = x$

$\langle p(z), z\rangle = \langle x, z\rangle = \langle x, x+y\rangle = \langle x, x\rangle + \langle x+y\rangle = \langle x, x\rangle$ (because $\langle x, y\rangle = 0$)

$\langle P^*(z), z\rangle = \langle z, P(z)\rangle = \langle z, x\rangle = \langle x+y, x\rangle = \langle x, x\rangle + \langle y, x\rangle = \langle x, x\rangle$

$\Rightarrow \quad \langle P(z), z\rangle = \langle P^*(z), z\rangle$ for all $z \in X$

$\Rightarrow \quad \langle P(z), z\rangle = \langle (P - P^*)(z), z\rangle = 0$ for all $z \in X$

$\Rightarrow \quad P - P^* = 0 \quad \Rightarrow \quad P = P^* \quad \Rightarrow \quad P$ is a Perpendicular projection on X .

Remarks

(1) From the above theorem if $M \perp N$, we have $N = M^{\perp}$, and hence $X = M \oplus M^{\perp}$.

(2) If P is a Perpendicular projection on a Hilbert space X over F with range M , then M is closed subspace of X .If N is kernel of p , then N is also closed subspace of X , and N nothing but $M^{\perp}$, i.e. $N = M^{\perp}$. Further if M is closed subspace of X , then $X = M \oplus M^{\perp}$. Therefore there exists a projection P on X with range M . This projection p is defined by $P(x+y) = x$, where $x \in M, y \in M^{\perp}$. Thus we see that in the case of a Hilbert space there exists one-to-one correspondence between projections on X and closed subspace of X .

(3) If P is a projection on a Hilbert space X over F with range M , then the(null space) kernel of P is uniquely determined and is always $M^{\perp}$. Thus will be one and only one projection on X with range M . Therefore instead of saying that P is a projection on X with range M , we shall simply say that P is the projection on M .

Theorem(7.4.12)

Let X be a Hilbert space over F , and let $P \in B(X)$. Then P is a Perpendicular projection on a closed subspace M of X iff $I - P$ is a Perpendicular projection on $M^{\perp}$.

Proof :

Suppose P is a Perpendicular projection on $X \quad \Rightarrow \quad P^2 = P, \quad P^* = P$

$\Rightarrow \quad (1-P)^* = I^* - P^* = I - P$ and

$(I-P)^2 = (I-P)(I-P) = I - P - P + P^2 = I - P - P - P = I - P$

$\Rightarrow \quad I - P$ is a Perpendicular projection on X

Now we shall show that if M is the range of P , then $M^{\perp}$ is the range of $I - P$.

Let N be the range of $I - P$. Then

$x \in N \quad \Rightarrow \quad (I-P)(x) = x \quad \Rightarrow \quad I(x) - P(x) = x \quad \Rightarrow \quad x - P(x) = x$

$\Rightarrow \quad P(x)=0 \quad \Rightarrow x \in \ker(P) \Rightarrow x \in M^{\perp} \Rightarrow \quad N \subset M^{\perp}$

Again

$x \in M^{\perp} \quad \Rightarrow \quad p(x)=0 \quad \Rightarrow \quad x-P(x)=x \quad \Rightarrow \quad (I-P)(x)=x \quad \Rightarrow \quad x \in N \quad \Rightarrow \quad M^{\perp} \subset N$

Hence $N=M^{\perp} \quad \Rightarrow \quad I-P$ is a Perpendicular projection on $M^{\perp}$

Conversely suppose $I-P$ is a Perpendicular projection on $M^{\perp}$.

$\Rightarrow \quad I-(I-P)$ is a Perpendicular projection on $(M^{\perp})^{\perp}$ (by first part)

$\Rightarrow \quad P$ is a Perpendicular projection on $(M^{\perp})^{\perp}$

Since M is closed subspace, then $(M^{\perp})^{\perp}=M \quad \Rightarrow \quad P$ is a Perpendicular projection on M.

Theorem(7.4.13)

Let X be a Hilbert space over F, and let P be a Perpendicular projection on the closed subspace M of X. Then $x \in M \Leftrightarrow P(x)=x \Leftrightarrow \|P(x)\|=\|x\|$

Proof :

First : we shall prove that $x \in M \Leftrightarrow P(x)=x$

Suppose $x \in M$. Then to show that $x \in M$

Let $P(x)=y$. Then we must show that $y=x$. We have

$P(P(x))=P(y) \quad \Rightarrow \quad P^{2}(x)=P(y) \quad \Rightarrow \quad P(x)=P(y) \quad \Rightarrow \quad P(x-y)=0$

$\Rightarrow x-y \in \ker(P) \quad \Rightarrow \quad x-y \in M^{\perp} \quad \Rightarrow \quad z=x-y$ where $z \in M^{\perp} \quad \Rightarrow \quad x=y+z$

Since $y=P(x) \quad \Rightarrow \quad y$ in the range of P, i.e. $y \in M$. Thus we have $x=y+z$ where $y \in M, z \in M^{\perp}$. But $x \in M$. So we can write $x=x+0$ where $x \in M, 0 \in M^{\perp}$.

Since $X=M \oplus M^{\perp}$. Therefore we must have $z=0$, $y=x$.

Conversely suppose $P(x)=x$

Since $p(x) \in M \quad \Rightarrow \quad x \in M$

Second : we shall prove that $P(x)=x \quad \Leftrightarrow \quad \|P(x)\|=\|x\|$

If $P(x)=x$, then obviously $\|P(x)\|=\|x\|$

Conversely suppose that $\|P(x)\|=\|x\|$. Then to show that $P(x)=x$

Since $x=P(x)+(I-P)(x) \quad \Rightarrow \quad \|x\|^{2}=\|p(x)+(I-p)(x)\|^{2} \qquad \cdots(1)$

Now $P(x) \in M$. Also p is a Perpendicular projection on M

$\Rightarrow \quad I-P$ is a Perpendicular projection on $M^{\perp}$. Therefore $(I-P)(x) \in M^{\perp}$

Therefore $P(x)$ and $(I-P)(x)$ are orthogonal vectors . then by Pythagorean theorem, we have $\|P(x)+(I-P)(x)\|^{2}=\|P(x)\|^{2}+\|(I-P)(x)\|^{2} \qquad \cdots(2)$

From (1) and(2), we get $\|x\|^{2}=\|P(x)\|^{2}+\|(I-P)(x)\|^{2}$

Since $\|P(x)\|=\|x\| \quad \Rightarrow \quad \|(I-P)(x)\|^{2}=0$

$\Rightarrow \quad \|(I-P)(x)\|=0 \quad \Rightarrow \quad (I-P)(x)=0 \quad \Rightarrow \quad I(x)-P(x)=0 \quad \Rightarrow \quad x-P(x)=0 \quad \Rightarrow \quad P(x)=x$

Theorem(7.4.14)

If P is a Perpendicular projection on a Hilbert space X. Then
(1) P is a positive , i.e. $P \geq 0$ (2) $0 \leq P \leq I$ (3) $\|P(x)\| \leq \|x\|$ for all $x \in X$ (4) $\|P\| \leq 1$

Proof :

Since P is a Perpendicular projection on $X \Rightarrow P^* = P$, $P^2 = P$

Let M be the range of P

(1) Let $x \in X \Rightarrow \langle P(x), x \rangle = \langle P^2(x), x \rangle = \langle P(P(x)), x \rangle = \langle P(x), P(x) \rangle = \|P(x)\|^2 \geq 0$

$\Rightarrow \langle P(x), x \rangle \geq 0$ for all $x \in X \Rightarrow P$ is positive .

(2) since P is a Perpendicular projection on X, then by part(1), we $I - P \geq 0 \Rightarrow P \leq I$

But $P \geq 0 \Rightarrow 0 \leq P \leq I$

(3) let $x \in X$, since M is the range of $P \Rightarrow M^{\perp}$ is the range of $I - P$

Since $P(x) \in M$, $(I-P)(x) \in M^{\perp} \Rightarrow P(x)$, $(I-P)(x)$ are orthogonal vectors.

So by Pythagorean theorem we have $\|P(x) + (I-P)(x)\|^2 = \|P(x)\|^2 + \|(I-P)(x)\|^2$

Since

$P(x) + (I-P)(x) = 0 \Rightarrow \|x\|^2 = \|P(x)\|^2 + \|(I-P)(x)\|^2 \Rightarrow \|x\|^2 \geq \|P(x)\|^2 \Rightarrow \|P(x)\| \leq \|x\|$

(4) we have $\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}$, but by part (3), $\|P(x)\| \leq \|x\|$ for all $x \in X$

$\Rightarrow \sup\{\|P(x)\| : \|x\| \leq 1\} \leq 1 \Rightarrow \|P\| \leq 1$

Invariance and Reducibility

Definition(7.4.15)

Let M be a subspace of a linear space X over a field F, and let $T \in L(X)$. We say that M is an invariant under T. If for all $x \in M$, then $T(x) \in M$ i.e. $T(M) \subset M$

Example(7.4.16)

Let X be a linear space over a field F, and let $T \in L(X)$. If M is a range of T ,and N is the kernel of T , $\{0\}, M$ and N are invariant under T .

Ans :

(1) since $T(0) = 0 \Rightarrow T(\{0\}) \subset \{0\} \Rightarrow \{0\}$ is an invariant under T .

(2) $M = \{T(x) : x \in X\}$

Let $x \in M \Rightarrow x \in X \Rightarrow T(x) \in M \Rightarrow T$

so that M is an invariant under T .

(3) $N = \ker(T)$. Then $N = \{x \in X : T(x) = 0\}$

Let $x \in N \Rightarrow T(x) = 0$. Since N is a subspace of $X \Rightarrow 0 \in N \Rightarrow T(x) \in N$

so that N is an invariant under T .

Theorem(7.4.17)

Let M be a closed subspace of a Hilbert space X over F, and let $T \in B(X)$. Then M is invariant under T iff $M^{\perp}$ is invariant under T^{*}.

Proof :

Suppose M is invariant under T

Let $y \in M^{\perp}$. To prove that $T^{*}(y) \in M^{\perp}$(i.e. $T^{*}(y) \perp M$)

Let $x \in M$, since M is invariant under $T \Rightarrow T(x) \in M$

Since $y \in M^{\perp} \Rightarrow \langle T(x), y \rangle = 0 \Rightarrow \langle x, T^{*}((y) \rangle = 0$.Thus $T^{*}(y) \perp M$

Conversely suppose that $M^{\perp}$ is invariant under T^{*}.

Since $M^{\perp}$ is closed subspace of X invariant under T^{*}, therefore by first case $(M^{\perp})^{\perp}$ is invariant under $(T^{*})^{*}$.

But $(M^{\perp})^{\perp} = M^{\perp\perp} = M$ and $(T^{*})^{*} = T^{**} = T$. Therefore M is invariant under T .

Theorem(7.4.18)

Let M be a closed subspace of a Hilbert space X over F, and let $T \in B(X)$. If P is the projection on M, then M is invariant under T iff $T \circ P = P \circ T \circ P$.

Proof :

Suppose M is invariant under T . Then to prove $T \circ P = P \circ T \circ P$

Let $x \in X$, then $P(x)$ is in the range of P , i.e. $P(x) \in M$

Since M is invariant under $T \Rightarrow T(P(x)) \in M$

Since P is the projection on $M \Rightarrow P(T(P(x))) = T(P(x)) \Rightarrow (P \circ T \circ P)(x) = (T \circ P)(x)$

We have $T \circ P = P \circ T \circ P$

Conversely : suppose that $T \circ P = P \circ T \circ P$. Then to prove M is invariant under T

Let $x \in M$

Since P is a Projection with rang M and $x \in M$, then $P(x) = x \Rightarrow T(P(x)) = T(x)$

Since $(T \circ P)(x) = (P \circ T \circ P)x) \Rightarrow T(P(x)) = P(T(P(x))) = P(T(x)) \Rightarrow T(x) = P(T(x))$

$\Rightarrow T(x) \in M$ .But P is the projection on M

Since $x \in M \Rightarrow T(x) \in M$. Therefore M is invariant under T .

Definition(7.4.19)

Let M be a closed subspace of a Hilbert space X over F, and let $T \in B(X)$. We say that T is reduced by M if both M and $M^{\perp}$ are invariant under T . If T is reduced by M, then sometimes we also say that M reduces T.

Theorem(7.4.20)

A closed subspace M of a Hilbert space X reduces an operator T iff M is invariant under both T and T^{*}

proof :

Suppose M reduces . Then by the definition of reducibility both M and $M^{\perp}$ are invariant under T .by theorem (8.75), if $M^{\perp}$ is invariant under T , then $(M^{\perp})^{\perp}$,i.e. M is invariant T^* . Thus M is invariant under both T and T^*

Conversely suppose that M is invariant under both T and T^*

Since M is invariant under T^*,therefore by theorem (17), $M^{\perp}$ is invariant under $(T^*)^*$, i.e. T . Thus both M and $M^{\perp}$ are invariant under T . Therefore M reduces T .

Theorem(7.4.22)

Let M be a closed subspace of a Hilbert space X over F , and let $T \in B(X)$. If P is the projection on M , then M is reduces under T iff $T \circ P = P \circ T$.

Proof :

Suppose M is reduces under T . Then to prove $T \circ P = P \circ T$

$\Rightarrow$ M is invariant under both T and T^* . $\Rightarrow T \circ P = P \circ T \circ P$ and $T^* \circ P = P \circ T^* \circ P$

$\Rightarrow T \circ P = P \circ T \circ P$ and $(T^* \circ P)^* = (P \circ T^* \circ P)^*$ $\Rightarrow T \circ P = P \circ T \circ P$ and

$P^* \circ T^{**} = P^* \circ T^{**} \circ P^*$

Since $T^{**} = T$ and since P is a projection, then $P^* = P$ $\Rightarrow T \circ P = P \circ T \circ P$ and

$P \circ T = P \circ T \circ P$

We have $T \circ P = P \circ T$

Conversely : suppose that $T \circ P = P \circ T$. Multiplying both sides by P on the left and then on the right by P we get $T \circ P^2 = P \circ T \circ P$ and $P \circ T \circ P = P^2 \circ T$

Since P is a projection, then $P^2 = P$ $\Rightarrow T \circ P = P \circ T \circ P$ and $P \circ T \circ P = P \circ T$

$\Rightarrow$ $(P \circ T \circ P)^* = (P \circ T)^*$ $\Rightarrow$ $P^* \circ T^* \circ P^* = T^* \circ P^*$ $\Rightarrow$ $P \circ T^* \circ P = T^* \circ P$

$\Rightarrow T \circ P = P \circ T \circ P$ and $T^* \circ P = P \circ T^* \circ P$ $\Rightarrow$ M is invariant under both T and T^*

$\Rightarrow$ M is reduces under T .

Orthogonal Projections

Definition(7.4.23)

Two perpendicular projection P and Q on a Hilbert space X are said to be orthogonal if $P \circ Q = 0$.

Theorem(7.4.24)

If M and N closed subspaces of a Hilbert space X and P and Q are the perpendicular projections on M and N respectively, then P and Q are orthogonal iff $M \perp N$

Proof :

Since P and Q are the perpendicular projections on X , then $P^* = P$ and $Q^* = Q$

Suppose that P and Q are orthogonal, i.e. $P \circ Q = 0$

Let $x \in M$ and $y \in N$

Since M is a range of P , then $P(x) = x$. Also since N is a range of Q , then $Q(y) = y$.

We have

$\langle x, y \rangle = \langle P(x), Q(y) \rangle = \langle x, P^*(Q(y)) \rangle$

Since $P^* = P \Rightarrow \langle x, y \rangle = \langle x, P(Q(y)) \rangle = \langle x, (P \circ Q)(y) \rangle$

Since $P \circ Q = 0 \Rightarrow \langle x, y \rangle = \langle x, 0(y) \rangle = \langle x, 0 \rangle = 0 \Rightarrow M \perp N$.

Conversely : suppose that $M \perp N$

Let $y \in N$, since $M \perp N \Rightarrow y \perp x$ for $x \in M \Rightarrow y \in M^{\perp} \Rightarrow N \subseteq M^{\perp}$

Let $z \in X \Rightarrow Q(z) \in N$, since $N \subseteq M^{\perp} \Rightarrow Q(z) \in M^{\perp}$ which is the null space of P .

Therefore

$P(Q(z)) = 0$ for all $z \in X$, then $P \circ Q = 0$.

Exercises(7)

7.1 If X is a Hilbert space, then X is reflexive. Prove that

7.2 Let X be a Hilbert space over F, and let $T \in B(X)$. Show that

(1)$\|T \circ T^*\| = \|T\|^*$ (2) $T = T_1 + iT_2$ such that $T_1, T_2 \in S(X)$ (3) If $\sim, \ \in F$, then $\sim T + \ T^* \in N(X)$

7.3 Let X be a Hilbert space over F, and let $T \in B(X)$. Show that

(1)$T \in N(X)$ iff $\|T^*(x)\| = \|T(x)\|$ for all $x \in X$

(2) If $T \in N(X)$, then $\|T \circ T\| = \|T\|^2$

(3) T can be uniquely expressed as $T = T_1 + iT_2$ where $T_1, T_2 \in S(X)$

(4) $T \in N(X)$ iff its real and imaginary parts commute.

(5) If $T \in N(X)$ and ª $\in F$, then $(T -$ ª$I) \in N(X)$.

(6) If $T \in N(X)$ and f is a polynomial with coefficients. Then the operator $f(T)$ is normal.

7.4 Show that : An operator T on a Hilbert space X is unitary iff it is an isometric isomorphism of X onto itself.

7.5 Show that : If T is an arbitrary operator on a Hilbert space X, and if $\sim, \ \in F$ such that $|\sim| = |\ |$, then $\sim T + \ T^*$ is normal.

7.6 If X is a finite dimensional Hilbert space , show that every isometric isomorphism of X into itself is unitary.

7.7 Show that the unitary operators on a Hilbert space X form a group.

7.8 Show that an operator T on a Hilbert space X is the unitary iff $T(\{e_n\})$ is complete orthonormal set whenever is.

7.9 If $P_1, P_2, \cdots, P_n$ are the projections on closed subspaces $M_1, M_2, \cdots, M_n$ of a Hilbert space X, then $P = P_1 + P_2 + \cdots + P_n$ is a perpendicular projection iff $P_i \circ P_j = 0$' whenever $i \neq j$. Also then P is a projection on $M = M_1 + M_2 + \cdots + M_n$.

7.10 If P and Q are the perpendicular projections on M and N respectively of a Hilbert space X. Show that $P \circ Q$ is a perpendicular projections iff $P \circ Q = Q \circ P$. In this case . Show that $P \circ Q$ is a perpendicular projections on $M \cap N$.