8. Spectral Theory

8.1 Matrix of Linear Transformation

Recall that a function between linear spaces is often referred to as a transformation. Let $T: X \to Y$ linear transformation where X and Y are finite dimensional linear spaces over a field F such that $\dim X = n$, $\dim Y = m$.

Let $\beta = \{x_1, x_2, \cdots, x_n\}$ be an ordered basis for X so that each vector in X is expressible as linear combination of the elements of β, i.e. for every $x \in X$ has unique representation

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad \alpha_i \in F, \quad i = 1, 2, \cdots, n$$

The vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is called the Coordinates Vector of x. Let $\beta' = \{y_1, y_2, \cdots, y_m\}$ be an ordered basis for Y so that each vector in Y is expressible as linear combination of the elements of β'.

Let us choose nm scalars $a_{ij} \in F$ where $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$

Since $x_1 \in X \Rightarrow T(x_1) \in Y \Rightarrow T(x_1)$ can be expressible as linear combination of m vectors in β'.i.e.

$$T(x_1) = a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m = \sum_{i=1}^{m} a_{i1} y_i$$

Also

$$T(x_2) = a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m = \sum_{i=1}^{m} a_{i2} y_i$$

$$T(x_n) = a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m = \sum_{i=1}^{m} a_{in} y_i$$

We can write the above n equations in symbolic form as under

$$T(x_j) = \sum_{i=1}^{m} a_{ij} y_i \quad , \quad j = 1, 2, \cdots, n$$

The coefficient matrix in the above expression is

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Then the matrix of $T: X \to Y$ with respect to the given basis β and β' is the transpose of the above coefficient matrix which is obtained by changing the rows into columns and columns into rows of the coefficient matrix,

Matrix of T with respect to basis β and β' is

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}=[a_{ij}]$$

Above matrix is $m\times n$ matrix consisting of m rows and n columns. The above matrix is symbolically written as $[T:\ ,\ ']$ or simply $[T]$.

Remark

If $X=Y$, then $T:X\to X$ and $m=n$ so that the matrix of T with respect to basis will be a $n\times n$ matrix and the rule for writing is same as expressed above.

Theorem(8.1.1)

Let $T:X\to Y$ linear transformation where X and Y are finite dimensional linear spaces over a field F such that $\dim X=n$, $\dim Y=m$. Let $=\{x_1,x_2,\cdots,x_n\}$ be an ordered basis for X , and let $'=\{y_1,y_2,\cdots,y_m\}$ be an ordered basis for Y . If $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ is the Coordinates Vector of $x\in X$ with respect to and $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ is the Coordinates Vector of $T(x)\in Y$ with respect to $'$. Then the matrix A of T with respect to the given basis and $'$ is satisfying $\gamma=\alpha A$

Proof :

Since $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ is the Coordinates Vector of $x\in X$ with respect to

$$\Rightarrow\quad x=\sum_{j=1}^{n}\alpha_j x_j \quad\Rightarrow\quad T(x)=T(\sum_{j=1}^{n}\alpha_j x_j)=\sum_{j=1}^{n}\alpha_j T(x_j)$$

Since $T(x_j)=\sum_{i=1}^{m}a_{ij}y_i \quad , \quad j=1,2,\cdots,n \quad\Rightarrow\quad T(x)=\sum_{j=1}^{n}\alpha_j(\sum_{i=1}^{m}a_{ij}y_i)=\sum_{i=1}^{m}(\sum_{j=1}^{n}\alpha_j a_{ij})y_i$

Since $\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ is the Coordinates Vector of $T(x)\in Y$ with respect to $'$

$$\Rightarrow\quad T(x)=\sum_{i=1}^{m}\gamma_i y_i \quad\Rightarrow\quad \sum_{i=1}^{m}\gamma_i y_i=\sum_{i=1}^{m}(\sum_{j=1}^{n}\alpha_j a_{ij})y_i$$

Since every $y\in Y$ has unique representation of linear combination of vectors in $'$. Thus

$$y_i=\sum_{k=1}^{m}\alpha_k a_{ki}$$

$$(\gamma_1,\gamma_2,\cdots,\gamma_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad\Rightarrow\quad \gamma=\alpha A$$

Example(8.1.2)

(1) Let $X=\mathbb{R}^2, Y=\mathbb{R}^3$ and $T:\mathbb{R}^2\to\mathbb{R}^3$ defined by $T(x,y)=(x,x+y,2x-y)$ for all $(x,y)\in\mathbb{R}^2$, then T is linear operator , the matrix A of T with respect to the given basis $=\{(1,0),(0,1)\}$

and $\ ' = \{(1,0,0),(0,1,0),(0,0,1)\}$ is $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$.

(2) Let $X = P_2(\mathbb{R}), Y = \mathbb{R}^2$ and $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by $T(a+bx+cx^2) = (2a, b-c)$ for all $a+bx+cx^2 \in P_2(\mathbb{R})$, then T is linear operator, the matrix A of T with respect to the given basis

$= \{5, 2x, x^2\}$ and $\ ' = \{(-1,0),(0,3)\}$ is $A = \begin{bmatrix} -10 & 0 \\ 0 & \frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}$.

Theorem (8.2.3)

If $\ = \{x_1, x_2, \cdots, x_n\}$ is an ordered basis for a finite dimensional linear space X over F. Then the function $T \to [T]$ which assigns to each operator T its matrix relative to is an isomorphism of algebra $L(X)$ onto the total matrix algebra $M_n(F)$.

Proof :

Define a function $f : B(X) \to M_n(F)$ by $f(T) = [T]$ for all $T \in B(X)$

Let $T_1, T_2 \in B(X)$, then $f(T_1) = [T_1] = [a_{ij}]_{n\times n}$ and $f(T_2) = [T_2] = [b_{ij}]_{n\times n}$

$T_1(x_j) = \sum_{i=1}^{m} a_{ij} y_i, \quad j = 1,2,\cdots,n, \quad T_2(x_j) = \sum_{i=1}^{m} b_{ij} y_i, \quad j = 1,2,\cdots,n$

Let $x = \sum_{j=1}^{n} a_j x_j$

To prove :

(1) f is one-one : Let $T_1, T_2 \in B(X)$ such that $f(T_1) = f(T_1)$

$\Rightarrow \ [a_{ij}]_{n\times n} = [b_{ij}]_{n\times n} \ \Rightarrow \ \sum_{i=1}^{n} a_{ij} x_i = \sum_{i=1}^{n} b_{ij} x_i, \quad j = 1,2,\cdots$

$\Rightarrow \ T_1(x_j) = T_2(x_j), \quad j = 1,2,\cdots,n \ \Rightarrow \ \sum_{j=1}^{n} a_j T_1(x_j) = \sum_{j=1}^{n} a_j T_2(x_j) \ \Rightarrow \ T_1(\sum_{j=1}^{n} a_j x_j) = T_2(\sum_{j=1}^{n} a_j x_j)$

$\Rightarrow \ T_1(x) = T_2(x)$ for all $x \in X \ \Rightarrow \ T_1 = T_2$. Hence f is one-one.

(2) f is onto : let $[c_{ij}]$ be any matrix in $M_n(F)$ the corresponding to this matrix three exists a linear operator $T : X \to X$ such that $T(x_j) = \sum_{i=1}^{n} a_{ij} x_i, \quad j = 1,2,\cdots,n$

Above defines the operators T is extended by linearity to whole of X then the resulting operator has $[c_{ij}]$ as its matrix relative to . Hence f is onto.

From (1),(2), we have f is bijective.

(3) f is preserves addition, i.e. $f(T_1 + T_2) = f(T_1) + f(T_2)$

$$(T_1+T_2)(x_j)=T_1(x_j)+T_2(x_j)=\sum_{i=1}^{n}a_{ij}x_i+\sum_{i=1}^{n}b_{ij}x_i=\sum_{i=1}^{n}(a_{ij}+b_{ij})x_i$$

$$f(T_1+T_2)=[T_1+T_2]=[a_{ij}+b_{ij}]_{n\times n}=[a_{ij}]_{n\times n}+[b_{ij}]_{n\times n}=[T_1]+[T_2]=f(T_1)+f(T_2)$$

(4) f is preserves scalar multiplication , i.e. $f(\alpha T)=\alpha f(T)$

$$(\alpha T))(x_j)=\alpha T(x_j)=\alpha\sum_{i=1}^{n}a_{ij}x_i=\sum_{i=1}^{n}\alpha a_{ij}x_i=\sum_{i=1}^{n}(c_{ij})x_i$$

$$f(\alpha T)=[\alpha T]=[c_{ij}]_{n\times n}=[\alpha a_{ij}]_{n\times n}=\alpha[a_{ij}]_{n\times n}=\alpha[T]=\alpha f(T)$$

(5) f is preserves multiplication, i.e. $f(T_1T_2)=f(T_1)f(T_2)$

$$(T_1T_2)(x_j)=T_1(T_2(x_j))=T_1(\sum_{k=1}^{n}b_{kj}x_k)=\sum_{k=1}^{n}b_{kj}T(x_k)=\sum_{k=1}^{n}b_{kj}(\sum_{i=1}^{n}a_{ik}x_i)=\sum_{i=1}^{n}(\sum_{k=1}^{n}a_{ik}x_i)$$

$$=\sum_{i=1}^{n}(\sum_{k=1}^{n}a_{ik}b_{kj})x_i+\sum_{i=1}^{n}b_{ij}x_i=\sum_{i=1}^{n}(c_{ij})x_i$$

$$f(T_1T_2)=[T_1T_2]=[c_{ij}]_{n\times n}=[\sum_{k=1}^{n}a_{ik}b_{kj}]=[a_{ij}]_{n\times n}[b_{ij}]_{n\times n}=[T_1][T_2]=f(T_1)f(T_2)$$

Hence is $f:B(X)\to M_n(F)$ by $f(T)=[T]$ for all $T\in B(X)$ an isomorphism

Matrices of identity and zero operators

If T be a linear operators on linear space whose matrix relative to basis $=\{x_1,x_2,\cdots,x_n\}$ be $[a_{ij}]$ then is $T(x_j)=\sum_{i=1}^{n}a_{ij}y_i \quad , \quad j=1,2,\cdots,n$

$$I(x_j)=x_j=0x_1+0x_2+\cdots+1\cdot x_j+\cdots+0x_n \quad\Rightarrow\quad I(x_j)=\sum_{i=1}^{n}(\delta_{ij})x_i \text{ where } \delta_{ij}=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases}$$

$\Rightarrow \quad [I]=[\delta_{ij}]$, i.e. unit matrix.

Again $0(x_j)=0=0x_1+0x_2+\cdots+0x_n \quad\Rightarrow\quad 0(x_j)=\sum_{i=1}^{n}(0_{ij})x_i$

$\Rightarrow \quad [0]=[0_{ij}]$, i.e. null matrix.

Matrix of an inverse operator

Theorem(8.1.4)

Let T be linear operator on a linear space X whose matrix relative to basis $=\{x_1,x_2,\cdots,x_n\}$is $[a_{ij}]$. Then T is non singular iff $[a_{ij}]$ is non singular and in this case $[a_{ij}]^{-1}=[T^{-1}]$

Proof :

Since T is non singular iff $T\circ T^{-1}=T^{-1}\circ T=I$

iff $[T][T^{-1}]=[T^{-1}][T]=[I]$

iff $[a_{ij}][T^{-1}]=[T^{-1}][a_{ij}]=\delta_{ij}$ unit matrix I_n

iff $[a_{ij}]$ is non singular and $[a_{ij}]^{-1}=[T^{-1}]$

8.2 Eigenvalues and Eigenvectors.

Definition(8.2.1)

Let X be a linear space over F, and let $T \in L(X)$.

(1) A scalar ª $\in F$ is called an eigenvalue of T, if there exists a non zero vector $x \in X$ such that $T(x) =$ ª x .

(2)A non zero vector $x \in X$ is called an eigenvector of T, if there exists ª $\in F$ such that $T(x) =$ ª x .

Form(1),(2) , we say that x is an eigenvector of T associated with eigenvalue ª .

Eigenvalues are some times also called characteristic values, proper values, or spectral values. Similarly eigenvectors are called characteristic vectors , proper vectors, or spectral vectors.

The set of all eigenvalues of T is called the spectrum of T and we shall denote it by ±(T).

Remark

If the linear space X has no non zero vectors at all, then T certainly has no eigenvectors. In this case the whole theory collapses into triviality. Therefore throughout the present lector we shall assume that $X \neq \{0\}$.

Examples(8.2.2)

(1) Let $X = \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (-y, x)$ for all $(x, y) \in \mathbb{R}^2$, then T is linear operator has no eigenvalue.

(2) Let $X = \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 2y, 3x + 2y)$ for all $(x, y) \in \mathbb{R}^2$, then T is linear operator have eigenvalues ª $= -1$, ª $= 4$.

(3) Let $X = \ell^2$ and $T : \ell^2 \to \ell^2$ defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$ for all $(x_1, x_2, x_3, \cdots) \in \ell^2$, then T is linear operator has no eigenvalue.

Theorem(8.2.3)

Let X be a linear space over F, and let $T \in L(X)$. If x is an eigenvector of T corresponding to the eigenvalue ª and ~ is any non zero scalar, then ~x is also an eigenvector of T corresponding to the same eigenvalue ª .

Proof:

Since x is an eigenvector of T corresponding to the eigenvalue ª , then $x \neq 0$ and $T(x) =$ ª x

Since ~ $\neq 0$ and $x \neq 0 \Rightarrow$ ~$x \neq 0 \Rightarrow T($~$x) =$ ~$T(x) =$ ~(ª $x) = ($~ª$)x = ($ª~$)x =$ ª $($~$x)$

Therefore ~x is an eigenvector of T corresponding to the eigenvalue ª .

Remark

Corresponding to an eigenvalue ª there may correspond more than one eigenvectors.

Theorem(8.2.4)

Let X be a linear space over F, and let $T \in L(X)$. If x is an eigenvector of T, then x cannot correspond to more than one eigenvalues of T .

Proof :

Let x be an eigenvector of T corresponding to two distinct eigenvalues λ_1 and λ_2 of T
$T(x)=\lambda_1 x$ and also $T(x)=\lambda_2 x$. Therefore we have $\lambda_1 x=\lambda_2 x \Rightarrow (\lambda_1-\lambda_2)x=0$
Since $x\neq 0 \Rightarrow \lambda_1-\lambda_2=0 \Rightarrow \lambda_1=\lambda_2$
and α is any non zero scalar, then αx is also an eigenvector of T corresponding to the same eigenvalue λ .

Definition (8.2.5)

Let X be a linear space over F , $T\in L(X)$ and let λ be an eigenvalue of T . The set consisting of all eigenvectors of T which correspond to eigenvalue λ together with the vector 0 is called eigenspace of T corresponding to the eigenvalue λ and is denoted by M_λ .

(1) Since by definition an eigenvector is a non zero vector, therefore the set M_λ necessarily contains some non zero vectors.

(2) Since by definition of M_λ a non zero vector x is in M_λ iff $T(x)=\lambda x$. Also it is given that the vector 0 is in M_λ . the vector 0 definitely satisfies the equation $T(x)=\lambda x$. Therefore

$$M_\lambda=\{x\in X : T(x)=\lambda x\}=\{x\in X : (T-\lambda I)(x)=0\}$$

Thus M_λ is null space(or kernel of) of linear operator $T-\lambda I$ on X .
Hence M_λ is a subspace of X .

(3) Let $x\in X$, since M_λ is a subspace of X and $\lambda\in F \Rightarrow \lambda x\in M_\lambda$
Since $x\in M_\lambda \Rightarrow T(x)=\lambda x \Rightarrow T(x)\in M_\lambda \Rightarrow M_\lambda$ is an invariant under T .

From(1),(2) and (3) , we have M_λ is a non zero subspace of X invariant under T .

(4) If X is normed space, and $T\in B(X)$ then M_λ is closed subspace of X
M_λ is called eigenspace of T corresponding to the eigenvalue λ

Characteristic equation of operator

Theorem(8.2.6)

Let $=\{x_1,x_2,\cdots,x_n\}$ be an ordered basis for a finite dimensional linear space X over F , and let T be a linear operator on X whose matrix with respect to be A and let $\lambda\in F$. Then λ is an eigenvalue of T iff $|A-\lambda I|=0$

Proof :

Suppose that λ is an eigenvalue of T , then there exist non zero vector $x\in X$ such that
$T(x)=\lambda x \Rightarrow T(x)=\lambda I(x) \Rightarrow T(x)-\lambda I(x)=0 \Rightarrow (T-\lambda I)(x)=0 \Rightarrow x\in \ker(T-\lambda I)$
Since $x\neq 0 \Rightarrow \ker(T-\lambda I)\neq\{0\} \Rightarrow T-\lambda I$ is not one–one $\Rightarrow T-\lambda I$ is not bijective.
i.e. $T-\lambda I$ is singular (not invertible)
Since A is the matrix of T with respect to the given basis , then $A-\lambda I$ is the matrix of $T-\lambda I$ with respect to the given basis , $\Rightarrow A-\lambda I$ is singular (not invertible) $\Rightarrow |A-\lambda I|=0$

Conversely : Suppose that $|A-\lambda I|=0$

$\Rightarrow$ $A-\lambda I$ is singular (not invertible) $\Rightarrow$ $T-\lambda I$ is singular (not invertible)

$\Rightarrow$ $\ker(T-\lambda I)\neq\{0\}$, there exists non zero $x\in X$ such that $(T-\lambda I)(x)=0$

$\Rightarrow$ $T(x)-\lambda I(x)=0$ $\Rightarrow$ $T(x)-\lambda x=0$ $\Rightarrow$ $T(x)=\lambda x$ $\Rightarrow$ $T(x)=\lambda x$

Remark

The equation $|A-\lambda I|=0$ is called **characteristic equation** of T where A is the matrix of T with respect to . Since $=\{x_1,x_2,\cdots,x_n\}$ and $A=[a_{ij}]_{n\times n}$, then $|A-\lambda I|=0$ is an equation of nth degree in λ .

Theorem(8.2.7)

A non zero eigenvectors $x_1,x_2,\cdots,x_n$ corresponding to eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ of linear operator T on a linear space X over F are linearly independent.

Proof :

We shall prove linear independent by induction method

If $n=1$

Let $\alpha_1 x_1=0$, since $x_1\neq 0$ $\Rightarrow$ $\alpha_1=o$. Thus the theorem is true for $n=1$

Suppose the theorem is true for $n=m$. i.e. $x_1,x_2,\cdots,x_m$ are linearly independent

We shall prove that $x_1,x_2,\cdots,x_m,x_{m+1}$ are linearly independent

Consider the relation $\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_mx_m+\alpha_{m+1}x_{m+1}=0$ (1)

$\Rightarrow$ $T(\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_mx_m+\alpha_{m+1}x_{m+1})=T(0)=0$

$\Rightarrow$ $\alpha_1T(x_1)+\alpha_2T(x_2)+\cdots+\alpha_mT(x_m)+\alpha_{m+1}T(x_{m+1})=0$

Since $T(x_i)=\lambda_ix_i$ for all $i=1,2,\cdots,n$ $\Rightarrow$ $\alpha_1\lambda_1x_1+\alpha_2\lambda_2x_2+\cdots+\alpha_m\lambda_mx_m+\alpha_{m+1}\lambda_{m+1}x_{m+1}=0$ (2)

Multiplying (1)by λ_{m+1} and substracting from (2)we get

$\alpha_1(\lambda_1-\lambda_{m+1})x_1+\alpha_2(\lambda_2-\lambda_{m+1})x_2+\cdots+\alpha_m(\lambda_m-\lambda_{m+1})x_m=0$

Since $x_1,x_2,\cdots,x_m$ are linearly independent and λ_i are all distinct and as such it follows from above that $\alpha_1=\alpha_2=\cdots=\alpha_m=0$. Putting in (1) we get $\alpha_{m+1}x_{m+1}=0$

Since $x_{m+1}\neq 0$ $\Rightarrow$ $\lambda_{m+1}=0$. Hence $x_1,x_2,\cdots,x_m,x_{m+1}$ are linearly independent

Thus the theorem is true for all n .

Corollary(8.2.8)

If T is a linear operator on an n dimensional linear space X over F , then T can not have more than n distinct eigenvalues

Proof :

Suppose that T has more than n distinct eigenvalues, then these will form a linearly independent subset of X which will contain more than more n vectors . But this is not possible as a n dimensional linear space can not have a linearly independent set containing more than n elements . Hence can not have more than n distinct eigenvalues.

Theorem(8.2.9)

If T be a self –adjoint operator on an n dimensional Hilbert space X over F, then the eigenvalues of T are real and the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

Proof :

(1) Let λ be the eigenvalue of T so that there exists non zero vector $x \in X$ such that $T(x) = \lambda x$

Since T is self-adjoint , then $\langle T(x), x\rangle$ is real

Now $\langle T(x), x\rangle = \langle \lambda x, x\rangle = \lambda \langle x, x\rangle = \lambda \|x\|^2 \Rightarrow \lambda = \dfrac{\langle T(x), x\rangle}{\|x\|^2}$

Since $\|x\|^2 \geq 0$ and $\langle T(x), x\rangle$ is real it follows that λ is real.

(2) Let λ_1, λ_2 be two distinct eigenvalues of T and x_1, x_2 be the corresponding eigenvectors so that $T(x_1) = \lambda_1 x_1$, $T(x_2) = \lambda_2 x_2$ where λ_1, λ_2 are real . To prove that $x_1 \perp x_2$

Since T is self-adjoint $\Rightarrow$ $T^* = T$, also since λ_1, λ_2 are real$\Rightarrow$ $\overline{\lambda_1} = \lambda_1$, $\overline{\lambda_2} = \lambda_2$

$\lambda_1 \langle x_1, x_2\rangle = \langle \lambda_1 x_1, x_2\rangle = \langle T(x_1), x_2\rangle = \langle x_1, T^*(x_2)\rangle = \langle x_1, T(x_2)\rangle = \langle x_1, \lambda_2 x_2\rangle = \overline{\lambda_2} \langle x_1, x_2\rangle = \lambda_2 \langle x_1, x_2\rangle$

$\Rightarrow \lambda_1 \langle x_1, x_2\rangle = \lambda_2 \langle x_1, x_2\rangle \Rightarrow (\lambda_1 - \lambda_2)\langle x_1, x_2\rangle = 0$

Since λ_1, λ_2 are distinct $\Rightarrow$ $\lambda_1 - \lambda_2 \neq 0$, therefore $\langle x_1, x_2\rangle = 0 \Rightarrow x_1 \perp x_2$, i.e. x_1, x_2 are orthogonal.

Remark

If T is non negative or positive, then the eigenvalues of T are non negative or positive respectively.

Theorem(8.2.10)

If T be an unitary operator on an n dimensional Hilbert space X over F, then the eigenvalues of T are real unimodular and the corresponding distinct eigenvectors are orthogonal.

Proof :

(1) Let λ be a eigenvalue of T , so that there exists non zero $x \in X$ such that $T(x) = \lambda x$

Also since T is an unitary operator, then $\langle T(x), T(x)\rangle = \langle x, x\rangle$

$\Rightarrow \langle \lambda x, \lambda x\rangle = \langle x, x\rangle \Rightarrow \lambda\overline{\lambda}\langle x, x\rangle = \langle x, x\rangle \Rightarrow |\lambda|^2 \|x\|^2 = \|x\|^2 \Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$

i.e. eigenvalues are unimodular.

(2) Let λ_1, λ_2 be two distinct eigenvalues of T and x_1, x_2 be the corresponding eigenvectors so that $T(x_1) = \lambda_1 x_1$, $T(x_2) = \lambda_2 x_2$, where λ_1, λ_2 are unimodular.

since T is an unitary operator, then $\langle T(x_1), T(x_2)\rangle = \langle x_1, x_2\rangle$

$\Rightarrow \langle \lambda_1 x_1, \lambda_2 x_2\rangle = \langle x_1, x_1\rangle \Rightarrow \lambda_1 \overline{\lambda_2}\langle x_1, x_2\rangle = \langle x_1, x_2\rangle$. Since $|\lambda_2|^2 = 1 \Rightarrow \lambda_2 \overline{\lambda_2} = 1 \Rightarrow \overline{\lambda_2} = \dfrac{1}{\lambda_2}$

$\Rightarrow \lambda_1 \dfrac{1}{\lambda_2}\langle x_1, x_2\rangle = \langle x_1, x_2\rangle \Rightarrow \lambda_1 \langle x_1, x_2\rangle = \lambda_2 \langle x_1, x_2\rangle \Rightarrow (\lambda_1 - \lambda_2)\langle x_1, x_2\rangle = 0$

Since λ_1, λ_2 are distinct $\Rightarrow$ $\lambda_1 - \lambda_2 \neq 0$, therefore $\langle x_1, x_2\rangle = 0 \Rightarrow x_1 \perp x_2$.

Theorem(8.2.11)

Let T be a normed operator on a finite dimensional Hilbert space X over F

(1) If $\lambda \in \sigma(T)$, then $T-\lambda I$ is normal

(2)Every eigenvector of T is also a eigenvector for T^*.

(3) the eigenspaces of T are pair wise orthogonal

Proof :

(1) Since T is normal $\Rightarrow\ T \circ T^* = T^* \circ T$

$(T-\lambda I)^* = T^* - \bar{\lambda} I^* = T^* - \bar{\lambda} I \quad \Rightarrow \quad (T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - \bar{\lambda} I) = TT^* - \lambda T^* - \bar{\lambda} T + \lambda\bar{\lambda}$

$\Rightarrow \quad (T-\lambda I)^*(T-\lambda I) = (T^*-\bar{\lambda} I)(T-\lambda I) = T^*T - \lambda T^* - \bar{\lambda} T + \lambda\bar{\lambda} \quad \Rightarrow \quad TT^* - \lambda T^* - \bar{\lambda} T + \lambda\bar{\lambda}$

$\Rightarrow \quad (T-\lambda I)(T-\lambda I)^* = (T-\lambda I)^*(T-\lambda I) \quad \Rightarrow \quad T-\lambda I$ is normal

(2) Let x be an eigenvector of T corresponding to eigenvalue $\lambda \quad \Rightarrow \quad T(x) = \lambda x$

$\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^*(T(x))\rangle = \langle x, T(T^*(x))\rangle = \langle T^*(x), T^*(x)\rangle = \|T^*(x)\|^2 \quad \Rightarrow \quad \|T(x)\| = \|T^*(x)\|$

Since $T-\lambda I$ is normal, therefore $x \in X$, we have

$\|(T-\lambda I)(x)\| = \|(T-\lambda I)^*(x)\| \quad \Rightarrow \quad \|T(x) - \lambda x\| = \|(T^* - \bar{\lambda} I)(x)\| = \|T^*(x) - \bar{\lambda} x\|$

Since $T(x) = \lambda x \Rightarrow \ 0 = \|T^*(x) - \bar{\lambda} x\|$. Hence it follows that $T^*(x) = \bar{\lambda} x$, therefore x is eigenvector of T^* and corresponding eigenvalue is $\bar{\lambda}$

(3)Let x_i and x_j belong to M_i and M_j the eigenspaces of T and the corresponding eigenvalues be λ_i and λ_j respectively so that $T(x_i) = \lambda_i x_i$, $T(x_j) = \lambda_j x_j$ and $T^*(x_j) = \overline{\lambda_j} x_j$ as T is normal

$\lambda_i \langle x_i, x_j\rangle = \langle \lambda_i x_i, x_j\rangle = \langle T(x_i), x_j\rangle = \langle x_i, T^*(x_j)\rangle = \langle x_i, \overline{\lambda_j} x_j\rangle = \overline{\overline{\lambda_j}} \langle x_i, x_j\rangle = \lambda_j \langle x_i, x_j\rangle$

$\Rightarrow \quad \lambda_i \langle x_i, x_j\rangle = \lambda_j \langle x_i, x_j\rangle \quad \Rightarrow \quad (\lambda_i - \lambda_j)\langle x_i, x_j\rangle = 0$

Since λ_i, λ_j are distinct $\Rightarrow\ \lambda_i - \lambda_j \neq 0$, therefore $\langle x_i, x_j\rangle = 0 \quad \Rightarrow \quad x_i \perp x_j$, i.e. M_i and M_j are pair wise orthogonal.

Theorem(8.2.12)

If T be a normal operator on an n dimensional Hilbert space X over F, then each eigenspace reduces T.

Proof :

Let x_i belong to M_i the eigenspace of T and the corresponding eigenvalue be λ_i so that $T(x_i) = \lambda_i x_i$

Since T is normal $\Rightarrow\ T^*(x) = \bar{\lambda} x$

Since M_i is a subspace $\Rightarrow\ \bar{\lambda} x \in M_i \Rightarrow\ T^*(x) \in M_i \quad \Rightarrow \quad M_i$ is invariant under T^*, but M_i is invariant under T. Hence M_i is reduces T.

8.3 Spectral Theorem for Normal Operators

Theorem(8.3.1) Spectral theorem for normal operators

Let T be an arbitrary linear operator on finite dimensional Hilbert space X, and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be eigenvalues of T with eigenspaces $M_1, M_2, \cdots, M_n$. Further $P_1, P_2, \cdots, P_n$ are perpendicular Projections on the spaces $M_1, M_2, \cdots, M_n$ respectively. Then the spectral theorem states that the following statements are equivalent.

(1) The subspaces $M_1, M_2, \cdots, M_n$ are pair wise orthogonal and span X

(2) $P_1, P_2, \cdots, P_n$ are pair wise orthogonal and (i) $P_1 + P_2 + \cdots + P_n = I$ (ii) $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n = T$

(3) T is a normal operator.

Proof :

$(1) \Rightarrow (2)$

Since $P_1, P_2, \cdots, P_n$ are perpendicular Projections on the spaces $M_1, M_2, \cdots, M_n$ respectively. Also the subspaces $M_1, M_2, \cdots, M_n$ are pair wise orthogonal and span X, i.e. $X = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Hence any $x \in X$ can by uniquely expressed as $x = x_1 + x_2 + \cdots + x_n$, where $x_i \in M_i$

Since M_i, M_j are subspaces of X and P_i, P_j are perpendicular projections on M_i, M_j respectively then M_i, M_j are orthogonal iff $P_i \circ P_j = 0$ iff $P_j \circ P_i = 0$

Since M_i, M_j are orthogonal, then $P_i \circ P_j = 0$, $i \neq j$

P_i is projection on M_i and $x = x_1 + x_2 + \cdots + x_n$, $x_i \in M_i$

Since $x_j \in M_j$ and $M_j \perp M_i \Rightarrow x_j \in M_i^{\perp}$. But $M_i^{\perp}$ is null space of P_i and hence $P_i(x_j) = 0$, $i \neq j$

Thus $P_i(x_i) = x_i$ for all i and $P_i(x_j) = 0$, $i \neq j \Rightarrow P_i(x) = P_i(x_1 + x_2 + \cdots + x_n) = P_i(x_i) = x_i$

Now $I(x) = x = x_1 + x_2 + \cdots + x_n = P_1(x) + P_2(x) + \cdots + P_n(x) = (P_1 + P_2 + \cdots + P_n)(x)$

Since above is true for x It follows that $P_1 + P_2 + \cdots + P_n = I$.

Now $x_i \in M_i$ the eigenspace of T corresponding to eigenvalue $\lambda_i \Rightarrow T(x_i) = \lambda_i x_i$

$T(x) = T(x_1 + x_2 + \cdots + x_n) = T(x_1) + T(x_2) + \cdots + T(x_n) = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$

$T(x) = \lambda_1 P_1(x) + \lambda_2 P_2(x) + \cdots + \lambda_n P_n(x) = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n)(x)$

Since above holds for all x we have $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$

$(2) \Rightarrow (3)$

Since $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n \Rightarrow T^* = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n)^* = (\lambda_1 P_1)^* + (\lambda_2 P_2)^* + \cdots + (\lambda_n P_n)^*$

Since $(\lambda_i P_i)^* = \overline{\lambda}_i P_i^*$ and $P_i^* = P_i = P_i^2 \Rightarrow T^* = \overline{\lambda}_1 P_1^* + \overline{\lambda}_2 P_2^* + \cdots + \overline{\lambda}_n P_n^* = \overline{\lambda}_1 P_1 + \overline{\lambda}_2 P_2 + \cdots + \overline{\lambda}_n P_n$

$T \circ T^* = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P)(\overline{\lambda}_1 P_1^* + \overline{\lambda}_2 P_2^* + \cdots + \overline{\lambda}_n P_n^*) = \sum_{i=1}^{n} \lambda_i \overline{\lambda}_i P_i^2 + \sum \lambda_i \overline{\lambda}_j P_i P_j$, $i \neq j$

$\Rightarrow T \circ T^* = \sum_{i=1}^{n} |\lambda_i|^2 P_i + 0 = \sum_{i=1}^{n} |\lambda_i|^2 P_i$ in a similar manner we can show that $T^* \circ T = \sum_{i=1}^{n} |\lambda_i|^2 P_i$

Hence $T \circ T^* = T^* \circ T$ and as such T is normal operator .

$(3) \Rightarrow (1)$

Theorem(5.3.2) Uniqueness of spectral resolution of a normal operator

The spectral resolution of a normal operator on a finite dimensional Hilbert space X is unique.

Another form. Let T be a normal operator on finite dimensional Hilbert space X . If $\sum_{i=1}^{n} \lambda_i P_i$ is the spectral form of T , then λ_i are all the distinct eigenvalues of T . If more $1 \le k \le n$ then there exists polynomials P_k with complex coefficients such that $P_k(\lambda_i) = 0$ whenever $i \ne k$ and $P_k(\lambda_k) = 1$. For all such polynomials $P_k(T) = Q_k$, i.e. each Q_k is a polynomial in T .

Proof :

Theorem(5.3.3)

Let T be a normal operator on a finite dimensional on Hilbert space X , then X has on orthonormal basis consisting of eigenvectors of T . Consequently the matrix of T relative to is a diagonal matrix.

5.4 Spectral theorem for Self adjoint Operators

Theorem(5.4.1)

Let T be a self-adjoint operator on finite dimensional Hilbert space X , then there exists n real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ and perpendicular projections $P_1, P_2, \cdots, P_n$,(where $n > 0$, and $n \le$ the dimension of X) such that

(1) λ_i , $i = 1, 2, \cdots, n$ are pair wise distinct

(2) $P_1, P_2, \cdots, P_n$ are pair wise orthogonal and different from zero.

(3) $P_1 + P_2 + \cdots + P_n = I$

(4) $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n = T$

Proof :

Theorem(5.4.2)

Let T be a self-adjoint operator on finite dimensional Hilbert space X . If $\sum_{i=1}^{n} \lambda_i P_i$ is the spectral form of T , then λ_i are all the distinct eigenvalues of T . If moreover $1 \le k \le n$ then there exists polynomials P_k with real coefficients such that $P_k(\lambda_i) = 0$ whenever $i \ne k$ and $P_k(\lambda_k) = 1$. For all such polynomials $P_k(T) = Q_k$, i.e. each Q_k is a polynomial in T .

Theorem(5.4.3)

Let T be a self-adjoint operator on finite dimensional Hilbert space X such that $T = \sum_{i=1}^{n} \lambda_i P_i$.

If S is any linear transformation on X , then S commutes with T iff S commutes with each P_i for $i = 1, 2, \cdots, n$

Proof :

Suppose that S commutes with each P_i , i.e. $S \circ P_i = P_i \circ S$ for all i . To prove that : $S \circ T = T \circ S$

Since $T = \sum_{i=1}^{n} \alpha_i P_i$

$$\Rightarrow \quad S \circ T = S \circ (\sum_{i=1}^{n} \alpha_i P_i) = \sum_{i=1}^{n} \alpha_i (S \circ P_i) = \sum_{i=1}^{n} \alpha_i (P_i \circ S) = \sum_{i=1}^{n} (\alpha_i P_i) \circ S = T \circ S$$

Conversely : Suppose that S commutes with T , i.e. $S \circ T = T \circ S$. To prove that : $S \circ P_i = P_i \circ S$ for all i