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On Soft 2-Inner Product Spaces

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Abstract

In this paper an idea of soft 2-inner product on soft linear spaces has been introduced and some of their properties are investigated. Soft 2-normed spaces, Soft 2-metric spaces, Soft 2-Banach spaces and Soft 2-Hilbert spaces are also studied.

Mathematics Subject Classification: 46AXX

1 Introduction

In 1999, Molodtsov [1] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc. Research works in soft set theory and its applications in various fields have been progressing rapidly since P. K. Maji([2],[3]) introduced several operations on soft sets and applied it to decision making problems. In the line of reduction and addition of parameters of soft sets some works have been done by Chen [4], Pei and Miao [5] , Kong et al. [6] , Zou and Xiao [7]. Aktas and Cagman [8] introduced the notion of soft group and discussed various properties. Jun ([9],[10]) investigated soft BCK/BCI { algebras and its application in ideal theory. F. Feng [11] worked on soft semirings, soft ideals and idealistic soft semirings M. I. Ali [12] and Shabir and Irfan Ali ([12],[13]) studied soft semigroups and soft ideals over a semi groupwhich characterize generalized fuzzy ideals and fuzzy ideals with thresholds of a semigroup.The idea of soft topological spaces was first given by M. Shabir, M. Naz [14] and mappings between soft sets were described by P. Majumdar, S. K. Samanta [15].F. Feng [16] worked on soft sets combined with fuzzy sets and rough sets.Recently in ([17],[18]) we have introduced a notion of soft real sets, soft real numbers,soft complex sets, soft complex numbers and some of their basicproperties have been investigated.

Some applications of soft real sets and soft real numbers have been presented in real life problems. In ([19],[20]), we introduced the concepts of 'soft metric', 'soft linear spaces', 'soft norm' on a 'soft linear spaces' and studied various properties of 'soft metric spaces' and 'soft normed linear spaces' in details.

The concept of 2-metric spaces, linear 2-normed spaces and 2-inner product spaces, introduced by S Gahler in 1963 , paved the way for a number of authors like, A White, Y J Cho, R Freese, C R Diminnie, to do work on possible applications of Metric geometry, Functional Analysis and Topology as a new tool. The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces

as well as an extensive list of the related references can be found in the book [21].

Here we give the basic definitions and the elementary properties of 2-inner product spaces.

In fact, in this paper we have introduced a notion of soft 2-inner product on soft linear space and studied some of its properties. In section 2, some preliminary results are given. In section 3, a notion of 'soft 2-inner product' on a 'soft linear space' is given and various properties of `soft 2-inner product spaces' are studied. We has been shown that every 'soft 2-inner product space' which satisfies (I6) is also a 'soft 2-normed linear space' which satisfy (N5). The definition of soft 2-Hilbert space is given some properties of soft 2-Hilbert spaces are investigated.

2 Preliminaries

Definition 2.1. [1]

Let U be an universe and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F,A) is called a soft set over U, where F is a mapping given by $F:A\to P(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon\in A, F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set(F,A) .

Definition 2.2. [16]

For two soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a soft subset of (G,B) if

(1) $A\subseteq B$ and

(2) For all $e\in A, F(e)\subseteq G(e)$. We write $(F,A)\tilde{\subset}(G,B)$.

(F,A) is said to be a soft superset of (G,B), if (G,B) is a soft subset of (F,A). We denote it by $(F,A)\tilde{\supset}(G,B)$.

Definition 2.3. [16]

Two soft sets (F,A) and (G,B) over a common universe U are said to be equal if (F,A) is a soft subset of (G,B) and (G,B) is a soft subset of (F,A).

Definition 2.4. [3]

The union of two soft sets (F,A) and (G,B) over the common universe U is the soft set (H,C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if \;\; e \in A - B \\ G(e) & if \; e \in B - A \\ F(e) \cup G(e) & if \; e \in A \cap B. \end{cases}$$

We express it as $(F,A) \tilde{\cup} (G,B) = (H,C)$.

The following definition of the intersection of two soft sets is given as that of the bi-intersection in [11].

Definition 2.5. [11]

The intersection of two soft sets $(F,A)$and (G,B) over the common universe U is the soft set (H,C); where $C = A \cap B$ and for all $e \in C; H(e) = F(e) \cap G(e)$. We write $(F,A) \tilde{\cap} (F,A) = (H,C)$.

Let X be an initial universal set and A be the non-empty set of parameters. In the above definitions the set of parameters may vary from soft set to soft set, but in our considerations, throught this paper all soft sets have the same set of parameters A.
The above definitions are also valid for these type of soft sets as a particular case of those definitions.

Definition 2.6. [16]

The complement of a soft set (F,A) is denoted by $(F,A)^c = (F^c, A)$ where $F^c: A \to P(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

Definition 2.7. [3]

A soft set (F,A) over U is said to be an absolute soft set denoted by $\breve{U}$ if for all $\varepsilon \in A, F(\varepsilon) = U$.

Definition 2.8. [3]

A soft set (F,A) over U is said to be a null soft set denoted by Φ if for all $\varepsilon \in A,\; F(\varepsilon) = \Phi$.

Definition 2.9. [14]

The difference (H,A) of two soft sets (F,A) and (G,A) over X, denoted by $(F,A)\backslash(G,A)$, is defined by $H(e) = F(e)\backslash G(e)$ for all $e \in A$.

Proposition 2.10. [14]

Let (F,A) and (G,A) be two soft sets over X. Then

i. $((F,A) \tilde{\cup} (G,A))^c = (F,A)^c \tilde{\cap} (G,A)^c$

ii. $((F,A) \tilde{\cap} (G,A))^c = (F,A)^c \tilde{\cup} (G,A)^c$.

Definition 2.11. [17]

Let X be a non-empty set and A be a non-empty parameter set. Then a function $\varepsilon : A \to X$ is said to be soft element of X. A soft element ε of X is said to belongs to a soft set B of X, which is denoted by $\varepsilon \tilde{\in} B$, if $\varepsilon(e) \in A(e), \forall\, e \in A$. Thus for a soft set A of X with respect to the index set A; we have $B(e) = \{\varepsilon(e), \varepsilon \tilde{\in} B\}, e \in A$.

It is to be noted that every singleton soft set (a soft set (F, A) for which $F(e)$ is a singleton set, $\forall e \in A$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in A$.

Definition 2.12. [17]

Let R be the set of real numbers and $\mathfrak{B}(R)$ the collection of all non-empty bounded subsets of R and A taken as a set of parameters. Then a mapping $F : A \to \mathfrak{B}(R)$ is called a soft real set. It is denoted by (F, A). If specifically (F, A) is a singleton soft set; then after identifying (F, A) with the corresponding soft element, it will be called a soft real number.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\overline{r}$, $\overline{s}$, $\overline{t}$ will denote a particular type of soft real numbers such that ,for all $\overline{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\overline{0}$ is the soft real number where $\overline{0}(\lambda) = 0$, for all $\lambda \in A$.
We now introduce some definitions and prove a theorem on soft real numbers which will be used in this paper

Definition 2.13. [20]

Let V be a vector space over a field K and let A be a parameter set. Let G be a soft set over (V, A). Now G is said to be a soft vector space or soft linear space of V over K if $G(\lambda)$ is a vector subspace of V, for all $\lambda \in A$.

Definition 2.14. [20]

Let F be a soft vector space of V over K. Let $G: A \to \wp(V)$ be a soft set over (V, A). Then G is said to be a soft vector subspace of F if

(i) For each $\lambda \in A, G(\lambda)$ is a vector subspace of V over K and

(ii) $F(\lambda) \supseteq G(\lambda),\ \forall \lambda \in A$

Definition 2.15. [20]

Let G be a soft vector space of V over K. Then a soft element of G is said to be a soft vector of G. In a similar manner a soft element of the soft set (K, A) is said to be a soft scalar, K being the scalar field.

Let X be a vector space over a field $K = R, X$ is also our initial universe set and A be a non-empty set of parameters. Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X, \forall\, \lambda \in A$. We use the notation $\tilde{x}, \tilde{y}, \tilde{z}$ to denote soft vectors of a soft vector space and $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\overline{r}$, $\overline{s}$, $\overline{t}$ will denote a particular type of soft real numbers such that $\overline{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\overline{0}$ is the soft real number such that $\overline{0}(\lambda) = 0$, for all $\lambda \in A$. Note that, in general, $\tilde{r}$ is not related to r.

Definition 2.16. [18]

Let C be the set of complex numbers and $\wp(C)$ be the collection of all non-empty bounded subsets of the set of complex numbers. A be a set of parameters. Then a mapping $F: A \to \wp(C)$is called a soft complex set. It is denoted by (F, A).

If in particular (F, A) is a singleton soft set; then identifying (F, A) with the corresponding soft element, it will be called a soft complex number.
The set of all soft complex numbers is denoted by $\mathbb{C}(A)$ and the set of all soft real numbers is denoted by $\mathbb{R}(A)$.

Definition 2.17.

Let $\check{X}$ be the absolute soft vector space of dimension greater than 1 i.e., $\check{X}(\lambda) = X$, $\forall \lambda \in A$. Then a mapping $\|.,.\|: SE(\check{X}) \times SE(\check{X}) \to \mathbb{R}(A)^*$ is said to be a soft 2-norm on the soft vector space $\check{X}$ if $\|.,.\|$ satisfies the following conditions:
(N1) $\|\tilde{x}, \tilde{y}\| = \overline{0} \Leftrightarrow \tilde{x}$ and $\tilde{y}$ are linearly dependent vectors;
(N2) $\|\tilde{x}, \tilde{y}\| = \|\tilde{y}, \tilde{x}\|$ for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$;
(N3) $\|\tilde{\alpha}.\tilde{x}, \tilde{y}\| = |\tilde{\alpha}|\|\tilde{x}, \tilde{y}\|$ for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ and for every soft scalar $\tilde{\alpha}$;
(N4) $\|\tilde{x} + \tilde{y}, \tilde{z}\| \tilde{\leq} \|\tilde{x}, \tilde{z}\| + \|\tilde{y}, \tilde{z}\|$ for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \check{X}$.
The soft vector space $\check{X}$ with a soft 2-norm $\|.,.\|$ on $\check{X}$ is said to be a soft 2-normed linear space and is denoted by $(\check{X}, \|.,.\|, A)$ or $(\check{X}, \|.,.\|)$. (1), (2), (3) and (4) are said to be soft 2-norm axioms.Someof the basic properties of soft 2-norms are that they are non-negative and $\|\tilde{x}, \tilde{y} + \tilde{\alpha}.\tilde{z}\| = \|\tilde{x}, \tilde{y}\|$for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \check{X}$ and for every soft scalar $\tilde{\alpha}$.

Theorem 2.18.

If a soft 2-norm $\|.,.\|$satisfies the condition
(N5) For $(\xi, \eta) \in X \times X$, and $\lambda \in A$, $\{\|\tilde{x}, \tilde{y}\|(\lambda): \tilde{x}(\lambda) = \xi \textit{ and } \tilde{y}(\lambda) = \eta\}$, is a singleton set.
And if for each $\lambda \in A$, $\|.,.\|_\lambda: X \times X \to R^*$ be a mapping such that for each $(\xi, \eta) \in X \times X$, $\|\xi, \eta\|_\lambda = \|\tilde{x}, \tilde{y}\|(\lambda)$, where $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi \textit{ and } \tilde{y}(\lambda) = \eta$. Then for each $\in A$, $\|.,.\|_\lambda$ is a 2-norm on X.

Definition 2.19.

A Sequence $\{\tilde{x}_n\}$ of soft elements $\tilde{x}$ in a soft 2-normed linear space $(\check{X}, \|.,.\|, A)$ is said to be convergent and converges to a soft element $\tilde{x}$ if $\lim_{n\to\infty} \|\tilde{x}_n - \tilde{x}, \tilde{y}\| = \overline{0}$ for all $\tilde{y} \tilde{\in} \check{X}$.
If $\tilde{x}_n$ converges to $\tilde{x}$, we write $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$. i.e., $\|\tilde{x}_n - \tilde{x}, \tilde{y}\| \to \overline{0}$ if and only if $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$.

Definition 2.20.

A Sequence $\{\tilde{x}_m\}$ in a soft 2-normed linear space $(\check{X}, \|.,.\|, A)$ is said to be a Cauchy sequence if there are $\tilde{y}, \tilde{z} \tilde{\in} \check{X}$ such that $\tilde{y}$ and $\tilde{z}$ are linearly independent,
$\lim_{m,n\to\infty} \|\tilde{x}_m - \tilde{x}_n, \tilde{y}\| = \overline{0}$ and $\lim_{m,n\to\infty} \|\tilde{x}_m - \tilde{x}_n, \tilde{z}\| = \overline{0}$.

Definition 2.21.

Let $(\breve{X}, \|.,.\|, A)$ be a soft 2-normed linear space. Then $\breve{X}$ is said to be complete if every Cauchy sequence in $\breve{X}$ converges to a soft element of $\breve{X}$.
Every complete soft 2-normed linear space is called a soft 2-Banach Space.

Definition 2.22.

A mapping $d: SE(\breve{X}) \times SE(\breve{X}) \times SE(\breve{X}) \to \mathbb{R}(A)^*$is said to be a soft 2-metric on the soft set $\breve{X}$ if d satisfies the following conditions:

(M1) for every pair of distinct soft points $\tilde{x}, \tilde{y}$ there exists a point $\tilde{z} \tilde{\in} \breve{X}$ such that $d(\tilde{x}, \tilde{y}, \tilde{z}) \neq \overline{0}$

(M2) $d(\tilde{x}, \tilde{y}, \tilde{z}) = \overline{0}$ only if at least two of the three points are same.

(M3) $d(\tilde{x}, \tilde{y}, \tilde{z}) = d(\tilde{x}, \tilde{z}, \tilde{y}) = d(\tilde{y}, \tilde{z}, \tilde{x})$ for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \breve{X}$

(M4) $d(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{\leq} d(\tilde{x}, \tilde{y}, \tilde{w}) + d(\tilde{x}, \tilde{w}, \tilde{z}) + d(\tilde{w}, \tilde{y}, \tilde{z}) \tilde{x}, \tilde{y}, \tilde{z}, \tilde{w} \tilde{\in} \breve{X}$.

The soft set $\breve{X}$ with a soft 2-metric d on $\breve{X}$ is said to be a soft 2-metric space and is denoted by $(\breve{X}, d, A)$ or $(\breve{X}, d)$.

Theorem 2.23.

If a soft 2-metric d satisfies the condition:

(M5) For $(\xi, \eta, \zeta) \in X \times X \times X$, and $\lambda \in A, \{d(\tilde{x}, \tilde{y}, \tilde{z})(\lambda): \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta, \tilde{z}(\lambda) = \zeta\}$ is a singleton set, and if for $\lambda \in A, d_\lambda: X \times X \to \mathbb{R}^+$ is defined by $d_\lambda(\xi, \eta, \zeta) = d(\tilde{x}, \tilde{y}, \tilde{z})(\lambda)$, where $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \breve{X}$ such that $\tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta, \tilde{z}(\lambda) = \zeta$. Then d_λ is a metric on$X$.

Proposition 2.24.

Let $(\breve{X}, \|.,.\|, A)$ be a soft 2-normed linear space. Let us define $d: SE(\breve{X}) \times SE(\breve{X}) \times SE(\breve{X}) \to \mathbb{R}(A)^*$ by $d(\tilde{x}, \tilde{y}, \tilde{z}) = \|\tilde{x} - \tilde{z}, \tilde{y} - \tilde{z}\|$, for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \breve{X}$. Then d is a soft 2-metric on $\breve{X}$. i.e., every soft 2-normed space which (N5) is also soft 2-metric space which satisfies (M5).

3 Soft 2-Inner Product and Soft Inner Product Spaces

Definition 3.1.

Let $\breve{X}$ be the absolute soft vector space with dimension greater than one i.e., $\breve{X}(\lambda) = X, \forall \lambda \in A$. Then a mapping $\langle .,.,. \rangle: SE(\breve{X}) \times SE(\breve{X}) \times SE(\breve{X}) \to \mathbb{C}(A)$ is said to be a soft 2-inner product on the soft vector space $\breve{X}$ if $\langle .,.,. \rangle$ satisfies the following conditions:

(I1) $\langle \tilde{x}, \tilde{x}, \tilde{z} \rangle \tilde{\geq} \overline{0}$, for all $\tilde{x}, \tilde{z} \tilde{\in} \breve{X}$and $\langle \tilde{x}, \tilde{x}, \tilde{z} \rangle = \overline{0}$ if and only if $\tilde{x}$ and $\tilde{z}$ are linearly dependent vectors;

(I2) $\langle \tilde{x}, \tilde{x}, \tilde{z} \rangle = \langle \tilde{z}, \tilde{z}, \tilde{x} \rangle$;

(I3) $\langle \tilde{y}, \tilde{x}, \tilde{z} \rangle = \overline{\langle \tilde{x}, \tilde{y}, \tilde{z} \rangle}$ where bar denote the complex conjugate of soft complex numbers;

(I4) $\langle \tilde{\alpha}. \tilde{x}, \tilde{y}, \tilde{z} \rangle = \tilde{\alpha}. \langle \tilde{x}, \tilde{y}, \tilde{z} \rangle$ for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \breve{X}$ and for every soft scalar $\tilde{\alpha}$;

(I5) $\langle \tilde{x} + \tilde{y}, \tilde{w}, \tilde{z} \rangle = \langle \tilde{x}, \tilde{w}, \tilde{z} \rangle + \langle \tilde{y}, \tilde{w}, \tilde{z} \rangle$ for all $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w} \tilde{\in} \breve{X}$.

The soft vector space $\breve{X}$ with a soft 2-inner product $\langle .,.,. \rangle$ on $\breve{X}$ is said to be a soft 2-inner product space and is denoted by $(\breve{X}, \langle .,.,. \rangle, A)$ or $(\breve{X}, \langle .,.,. \rangle)$. (I1), (I2), (I3), (I4) and (I5) are said to be soft 2-inner product axioms.

Example 3.2.

Let $X = l_2$. Then X is an 2-inner product space with respect to the 2-inner product $\langle x, y, z\rangle = \sum_{i=1}^{\infty} \xi_i \overline{\eta_i \zeta_i}$ for $x = \{\xi_i\}, y = \{\eta_i\}, z = \{\zeta_i\}$ of l_2. Let $\tilde{x}, \tilde{y}$ and $\tilde{z}$ be soft elements of the absolute soft vector space $\breve{X}$. Then $\tilde{x}(\lambda) = \{\xi_i^{\lambda}\}$, $\tilde{y}(\lambda) = \{\eta_i^{\lambda}\}$ and $\tilde{z}(\lambda) = \{\zeta_i^{\lambda}\}$are elements of l_2. The mapping $\langle .,.,.\rangle: SE(\breve{X}) \times SE(\breve{X}) \times SE(\breve{X}) \to \mathbb{C}(A)$ defined by $\langle \tilde{x}, \tilde{y}, \tilde{z}\rangle(\lambda) = \sum_{i=1}^{\infty} \xi_i^{\lambda} \overline{\eta_i^{\lambda} \zeta_i^{\lambda}} = \langle \tilde{x}(\lambda), \tilde{y}(\lambda), \tilde{z}(\lambda)\rangle, \forall \lambda \in A$, is a soft 2-inner product on the soft vector space $\breve{X}$.

Remark 3.3.

The soft 2-inner product $\langle .,.,.\rangle$as defined in Example 3.2, satisfies the condition

(I6) For $(\xi, \eta, \zeta) \in X \times X \times X$ and $\lambda \in A, \left\{\langle \tilde{x}, \tilde{y}, \tilde{z}\rangle(\lambda): \tilde{x}, \tilde{y}, \tilde{z} \,\tilde{\in}\, \breve{X}\right.$ such that $\tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta, \tilde{z}(\lambda) = \zeta \left.\right\}$ is a singleton set.

Proposition 3.4.

Let $(\breve{X}, \langle .,.,.\rangle, A)$ be a soft 2-inner product space, $\tilde{x}, \tilde{y}, \tilde{z} \,\tilde{\in}\, \breve{X}$ and $\tilde{\alpha}, \tilde{\beta}$ etc. be soft scalars. Then

(i) $\langle \tilde{\alpha}\tilde{x} + \tilde{\beta}\tilde{y}, \widetilde{w}, \tilde{z}\rangle = \tilde{\alpha}.\langle \tilde{x}, \widetilde{w}, \tilde{z}\rangle + \tilde{\beta}.\langle \tilde{y}, \widetilde{w}, \tilde{z}\rangle$;

(ii) $\langle \tilde{x}, \tilde{\alpha}\tilde{y}, \tilde{z}\rangle = \overline{\tilde{\alpha}}.\langle \tilde{x}, \tilde{y}, \tilde{z}\rangle$;

(iii) $\langle \tilde{x}, \tilde{\alpha}\tilde{y} + \tilde{\beta}\tilde{z}, \widetilde{w}\rangle = \overline{\tilde{\alpha}}.\langle \tilde{x}, \tilde{y}, \widetilde{w}\rangle + \overline{\tilde{\beta}}.\langle \tilde{x}, \tilde{z}, \widetilde{w}\rangle$.

Proof.

(i) $\langle \tilde{\alpha}\tilde{x} + \tilde{\beta}\tilde{y}, \widetilde{w}, \tilde{z}\rangle = \langle \tilde{\alpha}\tilde{x}, \widetilde{w}, \tilde{z}\rangle + \langle \tilde{\beta}\tilde{y}, \widetilde{w}, \tilde{z}\rangle = \tilde{\alpha}\langle \tilde{x}, \widetilde{w}, \tilde{z}\rangle + \tilde{\beta}\langle \tilde{y}, \widetilde{w}, \tilde{z}\rangle$.

(ii) $\langle \tilde{x}, \tilde{\alpha}\tilde{y}, \tilde{z}\rangle = \overline{\langle \tilde{\alpha}\tilde{y}, \tilde{x}, \tilde{z}\rangle} = \overline{\tilde{\alpha}}.\overline{\langle \tilde{y}, \tilde{x}, \tilde{z}\rangle} = \overline{\tilde{\alpha}}.\langle \tilde{x}, \tilde{y}, \tilde{z}\rangle$.

(iii) $\langle \tilde{x}, \tilde{\alpha}\tilde{y} + \tilde{\beta}\tilde{z}, \widetilde{w}\rangle = \overline{\langle \tilde{\alpha}\tilde{y} + \tilde{\beta}\tilde{z}, \tilde{x}, \widetilde{w}\rangle} = \overline{\langle \tilde{\alpha}\tilde{y}, \tilde{x}, \widetilde{w}\rangle} + \overline{\langle \tilde{\beta}\tilde{z}, \tilde{x}, \widetilde{w}\rangle}$

$$= \overline{\tilde{\alpha}}.\overline{\langle \tilde{y}, \tilde{x}, \widetilde{w}\rangle} + \overline{\tilde{\beta}}.\overline{\langle \tilde{z}, \tilde{x}, \widetilde{w}\rangle} = \overline{\tilde{\alpha}}.\langle \tilde{x}, \tilde{y}, \widetilde{w}\rangle + \overline{\tilde{\beta}}.\langle \tilde{x}, \tilde{z}, \widetilde{w}\rangle.$$

Theorem 3.5. (Schwarz inequality)

Let $(\breve{X}, \langle .,.,.\rangle, A)$ be a soft 2-inner product space satisfying (I6). Let $\tilde{x}, \tilde{y}, \tilde{z} \,\tilde{\in}\, \breve{X}$ and $\tilde{\alpha}, \tilde{\beta}$ etc. be soft scalars. Then $|\langle \tilde{x}, \tilde{y}, \tilde{z}\rangle| \,\tilde{\leq}\, \|\tilde{x}, \tilde{z}\|.\|\tilde{y}, \tilde{z}\|$.

Proof.

It is clear.

Theorem 3.6.

Let $(\breve{X}, \langle .,.,.\rangle, A)$ be a soft 2-inner product space satisfying (I5). Let us define $\|.,.\|: SE(\breve{X}) \times SE(\breve{X}) \to \mathbb{R}(A)^*$ by $\|\tilde{x}, \tilde{y}\| = \sqrt{\langle \tilde{x}, \tilde{x}, \tilde{y}\rangle}$ for all $\tilde{x}, \tilde{y} \,\tilde{\in}\, \breve{X}$. Then $\|.,.\|$ is a soft 2-norm on $\breve{X}$ satisfying (N5). i.e.,

Proof.

We have (N1), $\|\tilde{x},\tilde{y}\| = \overline{0} \Leftrightarrow \sqrt{\langle \tilde{x},\tilde{x},\tilde{y}\rangle} = \overline{0} \Leftrightarrow \langle \tilde{x},\tilde{x},\tilde{y}\rangle = \overline{0} \Leftrightarrow \tilde{x}$ and $\tilde{z}$ are linearly dependent vectors.

(N2) $\|\tilde{x},\tilde{y}\| = \sqrt{\langle \tilde{x},\tilde{x},\tilde{y}\rangle} = \sqrt{\langle \tilde{y},\tilde{y},\tilde{x}\rangle} = \|\tilde{y},\tilde{x}\|$.

(N3) $\|\tilde{\alpha}.\tilde{x},\tilde{y}\| = \sqrt{\langle \tilde{\alpha}.\tilde{x},\tilde{\alpha}.\tilde{x},\tilde{y}\rangle} = \sqrt{\tilde{\alpha}.\langle \tilde{x},\tilde{\alpha}.\tilde{x},\tilde{y}\rangle} = \sqrt{\tilde{\alpha}.\overline{\tilde{\alpha}}.\langle \tilde{x},\tilde{x},\tilde{y}\rangle} = |\tilde{\alpha}|.\sqrt{\langle \tilde{x},\tilde{x}|\tilde{y}\rangle} =$

$= |\tilde{\alpha}|.\|\tilde{x},\tilde{y}\|$ for all $\tilde{x},\tilde{y} \,\tilde{\in}\, \check{X}$ and for every soft scalar $\tilde{\alpha}$.

(N4) $\|\tilde{x}+\tilde{y},\tilde{z}\|^2 = \langle \tilde{x}+\tilde{y},\tilde{x}+\tilde{y},\tilde{z}\rangle = \langle \tilde{x},\tilde{x},\tilde{z}\rangle + \langle \tilde{x},\tilde{y},\tilde{z}\rangle + \langle \tilde{y},\tilde{x},\tilde{z}\rangle + \langle \tilde{y},\tilde{y},\tilde{z}\rangle$

$$= \|\tilde{x},\tilde{z}\|^2 + \langle \tilde{x},\tilde{y},\tilde{z}\rangle + \overline{\langle \tilde{y},\tilde{x},\tilde{z}\rangle} + \|\tilde{y},\tilde{z}\|^2$$
$$= \|\tilde{x},\tilde{z}\|^2 + \langle \tilde{x},\tilde{y},\tilde{z}\rangle + \overline{\langle \tilde{x},\tilde{y},\tilde{z}\rangle} + \|\tilde{y},\tilde{z}\|^2$$
$$= \|\tilde{x},\tilde{z}\|^2 + 2Re(\langle \tilde{x},\tilde{y},\tilde{z}\rangle) + \|\tilde{y},\tilde{z}\|^2$$

where Re is a soft real number.

Now, since $\langle .,.,.\rangle$ satisfies (I6), we have

$$|\langle \tilde{x},\tilde{y},\tilde{z}\rangle| \,\tilde{\leq}\, \|\tilde{x},\tilde{z}\|.\|\tilde{y},\tilde{z}\| \,\tilde{\leq}\, \|\tilde{x},\tilde{z}\|^2 + 2\|\tilde{x},\tilde{z}\|.\|\tilde{y},\tilde{z}\|) + \|\tilde{y},\tilde{z}\|^2 = (\|\tilde{x},\tilde{z}\| + \|\tilde{y},\tilde{z}\|)^2.$$

Hence, $\|\tilde{x}+\tilde{y},\tilde{z}\| \,\tilde{\leq}\, \|\tilde{x},\tilde{z}\| + \|\tilde{y},\tilde{z}\|$.

Therefore, $\|.,.\|$ is a soft 2-norm on $\check{X}$.

Remark 3.7.

From the above theorem it follows that every 'soft 2-inner product Space' which satisfies (I6) is also a 'soft 2-normed linear space' which satisfies (N5) with the soft 2-norm defined as above. With the help of this soft norm, we can introduce a 'soft 2-metric' on by the formula $d(\tilde{x},\tilde{y},\tilde{z}) = \|\tilde{x}-\tilde{z},\tilde{y}-\tilde{z}\| = \sqrt{\langle \tilde{x}-\tilde{z},\tilde{x}-\tilde{z},\tilde{y}-\tilde{z}\rangle}$. In a similar way as above, it can be proved that, this soft metric satisfies (M5).

Proposition 3.8.

Let $(\check{X},\langle .,.,.\rangle,A)$ be a soft 2-inner product space satisfying (5). Let $\tilde{x},\ \tilde{y},\tilde{z} \,\tilde{\in}\, \check{X}$ and $\tilde{\alpha},\ \tilde{\beta}$ etc. be soft scalars. Then

(i) $\|\tilde{x}+\tilde{y},\tilde{z}\|^2 + \|\tilde{x}-\tilde{y},\tilde{z}\|^2 = 2\|\tilde{x},\tilde{z}\|^2 + 2\|\tilde{y},\tilde{z}\|^2$*(Parallelogram law);*

(ii) $\langle \tilde{x},\tilde{y},\tilde{z}\rangle = \frac{1}{4}\left\{\|\tilde{x}+\tilde{y},\tilde{z}\|^2 - \|\tilde{x}+\tilde{y},\tilde{z}\|^2 + \bar{i}\left\|\tilde{x}+\bar{i}\tilde{y},\tilde{z}\right\|^2 - \bar{i}\left\|\tilde{x}-\bar{i}\tilde{y},\tilde{z}\right\|^2\right\}$ (Polarization identity).

Proof.

(i) $\|\tilde{x}+\tilde{y},\tilde{z}\|^2 = \langle \tilde{x}+\tilde{y},\tilde{x}+\tilde{y},\tilde{z}\rangle = \langle \tilde{x},\tilde{x},\tilde{z}\rangle + \langle \tilde{x},\tilde{y},\tilde{z}\rangle + \langle \tilde{y},\tilde{x},\tilde{z}\rangle + \langle \tilde{y},\tilde{y},\tilde{z}\rangle$

$$= \|\tilde{x},\tilde{z}\|^2 + \langle \tilde{x},\tilde{y},\tilde{z}\rangle + \langle \tilde{y},\tilde{x},\tilde{z}\rangle + \|\tilde{y},\tilde{z}\|^2 \quad (3.4)$$

$$\|\tilde{x}-\tilde{y},\tilde{z}\|^2 = \langle \tilde{x}-\tilde{y},\tilde{x}-\tilde{y},\tilde{z}\rangle = \langle \tilde{x},\tilde{x},\tilde{z}\rangle - \langle \tilde{x},\tilde{y},\tilde{z}\rangle - \langle \tilde{y},\tilde{x},\tilde{z}\rangle + \langle \tilde{y},\tilde{y},\tilde{z}\rangle$$
$$= \|\tilde{x},\tilde{z}\|^2 - \langle \tilde{x},\tilde{y},\tilde{z}\rangle - \langle \tilde{y},\tilde{x},\tilde{z}\rangle + \|\tilde{y},\tilde{z}\|^2 \quad (3.5)$$

After adding (3.4) and (3.5), then we obtain parallelogram law

(ii) By Subtracting (3.5) from (3.4), we have

$$\|\tilde{x}+\tilde{y},\tilde{z}\|^2-\|\tilde{x}-\tilde{y},\tilde{z}\|^2$$
$$=2\langle\tilde{x},\tilde{y},\tilde{z}\rangle+2\langle\tilde{y},\tilde{x},\tilde{z}\rangle \quad (3.6)$$

$$\left\|\tilde{x}+\bar{i}\tilde{y},\tilde{z}\right\|^2=\langle\tilde{x}+\bar{i}\tilde{y},\tilde{x}+\bar{i}\tilde{y},\tilde{z}\rangle=\langle\tilde{x},\tilde{x},\tilde{z}\rangle-\bar{i}\langle\tilde{x},\tilde{y},\tilde{z}\rangle+\bar{i}\langle\tilde{y},\tilde{x},\tilde{z}\rangle+\langle\tilde{y},\tilde{y},\tilde{z}\rangle$$
$$=\|\tilde{x},\tilde{z}\|^2-\bar{i}\langle\tilde{x},\tilde{y},\tilde{z}\rangle+\bar{i}\langle\tilde{y},\tilde{x},\tilde{z}\rangle+\|\tilde{y},\tilde{z}\|^2 \quad (3.7)$$

$$\left\|\tilde{x}-\bar{i}\tilde{y},\tilde{z}\right\|^2=\langle\tilde{x}-\bar{i}\tilde{y},\tilde{x}-\bar{i}\tilde{y},\tilde{z}\rangle=\langle\tilde{x},\tilde{x},\tilde{z}\rangle+\bar{i}\langle\tilde{x},\tilde{y},\tilde{z}\rangle-\bar{i}\langle\tilde{y},\tilde{x},\tilde{z}\rangle+\langle\tilde{y},\tilde{y},\tilde{z}\rangle$$
$$=\|\tilde{x},\tilde{z}\|^2+\bar{i}\langle\tilde{x},\tilde{y},\tilde{z}\rangle-\bar{i}\langle\tilde{y},\tilde{x},\tilde{z}\rangle+\|\tilde{y},\tilde{z}\|^2 \quad (3.8)$$

By multiplying (3.7), (3.8) by $\bar{i},-\bar{i}$ respectively

$$\bar{i}\left\|\tilde{x}+\bar{i}\tilde{y},\tilde{z}\right\|^2$$
$$=\bar{i}\|\tilde{x},\tilde{z}\|^2+\langle\tilde{x},\tilde{y},\tilde{z}\rangle-\langle\tilde{y},\tilde{x},\tilde{z}\rangle+\bar{i}\|\tilde{y},\tilde{z}\|^2 \quad (3.9)$$

$$-\bar{i}\left\|\tilde{x}-\bar{i}\tilde{y},\tilde{z}\right\|^2=-\bar{i}\|\tilde{x},\tilde{z}\|^2+\langle\tilde{x},\tilde{y},\tilde{z}\rangle-\langle\tilde{y},\tilde{x},\tilde{z}\rangle-\bar{i}\|\tilde{y},\tilde{z}\|^2 \quad (3.10)$$

After adding (3.6), (3.9) and (3.10) the right hand side becomes 4 $\langle\tilde{x},\tilde{y},\tilde{z}\rangle$ and that proves the identity.

Theorem 3.9.

Let $(\check{X},\langle.,.,.\rangle,A)$ be a soft 2-inner product space satisfying (5). Let $\tilde{x},,\tilde{y},\tilde{z}\,\tilde{\in}\,\check{X}$, if $\tilde{x}_n\to\tilde{x}$ and $\tilde{y}_n\to\tilde{y}$. Then $\langle\tilde{x}_n,\tilde{y}_n,\tilde{z}\rangle\to\langle\tilde{x},\tilde{y},\tilde{z}\rangle$

Proof.

$$\langle\tilde{x}_n,\tilde{y}_n,\tilde{z}\rangle=\langle\tilde{x}+(\tilde{x}_n-\tilde{x}),\tilde{y}+(\tilde{y}_n-\tilde{y}),\tilde{z}\rangle$$
$$=\langle\tilde{x},\tilde{y},\tilde{z}\rangle+\langle\tilde{x},\tilde{y}_n-\tilde{y},\tilde{z}\rangle+\langle\tilde{x}_n-\tilde{x},\tilde{y},\tilde{z}\rangle+\langle\tilde{x}_n-\tilde{x},\tilde{y}_n-\tilde{y},\tilde{z}\rangle$$
$$|\langle\tilde{x}_n,\tilde{y}_n,\tilde{z}\rangle-\langle\tilde{x},\tilde{y},\tilde{z}\rangle|=|\langle\tilde{x},\tilde{y}_n-\tilde{y},\tilde{z}\rangle+\langle\tilde{x}_n-\tilde{x},\tilde{y},\tilde{z}\rangle+\langle\tilde{x}_n-\tilde{x},\tilde{y}_n-\tilde{y},\tilde{z}\rangle|$$
$$\tilde{\leq}\,|\langle\tilde{x},\tilde{y}_n-\tilde{y},\tilde{z}\rangle|+|\langle\tilde{x}_n-\tilde{x},\tilde{y},\tilde{z}\rangle|+|\langle\tilde{x}_n-\tilde{x},\tilde{y}_n-\tilde{y},\tilde{z}\rangle|$$
$$\tilde{\leq}\,\|\tilde{x},\tilde{z}\|\|\tilde{y}_n-\tilde{y},\tilde{z}\|+\|\tilde{x}_n-\tilde{x},\tilde{z}\|\|\tilde{y},\tilde{z}\|+\|\tilde{x}_n-\tilde{x},\tilde{z}\|\|\tilde{y}_n-\tilde{y},\tilde{z}\|$$

Since $\|\tilde{x}_n-\tilde{x},\tilde{z}\|\to\bar{0}$ and $\|\tilde{y}_n-\tilde{y},\tilde{z}\|\to\bar{0}$ as $n\to\infty$. Then $|\langle\tilde{x}_n,\tilde{y}_n,\tilde{z}\rangle-\langle\tilde{x},\tilde{y},\tilde{z}\rangle|\to\bar{0}$, hence $\langle\tilde{x}_n,\tilde{y}_n,\tilde{z}\rangle\to\langle\tilde{x},\tilde{y},\tilde{z}\rangle$.

Corollary 3.10.

Let $(\check{X},\langle.,.,.\rangle,A)$ be a soft 2-inner product space satisfying (5). If $\tilde{x}_n\to\tilde{x}$, then $\|\tilde{x}_n,\tilde{y}\|\to\|\tilde{x},\tilde{y}\|$

Proof.

Since $\tilde{x}_n\to\tilde{x}$, then $\langle\tilde{x}_n,\tilde{x}_n,\tilde{y}\rangle\to\langle\tilde{x},\tilde{x},\tilde{y}\rangle$(by Theorem 3.9). Hence $\|\tilde{x}_n,\tilde{y}\|^2\to\|\tilde{x},\tilde{y}\|^2$, so $\|\tilde{x}_n,\tilde{y}\|\to\|\tilde{x},\tilde{y}\|$.

Definition 3.11.

A soft 2-inner product space which satisfies (I6), is said to be complete if it is complete with respect to the soft 2-metric defined by soft 2-inner product.
A complete soft 2-inner product space is said to be a soft 2-Hilbert space.

Theorem 3.12.

Every soft 2-Hilbert space is soft 2-Banach space.

Proof.

Let $(\check{X}, \langle .,.,. \rangle, A)$ be a soft 2-Hilbert space. Then $\check{X}$ is a complete.
But $\check{X}$ is soft 2-normed space with the soft 2-norm $\|\tilde{x}, \tilde{y}\| = \sqrt{\langle \tilde{x}, \tilde{x}, \tilde{y} \rangle}$ for all $\tilde{x},\ \tilde{y} \,\tilde{\in}\, \check{X}$. So $(\check{X}, \|.,.\|, A)$ soft 2-Banach space.

Theorem 3.13.

A soft 2-Banach space which satisfies (N5) is a soft 2-Hilbert space if and only if the parallelogram law holds.

References

[1] D. Molodtsov, *Soft set theory first results*, Comput. Math. Appl. 37, 19- 31,(1999).
[2] P.K. Maji, R. Biswas and A. R. Roy, *An application of soft sets in a decision making problem*, Comput. Math. Appl. 44 ,107-1083,(2002).
[3] P.K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Comput. Math. Appl. 45 , 555 - 562,(2003).
[4] D. Chen, *The parametrization reduction of soft sets and its applications*, Comput. Math. Appl. 49 , 757-763, (2005).
[5] D. Pie, D. Miao, *From soft sets to information systems*, Granular Computing, , IEEE, International Conference, Volume - 2. 617- 622,(2005).
[6] Z. Kong, L. Gao, L. Wong and S. Li, *The normal parameter reduction of soft sets and its algorithm*, Comput. Math. Appl. 56, 3029- 3037, (2008).
[7] Y. Zou, Z. Xiao, *Data analysis approaches of soft sets under incomplete information*, Knowledge-Based Systems. 21, 941- 945,(2008).
[8] H. Aktas, N. Cagman, *Soft sets and soft groups*, Inform. Sci. 177 , 2226 -2735, (2007).
[9] Y.B. Jun, Soft BCK/BCI { algebras, Comput. Math. Appl. 56 , 1408-1413, (2008).
[10] Y.B. Jun, C.H. Park, *Applications of soft sets in ideal theory of (BCK/BCI) algebras*, Inform. Sci. 178 , 2466 - 2475, (2008).
[11] F. Feng, Y. B. Jun, X. Zhao, Soft semi-rings, Comput. Math. Appl. 56 , 2621- 2628 ,(2008).
[12] M.I. Ali, F. Feng, X. Liu, W.K. Min, M. Shabir, *On some new operations in soft set theory*, Comput. Math. Appl. 57 , 1547-1553, (2009).
[13] M. Shabir, M. Irfan Ali, *Soft ideals and generalized fuzzy ideals in semi-groups*, New Math. Nat. Comput. 5, 599 – 615,(2009).
[14] M. Shabir, M. Naz, *On soft topological Spaces*, Comput. Math. Appl. 61, 1786 - 1799.(2011).
[15] P. Majumdar, S.K. Samanta, *On soft mappings*, Comput. Math. Appl. 60 , 2666 -2672, (2010).
[16] F. Feng, C. X. Li, B. Davvaz, M. I. Ali, *Soft sets combined with fuzzy sets and rough sets*: a tentative approach, Soft Computing 14 , 8999-9911, (2010).
[17] Sujoy Das, S.K. Samanta, Soft Real Sets, *Soft Real Numbers and Their Properties*, J. Fuzzy Math. 20 (3) , 551-576, (2012).
[18] Sujoy Das, S.K. Samanta, *On Soft Complex Sets and Soft Complex Numbers*, To be appeared in J. Fuzzy Math. 20 (4) , (2012).
[19] Sujoy Das, S.K. Samanta, *On Soft Metric Spaces*, J. Fuzzy Math. Accepted, 21 (3), (2013).
[20] Sujoy Das, PinakiMajumdar, S.K. Samanta, *On Soft Linear Spaces and Soft Normed Linear Spaces*, Communicated to The Ukrainian Mathematical Journal, (2013).
[21] Y. J. Cho, C. S. Lin, S. S. Kim and A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, Inc., New York, (2001).

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حول فضاءات هلبرت الابتدائية الثنائية الناعمة

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المستخلص

في هذا البحث تم عرض فكرة الضرب الداخلي الثنائي الناعم على الفضاءات الخطية الناعمة وتم التحقيق في بعض خصائصها. الفضاءات المعيارية الثنائية الناعمة، الفضاءات المترية الثنائية الناعمة، فضاءات بناخ الثنائية الناعمة و فضاءات هلبرت الثنائية الناعمة تم دراستها ايضاً.