On Fuzzy Measure on Fuzzy sets

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Abstract: In this paper, we study the fuzzy measure on fuzzy sets and prove some new properties.

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1. Introduction

The fuzzy measure, defined on σ-field, was introduced by Sugeno [4]. Ralescu and Adams [10] generalized the concepts of fuzzy measure and fuzzy integral to the case that the value of a fuzzy measure can be infinite, and to realize an approach from Subjective.

Wang [7,11]and Kruse [17] studied some structural characteristics of fuzzy measures and proved several theorem about fuzzy measure.

Wang [7, 11] introduced the concept of 'autocontinuity of a set function', used it with regard to the above-mentioned researches, and obtained a series of new results.

The notion of fuzzy measure was extended by Avallone and Barbieri, Jiang and Suzuki [14] Narukawa and Murofushi [8]. Ralescu and Adams [10] as a set function which was defined on σ-field with values in [0,∞]. After that, many authors studied the fuzzy measure and proved some results about it as Guo and Zhang [8] Kui [13], Li and Yasuda [6] Lushu and Zhaohu [15] Minghu [16].

In this paper, we mention the definition of Fuzzy Measure on Fuzzy Set with some Properties, and prove some new relations deal with fuzzy measure.

Definition (1): [18, 19]
Let Ω be an empty set, a fuzzy set A in Ω (or a fuzzy subset in Ω) is a function from Ω into I, i.e. A ∈ IΩ. A(x) is interpreted as the degree of membership of element x in a fuzzy set A for each x ∈ Ω, a fuzzy set A in Ω is can be represented by the set of pairs:

\[ A = \{ (x, A(x)) : x \in \Omega \} \]

Note that every ordinary set is fuzzy set, i.e. P(Ω) ⊆ IΩ.

Definition (2): [1, 2]
A family \( F \) of fuzzy sets in a set \( \Omega \) is called a fuzzy \( \sigma \) -field on a set \( \Omega \) if,
1. \( \emptyset, \Omega \in \mathcal{F} \).
2. If \( A \in \mathcal{F} \), then \( A^c \in \mathcal{F} \).
3. If \( \{ A_n \} \subseteq \mathcal{F}, n = 1,2,3,\ldots \), then \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \).

Evidently, an arbitrary \( \sigma \) -field must be fuzzy \( \sigma \) -field.

A fuzzy measurable Space is a pair (Ω, \( \mathcal{F} \)), where Ω is a set and \( \mathcal{F} \) is a fuzzy \( \sigma \) -field on Ω. A fuzzy set A in Ω is called fuzzy measurable (fuzzy measurable with respect to the fuzzy \( \sigma \) -field ) if \( A \in \mathcal{F} \), i.e. any member of \( \mathcal{F} \) is called a fuzzy measurable set.

Definition (3) [3]:
Let (Ω, \( \mathcal{F} \)) be a fuzzy measurable space. A set function \( \mu : \mathcal{F} \to [0,\infty] \) is said to be
1. (Finite if, \( \mu(A) < \infty \) for each \( A \in \mathcal{F} \).
2. Semi-finite, if for each \( A \in \mathcal{F} \) with \( \mu(A) = \infty \), there exists \( B \in \mathcal{F} \) with \( B \subseteq A \) and \( 0 < \mu(B) < \infty \).
(3) Bounded, if
\[
\sup(|\mu(A)| : A \in \mathcal{F}) < \infty
\]
(4) \(\sigma\)-finite, if for each \(A \in \mathcal{F}\), there is a sequence \(\{A_n\}\) of sets in \(\mathcal{F}\) such that
\[
A = \bigcup_{n=1}^{\infty} A_n
\]
And \(\mu(A_n) < \infty\) for all \(n\).
(5) Additive if,
\[
\mu(A \cup B) = \mu(A) + \mu(B)
\]
whenever \(A, B \in \mathcal{F}\) and \(A \cap B = \emptyset\).
(6) Finitely additive if,
\[
\mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k)
\]
whenever \(A_1, A_2, \ldots, A_n\) are disjoint sets in \(\mathcal{F}\).
(7) \(\sigma\)-additive (sometimes called Completely additive, or A countably additive) if,
\[
\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)
\]
whenever \(\{A_n\}\) is a sequence of disjoint sets in \(\mathcal{F}\).
(8) Measure, if \(\mu\) is \(\sigma\)-additive and \(\mu(A) \geq 0\) for all \(A \in \mathcal{F}\).
(9) Probability, if \(\mu\) is a measure and \(\mu(\Omega) = 1\).
(10) Continuous from below at \(A \in \mathcal{F}\), if \(\lim_{n \to \infty} \mu(A_n) = \mu(A)\), whenever \(\{A_n\}\) is a sequence of sets in \(\mathcal{F}\), and \(A_n \uparrow A\).
(11) Continuous from above at \(A \in \mathcal{F}\), if \(\lim_{n \to \infty} \mu(A_n) = \mu(A)\), whenever \(\{A_n\}\) is a sequence of sets in \(\mathcal{F}\), and \(A_n \downarrow A\).
(12) Continuous at \(A \in \mathcal{F}\), if it is continuous both from below and from above at \(A\).

Definition (4): [4]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \to [0, \infty]\) is said to be a fuzzy measure on \((\Omega, \mathcal{F})\) if it satisfies the following properties:
(1) \(\mu(\emptyset) = 0\)
(2) If \(A, B \in \mathcal{F}\) and \(A \subseteq B\), then \(\mu(A) \leq \mu(B)\)

Definition (5): [5]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \to [0, \infty]\) is called:
(1) Upper semi continuous fuzzy measure if and only if
\[
\lim_{n \to \infty} \mu(A_n) = \mu(\bigcup_{i=1}^{\infty} A_i)
\]
whenever \(\{A_n\}\) is increasing sequence.

(2) Lower semi continuous fuzzy measure if and only if
\[
\lim_{n \to \infty} \mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)
\]
whenever \(\{A_n\}\) is decreasing sequence.
(3) Semi continuous fuzzy measure if it is both upper and lower semi continuous fuzzy measure.
(4) Regular if and only if \(\Omega \in \mathcal{F}\) and \(\mu(\Omega) = 1\).

Definition (6): [5]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \to [0, \infty]\) is said to be
1. Exhaustive if \(\mu(A_n) \to 0\) whenever \(\{A_n\}\) is infinite sequence of disjoint sets in \(\mathcal{F}\).
2. Order-continuous if \(\mu(A_n) \to 0\), whenever \(A_n \in \mathcal{F}\), \(n = 1, 2, \ldots\) and \(A_n \downarrow \emptyset\).

Definition (7): [6]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \to [0, \infty]\) is said to be
Null-additive if \(\mu(A \cup B) = \mu(A) + \mu(B)\) whenever \(A, B \in \mathcal{F}\) and \(A \cap B = \emptyset\).

Definition (8): [6, 7]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \to [0, \infty]\) is said to be
Null-additive if \(\mu(A \cup B) = \mu(A)\) whenever \(A, B \in \mathcal{F}\) such that \(A \cap B = \emptyset\), and \(\mu(B) = 0\).

Definition (9): [8]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \to [0, \infty]\) is said to be
Weakly null-additive, if for any \(A, B \in \mathcal{F}\) such that \(A \cap B = \emptyset\), and \(\mu(B) = 0\).

Remark (10):
The concept of null-null additive stems from a wings textbook which the book[8] derived from, in which it is said to be weak null additive. But we consider that it is more precise and vivid to call it “null-null additive”.

Definition (11):
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \to [0, \infty]\) is said to be
Finitely weakly null-additive, if for any \(\{A_i\} \subseteq \mathcal{F}, \mu(A_i) = 0\)
\[
\lim_{i=1}^{\infty} \mu(\bigcup_{i=1}^{n} A_i) = 0
\]
for all \(i = 1, \ldots, n\) and a sequence of disjoint sets in \(\mathcal{F}\).
Definition (12): [6]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \rightarrow [0, \infty)\) is said to be

Countably weakly null-additive, if for any \(\{A_n\} \subset \mathcal{F}, \mu(A_n) = 0\)

, for all \(n \geq 1 \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0\)

Definition (13): [6]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \rightarrow [0, \infty)\) is said to be

null-continuous, if for every increasing sequence \(A_n \subset \mathcal{F}\) such that \(\mu(A_n) = 0\) for all \(n \geq 1\).

Definition (14): [9]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \rightarrow [0, \infty)\) is said to be

null-subtractive, if we have

\(\mu(A \cap B^c) = \mu(A)\), whenever \(A, B \in \mathcal{F}\) and \(\mu(B) = 0\).

Definition (15): [9]
Let \(A \in \mathcal{F}\), \(\mu(A) < \infty\). \(\mu\) is called pseudo-null-subtractive with respect to \(A\), if for any \(B \in A \cap \mathcal{F}\), we have

\(\mu(B \cap C) = \mu(B),\) whenever \(C \in \mathcal{F}\), \(\mu(A \cap C) = \mu(A)\), where

\(A \cap \mathcal{F} = \{A \cap D : D \in \mathcal{F}\}\).

Definition (16): [9]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \(\mu : \mathcal{F} \rightarrow [0, \infty)\) is said to be

auto continuous from above (resp. autocontinuous from below), if \(\mu(B_n) \rightarrow 0\) implies \(\mu(A \cup B_n) \rightarrow \mu(A)\) (resp. \(\mu(A \cap B_n) \rightarrow \mu(A)\)) whenever \(A \in \mathcal{F}\), \(\{B_n\} \subset \mathcal{F}\).

Definition (17): [9]
Let \(A \in \mathcal{F}\), \(\mu(A) < \infty\). \(\mu\) is called pseudo-autocontinuous from above with respect to \(A\), if for any \(\{B_n\} \subset \mathcal{F}\), when

\(\mu(B_n \cap A) \rightarrow \mu(A)\), then

\(\mu(B_n \cap A) \cup C \rightarrow \mu(C)\), (resp. \(\mu(B_n \cap C) \rightarrow \mu(C)\) whenever \(C \in A \cap \mathcal{F}\).

\(\mu\) is called pseudo-autocontinuous with respect to \(A\) if it is both autocontinuous from above and autocontinuous from below.

2. Main results

Theorem (1):
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space, if \(\mu\) is \(\sigma\)-additive then

\(\mathcal{F}^\ast = \{A \in \mathcal{F} : B \subseteq \Omega \text{ and } \mu(B) = 0\}\) is fuzzy \(\sigma\)-field on \(\Omega\).

Proof:

(1) Since \(\Omega \Delta \emptyset = \Omega, \emptyset \subseteq \Omega\) and \(\mu(\emptyset) = 0\)

\(\emptyset \subseteq \Omega\) and \(\mu(\emptyset) = 0\), we have \(\Omega \in \mathcal{F}^\ast\)

(2) Let \(V \in \mathcal{F}^\ast\); we have

\(V = A \Delta B, A \in \mathcal{F}, B \subseteq \Omega \text{ with } \mu(B) = 0\)

\(\Rightarrow V^c = (A \Delta B)^c = [(A / B) \cup (B / A)]^c\)

\(= (A^c \cap B^c) \cup (A \cap B)\)

\(= A^c \Delta B, \text{where } B \subseteq \Omega \text{ with } \mu(B) = 0\)

Since \(A \in \mathcal{F}\) and \(\mathcal{F}\) is a fuzzy \(\sigma\)-field, we have

\(A^c \in \mathcal{F}\) and \(B \subseteq \Omega \text{ with } \mu(B) = 0\)

\(\Rightarrow V^c = A^c \Delta B, A^c \in \mathcal{F}\) and \(B \subseteq \Omega \text{ with } \mu(B) = 0\)

\(\Rightarrow V^c \in \mathcal{F}^\ast\)

(3) Let \(\{V_n\}\) be a sequence of sets in \(\mathcal{F}^\ast\) with

\(V_n = A_n \Delta B_n\). \(A_n \in \mathcal{F}, B_n \subseteq \Omega\) and \(\mu(B_n) = 0\)

for all \(n\). We have

\(\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}\)

\(\mathcal{F}\) is fuzzy \(\sigma\)-field

\(\bigcup_{n=1}^{\infty} B_n \subseteq \Omega \text{ and } \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = 0\)

So

\(\bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} (A_n \Delta B_n)\)

\(= \bigcup_{n=1}^{\infty} (A_n / B_n) \cup (B_n / A_n)\)

\(= \bigcup_{n=1}^{\infty} A_n \Delta B_n\)

\(\Rightarrow \bigcup_{n=1}^{\infty} V_n \in \mathcal{F}^\ast\)

Consequently \(\mathcal{F}^\ast\) is fuzzy \(\sigma\)-field on \(\Omega\).

Remark (2):
The union of a collection of fuzzy \(\sigma\)-field need not be fuzzy \(\sigma\)-field as in the following example.

Example (3):
Let \(A, B, C, D\) are fuzzy sets and \(\Omega = \{A(x), B(x), C(x), D(x)\}\), such that

\(A(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 0 & 1/2 < x \leq 1 \end{cases}\)

\(B(x) = \begin{cases} 2x & 1/4 < x \leq 1/2 \\ 1 & 1/2 < x \leq 1 \end{cases}\)

\(C(x) = \begin{cases} 1 - 2x & 0 \leq x \leq 1/2 \\ 1 & 1/2 < x \leq 1 \end{cases}\)
Let $\mathcal{F}_1 = \emptyset, A(x), C(x), \Omega \}$. $\mathcal{F}_2 = \{\emptyset, B(x), D(x), \Omega\}$ are two fuzzy $\sigma$-fields, but $\mathcal{F}_1 \cup \mathcal{F}_2$ is not fuzzy $\sigma$-field.

**Solution:**

First we must prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ are fuzzy $\sigma$-fields.

$\mathcal{F}_1$ is fuzzy $\sigma$-field:

1. $\emptyset, \Omega \in \mathcal{F}_1$.
2. Let $A(x) \in \mathcal{F}_1$, prove $A^c(x) \in \mathcal{F}_1$.

From Definition (1.1.6) we get on

$A^c(x) = 1 - A(x)$

$= 1 - \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 0 & 1/2 < x \leq 1 \\ 1 & 1/2 < x \leq 1 \end{cases}$

But $C(x) \in \mathcal{F}_1$ implies $A^c(x) \in \mathcal{F}_1$.

(ii) Let $C(x) \in \mathcal{F}_1$, to prove $C^c(x) \in \mathcal{F}_1$.

$C(x) = 1 - C(x)$

$= \begin{cases} 1 & 0 \leq x \leq 1/2 \\ 1 - 2x & 1/2 < x \leq 1 \\ 0 & 1/2 < x \leq 1 \end{cases}$

But $A(x) \in \mathcal{F}_1$ implies $C^c(x) \in \mathcal{F}_1$.

(iii) It is clear that $\emptyset^c = \Omega \in \mathcal{F}_1$.

And $\mathcal{F}_2 = \emptyset \in \mathcal{F}_2$.

(i) If $0 \leq x \leq 1/2$ then $A(x) \in \mathcal{F}_1$.

(ii) If $A(x) = \emptyset$, then $A(x) \in \mathcal{F}_1$.

(iii) $1/2 < x \leq 1$.

$\therefore$ $A \cup C(x) = C(x) \in \mathcal{F}_1$.

$\therefore$ $A$ is fuzzy $\sigma$-field.

In the same way we can prove that $\mathcal{F}_2$ is fuzzy $\sigma$-field.

Now to prove that $\mathcal{F}_1 \cup \mathcal{F}_2$ is not fuzzy $\sigma$-field.

So $\mathcal{F}_1 \cup \mathcal{F}_2 \neq \{\emptyset, A(x), B(x), C(x), D(x), \Omega\}$

$A(x) = \begin{cases} 2x & 0 \leq x \leq 1/4 \\ 0 & 1/4 < x \leq 1/2 \\ 1/2 < x \leq 1 \end{cases}$

$B(x) = \begin{cases} 0 & 0 \leq x \leq 1/4 \\ 2x & 1/4 < x \leq 1/2 \\ 1 & 1/2 < x \leq 1 \end{cases}$

(i) if $0 \leq x \leq 1/4$

$\Rightarrow (A \cup B)(x) = \max(A(x), B(x)) = \max(2x, 0) = 2x$

(a) If $x = 0 \Rightarrow A \cup B(x) = 0 \in \mathcal{F}_1$.

(b) If $x = \frac{1}{2} \Rightarrow (A \cup B)(x) = 1/2 \notin \mathcal{F}_1$.

$\therefore$ $\mathcal{F}_1 \cup \mathcal{F}_2$ is not fuzzy $\sigma$-field.

**Theorem (4):**

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space, suppose that $\mathcal{F}^*$ is $\sigma$-field and $\mu^*$ is a measure on $(\Omega, \mathcal{F})$, for any $A \in \mathcal{F}$ such that $\mu(B) = \mu^*(A \cap B)$. For any $B \in \mathcal{F}^*$ is fuzzy measure on $(\Omega, \mathcal{F})$.

**Proof:**

(1) Since $\mathcal{F}^*$ is $\sigma$-field implies $\emptyset \in \mathcal{F}^*$

$\therefore \mu(\emptyset) = \mu^*(A \cap \emptyset) = \mu^*(\emptyset) = 0$.

(2) Let $A_1, A_2 \in \mathcal{F}$, if $A_1 \in A_2$, then

$\mu(A_1) = \mu^*(A_1 \cap B) \leq \mu^*(A_2 \cap B) = \mu(A_2)$

$\therefore \mu$ is fuzzy measure on $(\Omega, \mathcal{F})$.

**Theorem (5):**

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that there is $B \in \mathcal{F}$ with $0 < \mu(B) < \infty$, define $\mu^* : \mathcal{F} \rightarrow [0, \infty]$ by 

$\mu^*(A) = \frac{\mu(A \cap B)}{\mu(B)}$, then $(\Omega, \mathcal{F}, \mu^*)$ is fuzzy measure space.

**Proof:**

$\mu^*(\emptyset) = \frac{\mu(\emptyset \cap B)}{\mu(B)} = 0$.

Let $A, B \in \mathcal{F}$, if $A \subseteq B$, we have $\mu(A) \leq \mu(B)$

Since $A \subseteq B$, hence $A \cap B = A$

$\Rightarrow \mu(A \cap B) = \mu(A) \Rightarrow \mu(A \cap B) = \mu(A) \leq \mu(B)$

$\Rightarrow \mu(A \cap B)/\mu(B) \leq \mu(B \cap B)/\mu(B)$

$\Rightarrow \mu^*(A) \leq \mu^*(B)$.

Consequently $\mu^*$ is a fuzzy measure.

**Theorem (6):**

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space, $\mu, \nu$ be fuzzy measures on $\Omega$, then $\mu + \nu$ which denoted by

$(\mu + \nu)(A) = \mu(A) + \nu(A)$

is fuzzy measure on $\Omega$.

**Proof:**

(1) Since $\mu, \nu$ be two fuzzy measures

$\Rightarrow (\mu + \nu)(\emptyset) = 0$.

(2) Let $A, B \in \mathcal{F}$, if $A \subseteq B$, we have

$(\mu + \nu)(A) = \mu(A) + \nu(A) \leq \mu(B) + \nu(B)$

$= (\mu + \nu)(B)$.

So $\mu + \nu$ is fuzzy measure.
Corollary (1):
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, \(\mu\) be a fuzzy measure on \(\Omega\), and \(\alpha > 0\), define a set function \(\alpha \mu(A) = \alpha \mu(A)\) on \(\Omega\).

**Proof:**
(1) Since \(\mu\) is a fuzzy measure, we have \(\alpha \mu(\emptyset) = \alpha \mu(\emptyset) = 0\).
(2) Let \(A, B \in \mathcal{F}\), if \(A \subseteq B\), we have \(\mu(A) \leq \mu(B)\)
\[\Rightarrow (\alpha \mu(A)) = \alpha \mu(A) \leq \alpha \mu(B) = (\alpha \mu(B)).\]
So \(\alpha \mu\) is fuzzy measure.

**Remark (7):**
The points (1) and (2) from Definition (5) explain fuzzy measure is upper semi continuous and lower semi continuous; the following results take us to the converse direction.

**Theorem (8):**
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space and let \(\mu\) be a function: \(\mathcal{F} \rightarrow \mathbb{R}_+\), if \(\mu\) is additive, non-decreasing and upper semi continuous, then \(\mu\) is fuzzy measure.

**Proof:**
(1) Since \(A = A \cup \emptyset\),
Also \(\mu\) is additive we have
\[\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)\]
\[\Rightarrow \mu(\emptyset) = 0\]
(1) Let \(A, B \in \mu\), if \(A \subseteq B\), we have \(B = A \cup (B \setminus A)\)
and \(A \cap (B \setminus A) = \emptyset\)
Since \(\mu\) is additive we have, we obtain
\[\mu(B) = \mu(A) + \mu(B \setminus A)\]
Consequently
\[\mu(B \setminus A) = \mu(B) - \mu(A)\]
In addition, \(\mu(B \setminus A) \geq 0\)
Hence
\[\mu(A) \leq \mu(B)\]
Then \(\mu\) is fuzzy measure.

**Theorem (9):**
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, let \(\{A_n\}\) be a sequence of disjoint fuzzy set in \(\mathcal{F}\) and it is decreasing, if \(\mu(A_n) < \infty\) and \(\mu\) is lower semi continuous fuzzy measure at \(\emptyset\), then \(\lim_{n \to \infty} \mu(A_n) = 0\).

**Proof:**
Since \(\{A_n\}\) is lower continuous fuzzy measure at \(\emptyset\), we have
\[\lim_{n \to \infty} \mu(A_n) = \mu(\emptyset)\]
But \(\mu(\emptyset) = 0\)
Consequently, we have
\[\lim_{n \to \infty} \mu(A_n) = 0\]

**Theorem (10):**
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, and for any \(A \in \mathcal{F}\), \(\mu(A) \neq 0\), then \(\mu\) is null additive.

**Proof:**
If there exists some set \(B \in \mathcal{F}\) such that \(\mu(B) = 0\), then \(B = \emptyset\).
Consequently, for any \(A \in \mathcal{F}\), we have \(\mu(A \cup B) = \mu(A)\).

**Theorem (11):**
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, if \(\mu\) is autocontinuous from below, then it is null-subtractive.

**Proof:**
Let \(A, B_n \in \mathcal{F}\)
Since if \(\mu\) is autocontinuous from below, we have
\[\lim_{n \to \infty} \mu(B_n) = 0\]
Also we have
\[\mu(A \cap B_n^c) \to \mu(A)\]
Consequently \(\mu\) is null-subtractive.

**Theorem (12):**
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, if \(\mu\) is pseudo-autocontinuous from below with respect to \(A\), then it is pseudo-null-subtractive with respect to \(A\).

**Proof:**
Let \(A, B_n \in \mathcal{F}\)
Since if \(\mu\) is pseudo-autocontinuous from below, we have \(\mu(A) < \infty\). And \(C \in A \cap \mathcal{F}\)
\[\mu(B_n \cap C) \to \mu(C)\]
Consequently, \(\mu\) is pseudo-null-subtractice with respect to \(A\).

**Theorem (13):**
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, if \(\mu\) is upper semi continuous fuzzy measure and countably weakly null additive then \(\mu\) is exhaustive.

**Proof:**
Let \(\{A_n\}\) be a disjoint of sequence of sets in \(\mathcal{F}\). Since \(\mu\) is countably weakly null additive
\[\mu(A_n) = 0\] for all \(n \geq 1\)
\[\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0\]
Also \(\mu\) is upper semi continuous
\[\Rightarrow \lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)\]
\[\Rightarrow \lim_{n \to \infty} \mu(A_n) = 0\]
\(\therefore\) \(\mu\) is exhaustive.

**Theorem (14):**
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space, if \(\mu\) is countably weakly null additive then \(\mu\) is null-continuous.
Proof:
Let \( \{A_n\} \) be a increasing sequence of sets in \( F \), such that
\[ \mu(\bigcup_{n=1}^{\infty} A_n) = 0, \quad \text{for all } n \geq 1 \]
Since \( \mu \) is countably weakly null additive
\[ \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = 0 \]
\( \therefore \mu \) is null-continuous.

References