The Completion of $\oplus$-measure

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1- Abstract

The theory of measure is an important subject in mathematics; in Ash [4,5] discusses many details about measure and proves some important results in measure theory.

In 1986, Dimiev [7] defined the operation addition and multiplication by real numbers on a set $E = (-\infty, 1) \subseteq \mathbb{R}$, he defined the operation multiplication on the set $E$ and prove that $E$ is a vector space over $\mathbb{R}$ and for any $a > 1$ $E_a$ is field, also he defined the fuzzifying functions on arbitrary set $X$.

In 1989, Dimiev [6] discussed the field $E_a$ as in [7] and defined the operations addition, multiplication and multiplication by real number on a set of all fuzzifying functions defined on arbitrary set $X$, and also defined $\oplus$-measure on a measurable space and proved some results about it.

we mention the definition of the field $E_a$, and the fuzzifying functions on the arbitrary set $X$ also we mention the definition of the operations.

Definition (1.1.1) [7]:

Let $(\mathbb{R}, +, \cdot)$ be a field of real numbers with usual order and $E = (-\infty, 1) \subseteq \mathbb{R}$, we introduce the operations addition $\oplus$ and scalar multiplication $\odot$ on the set $E$ as follows:

For any $x, y \in E$ and $\lambda \in \mathbb{R}$ we have

$x \oplus y = x + y - xy$,

$\lambda \odot x = 1 - (1 - x)^\lambda$.

Proposition (1.2) [7]:

The set $E$ with the operations $\oplus$, $\odot$ and the relation order, represent ordered linear space.

Definition (1.3) [6]:

Let $a > 1$, we introduce an operation multiplication on the set $E$ as follows

For any $x, y \in E$ we have $x \circ y = 1 - a^{\log_a(1-x)\log_a(1-y)}$.
Proposition (1.4) [6]:
The set $E$ with the operations $\oplus, \circ$ is a field which is denoted by $E_a$.

Remark (1.5):
Let $x, y \in E_a$, we denote $x \Theta y = x \oplus (-w) \circ y$ and $\Theta_ax = (-w) \circ x$ where $w = 1 - a^{-1}$ the unit element in the field $E_a$.

Definition (1.6)[6]:
Let $X$ be arbitrary set, the map $f : X \rightarrow E_a$ is said to be $E_a$-valued fuzzifying function.

2- $\oplus$ - Measure:
In this section we mention the definition of $\oplus$–measure on a measurable space and proved some results about it, also we defined $\oplus$–outer measure and proved some results about it.

Definition (2.1)[5]:
A collection $F$ of subsets of a set $\Omega$ is said to be:
a) $\sigma$-ring if
1- $\phi \in F$, where $\phi$ is empty set.
2- if $A,B \in F$ then $A|B \in F$.
3- if $\{A_n\}$ is a sequence of sets in $F$ then $\bigcup_{n=1}^{\infty} A_n \in F$.
b) $\sigma$-field (or $\sigma$-algebra) if
1- $\Omega \in F$.
2- if $A \in F$ then $A^c \in F$.
3- if $\{A_n\}$ is a sequence of sets in $F$ then $\bigcup_{n=1}^{\infty} A_n \in F$. A measurable space is a pair $(\Omega,F)$ where $\Omega$ is a set and $F$ is $\sigma$-ring or $\sigma$-field and a measurable set is a subset $A$ of $\Omega$ such that $A \in F$.

Definition (2.2) [6]:
Let $(\Omega, F)$ be a measurable space, a fuzzifying function $\mu : F \rightarrow E_a$ is said to be:
1- $\oplus$-additive if $\mu(A \cup B) = \mu(A) \oplus \mu(B)$ for every disjoint sets $A, B$ in $F$.
2- Accountability $\oplus$-additive if $\mu(\bigcup_{n=1}^{\infty} A_n) = \oplus_{n=1}^{\infty} \mu(A_n)$ for every disjoint sequence $\{A_n\}$ of sets of $F$.
3- $\oplus$-measure, if $\mu$ is accountability $\oplus$- additive and non-negative
The triple $(\Omega,F,\mu)$ is called a space with $\oplus$-measure.

Theorem (2.3):
Let $(\Omega,F,\mu)$ be a space with $\oplus$- measure and $A, B \in F$ then:
1- $\mu(\phi) = 0$.
2- $\mu(A) = \mu(A \cap B) \oplus \mu(A \cap B^c)$.
3- $\mu(A \cup B) \oplus \mu(A \cap B) = \mu(A) \oplus \mu(B)$.
4- if $A \subseteq B$ then:
   (a) $\mu(B|A) = \mu(B) \oplus (-w) \circ \mu(A)$.
   (b) $\mu(A) \leq \mu(B)$.
Proof:
1- Since $A = A \cup \emptyset$ and $A \cap \emptyset = \emptyset$.
   
   $\mu(A) = \mu(A \cup \emptyset) = \mu(A) \oplus \mu(\emptyset)$.

   Since $E_a$ is a field $\Rightarrow \mu(\emptyset) = 0$.

2- Since $A = (A \cap B) \cup (A \cap B^c)$.

   and $(A \cap B) \cap (A \cap B^c) = \emptyset$.

   $\Rightarrow \mu(A) = \mu((A \cap B) \cup (A \cap B^c))$.

   $= \mu(A \cap B) \oplus \mu(A \cap B^c)$.

3- Since $A \cup B = (A \cap B^c) \cup B$ and $(A \cap B^c) \cap B = \emptyset$.

   $\Rightarrow \mu(A \cup B) = \mu((A \cap B^c) \cup B)$

   $= \mu(A \cap B^c) \oplus \mu(B)$.

   $\mu(A \cup B) \oplus \mu(A \cap B) = (\mu(A \cap B^c) \oplus \mu(B)) \oplus \mu(A \cap B)$.

   $= (\mu(A \cap B^c) \oplus \mu(A \cap B)) \oplus \mu(B)$.

   $= \mu(A) \oplus \mu(B)$.

4- (a) Since $A \subseteq B \Rightarrow B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \emptyset$.

   $\mu(B) = \mu(A \cup (B \setminus A))$.

   $= \mu(A) \oplus \mu(B \setminus A)$.

   Since $E_a$ is a field $\Rightarrow \mu(B \setminus A) = \mu(B) \oplus (\omega) \cdot \mu(A)$.

   (b) Since $\mu(B \setminus A) \geq \omega$ from (a) we get that $\mu(A) \leq \mu(B)$.

Definition (2.4):
Let $(\Omega, \mathcal{F})$ be a measurable space and let the fuzzifying $\mu : \mathcal{F} \rightarrow E_a$ be a $\oplus$– additive, we say that $\mu$ is :

1. $\oplus$–continuous from below at $A \in \mathcal{F}$ if $\mu(A_n) \rightarrow \mu(A)$.

   For every non–decreasing sequence $\{A_n\}$ of sets in $\mathcal{F}$ which converge to $A$ (i.e $A_n \uparrow A$).

2. $\oplus$–continuous from below at $A \in \mathcal{F}$ if $\mu(A_n) \rightarrow \mu(A)$.

   For every non-increasing sequence $\{A_n\}$ of sets in $\mathcal{F}$ converge to $A$ (i.e $A_n \downarrow A$).

3. $\oplus$–continuous at $A \in \mathcal{F}$ if it is continuous at $A$ from above and from below.

Theorem (2.5):
Let $\mu$ be $\oplus$– additive fuzzifying function on measurable space $(\Omega, \mathcal{F})$, then the following are valid.

1- If $\mu$ is countable $\oplus$–additive, then $\mu$ is $\oplus$–continuous at $A$ for all $A \in \mathcal{F}$.

2- If $\mu$ is $\oplus$–continuous from below at every $A \in \mathcal{F}$, then $\mu$ is countable $\oplus$–additive.

3- If $\mu$ is continuous from above at $\emptyset$ then $\mu$ is countable $\oplus$–additive.
Proof:

1- Let \( \{A_n\} \) be an increasing sequence of sets in \( \mathcal{F} \) which converge to \( A \), i.e \( A_n \uparrow A \).

(a) Let \( B_1 = A_1, B_n = A_n |_{A_{n-1}} \) \( \forall n \geq 2 \).

\[
B_n \cap B_m = \varnothing, \forall n \neq m \quad \text{and} \quad \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = A.
\]

\[
\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu(A_1) \oplus \left( \bigoplus_{N=2}^{\infty} \mu(B_n) \right).
\]

\[
\Rightarrow \mu(A_1) \oplus \left( \bigoplus_{n=2}^{\infty} \mu(A_n |_{A_{n-1}}) \right).
\]

\[
\mu(A) = \mu(A_1) \oplus \lim_{K \to \infty} \left( \bigoplus_{n=2}^{K} \mu( A_n |_{A_{n-1}}) \right) = \lim_{K \to \infty} \mu(A_K).
\]

\[
\Rightarrow \mu \text{ is } \oplus-\text{continuous from below at } A \in \mathcal{F}.
\]

(b) Suppose that \( A_n \downarrow A \rightarrow A_1|_{A_n} \uparrow A|_{A} \).

\[
\Rightarrow \mu(A_1|_{A_n}) \rightarrow \mu(A_1|_{A}) \Rightarrow \mu(A_n) \rightarrow \mu(A).
\]

So \( \mu \text{ is } \oplus-\text{continuous from above at } A \in \mathcal{F} \).

From (a) and (b) we get that \( \mu \text{ is } \oplus-\text{continuous at } A \in \mathcal{F} \).

2- Let \( \{A_n\} \) be a disjoint sequence of sets in \( \mathcal{F} \), and \( A = \bigcup_{n=1}^{\infty} A_n \).

Put \( B_n = \bigcup_{i=1}^{n} A_i \Rightarrow B_n \in F \Rightarrow B_n \uparrow A \).

Since \( \mu \text{ is } \oplus-\text{continuous from below at } A \in \mathcal{F} \).

\[
\Rightarrow \mu(B_n) \rightarrow \mu(A).
\]

Since \( \mu \text{ is } \oplus-\text{additive} \Rightarrow \mu(B_n) = \mu\left(\bigcup_{i=1}^{n} A_i\right) = \bigoplus_{i=1}^{n} \mu(A_i).
\]

\[
\Rightarrow \bigoplus_{i=1}^{n} \mu(A_i) \rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigoplus_{n=1}^{\infty} \mu(A_n).
\]

So \( \mu \text{ is countable } \oplus-\text{additive} \).

3- In the notation of (2) put \( C_n = A|_{B_n} \Rightarrow C_n \in \mathcal{F}, n=1,2, \ldots \).

\[
\Rightarrow C_n \downarrow \varnothing.
\]

\[
\Rightarrow \mu(C_n) \rightarrow \mu(\varnothing) = 0 \Rightarrow \mu(A|_{B_n}) \rightarrow 0.
\]

\[
\mu(A) = \bigoplus_{i=1}^{n} \mu(A_i) \oplus \mu(C_n).
\]

So that \( \mu(A) = \bigoplus_{i=1}^{\infty} \mu(A_i) \).
3- The completion of $\Theta$-measure

In this section we construct the completion of $\Theta$-measure.

Definition (3.1)

Let $(\Omega, F)$ be a measurable space with $F$ a $\sigma$-ring and $\mu$ is $\Theta$-measure on $F$, $E \in F$ is said to be $\mu$-null set if $\mu(E) = 0$. The $\Theta$-measure $\mu$ is said to be complete on $F$ if $F$ contains the subsets of every $\mu$-null sets.

Theorem (3.2):

Let $(\Omega, F, \mu)$ be a space with $\Theta$-measure where $F$ is $\sigma$-ring and $N_\mu = \{ E : E \subseteq A \in F \text{ and } \mu(A) = 0 \}$ then $N_\mu$ is a $\sigma$-ring.

Proof:

1- Clearly $\varnothing \in N_\mu$.

2- Let $E_1, E_2 \in N_\mu \Rightarrow$ there exists $A_1, A_2 \in F$ such that $E_1 \subseteq A_1, E_2 \subseteq A_2$ and $\mu(A_1) = 0, \mu(A_2) = 0$.

$E_1|E_2 \subseteq E_1 \subseteq A_1 \in F$ So $E_1|E_2 \in N_\mu$.

3- Let $\{E_i\}$ be a sequence of sets in $N_\mu$ $i = 1, 2, \ldots \Rightarrow$ there exist a sequence $\{A_i\}$ $i = 1, 2, \ldots$ of sets in $F$ such that $E_i \subseteq A_i$ and $\mu(A_i) = 0$.

$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} A_i$ Since $F$ is $\sigma$-ring $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$.

$\mu(\bigcup_{i=1}^{\infty} A_i) \leq \bigoplus_{i=1}^{\infty} \mu(A_i) = 0 \Rightarrow \mu(\bigcup_{i=1}^{\infty} A_i) = 0$.

So $\bigcup_{i=1}^{\infty} E_i \in N_\mu$ therefore $N_\mu$ is $\sigma$-ring.

Theorem (3.3):

Let $(\Omega, F, \mu)$ be a space with $\Theta$-measure where $F$ is a $\sigma$-ring, define $\overline{F} = \{(E \cup E_i) - E_2 : E \in F, E_i, E_2 \in N_\mu \}$ then $A \in \overline{F}$ iff there exist sets $M, N \in F$ such that $M \subseteq A \subseteq N$ and $\mu(N - M) = 0$.

Proof:

Let $M, N \in F$ and $M \subseteq A \subseteq N$ such that $\mu(N - M) = 0$, so $A = (N \cup \varnothing)-(N-A)$.

Since $N - A \subseteq N-M \in F$ and $\mu(N - M) = 0$.

$\Rightarrow N - A \in N_\mu$.

Therefore $A \in \overline{F}$.

Suppose that $A \in \overline{F}$.

Then $A = (E \cup E_i) - E_2$, $E \in F, E_i, E_2 \in N_\mu$.

Therefore there exist $A_1, A_2 \in F$ such that $\mu(A_1) = 0$ and $E_i \subseteq A_1, E - A_2 \subseteq A \subseteq E \cup A_i$.

$E \cup A_i, E - A_2 \in F$ and

$\mu((E \cup A_i) - (E - A_2)) = \mu((A_1 + E) \cup (A_2 \cap E)).$

$= \mu((A_1 - E)) \oplus \mu(A_2 \cap E).$

Since $A_1 - E \subseteq A_i$ and $A_2 \cap E \subseteq A_2$.

$\Rightarrow \mu(A_1 - E) = 0 \rightarrow \mu(A_2 \cap E) = 0$.

So $\mu((E \cup A_i) - (E - A_2)) = 0$.
Corollary (3.4):
Let \((\Omega, \mathcal{F}, \mu)\) be a space with \(\oplus\)-measure where \(\mathcal{F}\) is \(\sigma\)-ring then \(A \in \mathcal{F}\) iff \(A = E \cup M\), \(E \in \mathcal{F}\) and \(M \in \mathcal{N}\).

Proof:
Suppose that \(A \in \mathcal{F}\).
By theorem (1.3.3) there exist \(M, N \in \mathcal{F}\) such that \(N \subset A \subset M\) and \(\mu(M - N) = 0\)
\[A = N \cup (A - N), N \in \mathcal{F}.
\]
Since \(A - N \subset M - N \in \mathcal{F}\) and \(\mu(M - N) = 0\).
\[\Rightarrow A - N \in \mathcal{N}.
\]
Conversely suppose \(A = E \cup M\), \(E \in \mathcal{F} \wedge M \in \mathcal{N}\).
\[\Rightarrow A \in \mathcal{F}.
\]

Corollary (3.5):
Let \((\Omega, \mathcal{F}, \mu)\) be a space with \(\oplus\) -measure where \(\mathcal{F}\) is \(\sigma\)-ring then \(A \in \mathcal{F}\) iff \(A = E - D\) with \(E \in \mathcal{F}\) and \(D \in \mathcal{N}\).

Proof:
Suppose that \(A \in \mathcal{F}\).
\[\Rightarrow \text{There exist } M, N \in \mathcal{F} \text{ such that } M \subset A \subset N.
\]
and \(\mu(N - M) = 0\).
\[A = N - (N - A), N \in \mathcal{F}.
\]
Since \(N - A \subset N - M \in \mathcal{F}\) and \(\mu(N - M) = 0\).
\[\Rightarrow N - A \in \mathcal{N}.
\]
Conversely suppose that \(A = E - D\) where \(E \in \mathcal{F}\) \(\wedge\) \(D \in \mathcal{N}\).
\[\Rightarrow A = (E \cup \phi) - D, \phi \in \mathcal{N}.
\]
\[\Rightarrow A \in \mathcal{F}.
\]

Theorem (3.6):
Let \((\Omega, \mathcal{F}, \mu)\) be a space with \(\oplus\) -measure where \(\mathcal{F}\) is a \(\sigma\)–ring then \(\mathcal{F}\) is \(\sigma\)-ring.

Proof:
1-clearly \(\phi \in \mathcal{F}\).
2-Let \(\{A_i\}_{i=1}^\infty\) be a sequence of sets such that \(A_i \in \mathcal{F} \Rightarrow A_i = M_i \cup N_i\) where \(M_i \in \mathcal{F}\) and \(N_i \in \mathcal{N}\).
\[\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty (M_i \cup N_i).
\]
\[= (\bigcup_{i=1}^\infty M_i) \cup (\bigcup_{i=1}^\infty N_i).
\]
Since \(\mathcal{F}\) and \(\mathcal{N}\) are \(\sigma\)-ring.
\[\Rightarrow \bigcup_{i=1}^\infty M_i \in \mathcal{F}_1
\]
\[\bigcup_{i=1}^\infty N_i \in \mathcal{N}.
\]
So \( \bigcup_{i=1}^{\infty} A_i \in \overline{F} \).

3- Let \( A, B \in \overline{F} \) from Corollary (1.3.4) we obtain \( A = M_1 \cup N_1 \quad B = M_2 \cup N_2 \).

\[
A - B = (M_1 \cup N_1) - (M_2 \cup N_2).
\]

\[
= ((M_1 - M_2) - N_2) \cup ((N_1 - M_2) - N_2).
\]

\[
= [(M_1 - M_2) - E_2] \cup (E_2 - N_2) \cap (M_1 - M_2)) \cup ((N_1 - M_2) - N_2)
\]

\[N_2 \subset E_2 \in \mathcal{F}, \quad \mu(E_2) = 0\]

\[
A - B \in \overline{F}.
\]

Therefore \( \overline{F} \) is \( \sigma \)-ring.

**Theorem (3.7):**

Let \( (\Omega, \mathcal{F}, \mu) \) be a space with \( \oplus \)-measure and \( \overline{\mu}: \overline{F} \to E_\mu \) defined as follows

\[
\overline{\mu}(A) = \mu(M) \text{ where } A = (M \cup N), \ M \in \mathcal{F} \text{ and } N \in N_\mu.
\]

Then \( \overline{\mu} \) is complete \( \oplus \)-measure on \( \overline{F} \), where is restriction to \( F \) is \( \mu \).

**Proof:**

1- \( \overline{\mu}(\varnothing) = \mu(\varnothing) = 0 \).

2- Let \( \{A_i\} \) be a sequence of sets in \( \overline{F}, \ i = 1, 2, \ldots \)

\( \Rightarrow \) There exist a sequence of sets \( \{E_i\} \) in \( F \) and a sequence of sets \( \{N_i\} \) in \( N_\mu \) such that \( A_i = E_i \cup N_i \).

\[
\overline{\mu}(\bigcup_{i=1}^{\infty} A_i) = \overline{\mu}(\bigcup_{i=1}^{\infty} (E_i \cup N_i)).
\]

\[
= \overline{\mu}((\bigcup_{i=1}^{\infty} E_i) \cup (\bigcup_{i=1}^{\infty} N_i))
\]

\[
= \mu((\bigcup_{i=1}^{\infty} E_i) \cap \bigcup_{i=1}^{\infty} (N_i))
\]

\[
= \mu(\bigcup_{i=1}^{\infty} E_i) = \oplus_{i=1}^{\infty} \mu(E_i) = \oplus_{i=1}^{\infty} \mu(A_i)
\]

So \( \overline{\mu} \) is \( \oplus \)-measure on \( \overline{F} \).

3- Let \( A \in F, \ A = A \cup \varnothing, \varnothing \in N_\mu \).

\[
\overline{\mu}(A) = \overline{\mu}(A \cup \varnothing) = \mu(A).
\]

\( \mu \) is \( \oplus \)-restriction of \( \overline{\mu} \) to \( F \).
4- Let $E \in \overline{F}$ and $\overline{\mu}(E) = 0$, $A \subset E$.

$E = M \cup N$, $M \in \mathcal{F}, N \in N_\mu$.

$\overline{\mu}(E) = \mu(M) \Rightarrow \mu(M) = 0$.

Since $N \in N_\mu \Rightarrow$ There exists $E_i \in \mathcal{F}$ such that $N \subset E_i$ and $\mu(E_i) = 0$, since $\mu(E_i) = \mu(M) = 0 \Rightarrow M, E \in N_\mu$.

A $\subset E = M \cup N \subset M \cup E_i \Rightarrow A \subset M \cup E_i \in \mathcal{F}$, $\mu(M \cup E_i) = \mu(M) \oplus \mu(E_i) = 0 \Rightarrow A \in N_\mu$.

$A = (M \cup E_i) - ((M \cup E_i) - A)$, $M \cup E_i \in \mathcal{F}, (M \cup E_i) - A \in N_\mu \Rightarrow A \in \overline{F}$ $\Rightarrow \overline{\mu}$ is complete on $\overline{F}$.

5- To show that the definition of $\overline{\mu}$ is well defined.

Let $A \in \overline{F} \Rightarrow A = M \cup N$, $M \in \mathcal{F}$ and $N \in N_\mu$.

$\Rightarrow \exists E \in \mathcal{F}$ $N \subset E$ and $\mu(E) = 0$.

The relations $M \cup N = (M - E) \cup (E \cap (M \cup N))$.

and $M \Delta N = (M - E) \cup (E \cap (M \Delta N))$ show that

the class $\overline{F}$ may also be denoted as there class of the form $M \Delta N, M \in \mathcal{F}$ and $N \in N_\mu, \overline{\mu}(M \Delta N) = \overline{\mu}(M \cup N) = \mu(M)$.

Let $F_1 \Delta N_1 = F_2 \Delta N_2$.

$F_i \in \mathcal{F}$, $N_i \subseteq E_i \subset \mathcal{F}$, $\mu(E_i) = 0$ $i=1,2$.

Then $F_1 \Delta F_2 = N_1 \Delta N_2$.

Therefore $\mu(F_1 \Delta F_2) = 0 \Rightarrow \mu(F_1) = \mu(F_2) \Rightarrow \overline{\mu}(F_1 \Delta N_1) = \overline{\mu}(F_2 \Delta N_2)$.

So the definition of $\overline{\mu}$ is well defined.
References