Abstract
In this paper we give some definitions and properties of spectral theory in fuzzy Hilbert spaces also we introduce definitions Invariant under a linear operator $T$ on fuzzy normed spaces and reduced linear operator on fuzzy Hilbert spaces and we prove theorems related to eigenvalue and eigenvectors, eigenspace in fuzzy normed. Invariant reduced in fuzzy Hilbert spaces and show relationship between them.

Keywords: fuzzy normed spaces, fuzzy Hilbert spaces, eigenvalue and eigenvectors, eigenspace in fuzzy normed, linear operator on fuzzy normed spaces


1. Introduction
The theory of a fuzzy sets was introduced by L. A. Zadeh [1] in 1965. After the pioneer work of Zadeh, many researchers have extended this concept in various branches, many other mathematicians have studied fuzzy normed space from several points of view [2], [7]. Fuzzy Hilbert spaces is an extension to the Hilbert space. The definition of a fuzzy Hilbert space has been introduced by M. Goudarzi and S. M. Vaezpour [9] in 2009.

2. Preliminaries
Definition (2.1): [3] Let $*$ be a binary operation on the set $I$, i.e. $*: I \times I \rightarrow I$ is a function. Then $*$ is said to be $t$-norm (triangular-norm) on the set $I$ if the following axioms are satisfied:

1. $a * 1 = a$, for all $a \in I$.
2. $*$ is commutative (i.e. $a * b = b * a$, for all $a, b \in I$).
3. $*$ is monotone (i.e. if $b, c \in I$ such that $b \leq c$, then $a * b \leq a * c$, for all $a \in I$).
4. $*$ is associative (i.e. $a * (b * c) = (a * b) * c$, for all $a, b, c \in I$).

If, in addition, $*$ is continuous then $*$ is called a continuous $t$-norm.

Definition (2.2): [2] Let $X$ be a vector space over $F$, $*$ be a continuous $t$-norm on $I$, a function $N: X \times (0, \infty) \rightarrow [0,1]$ is called fuzzy norm if it satisfies the following conditions: for all $x, y \in X$ and $t, s > 0$,

$(N.1) \quad N(x, t) > 0,$
$(N.2) \quad N(x, t) = 1$ if and only if $x = 0,$
$(N.3) \quad N(\alpha x, t) = N \left( x, \frac{t}{|\alpha|} \right)$, for all $\alpha \neq 0,$
$(N.4) \quad N(x, t) \ast N(y, s) \leq N(x + y, t + s),$
$(N.5) \quad N(x,.) : (0, \infty) \rightarrow [0,1]$ is continuous,
$(N.6) \quad \lim_{t \rightarrow \infty} N(x, t) = 1.$

$(X, N, *)$ is called fuzzy normed space.

Remark (2.3): [8]

1. For any $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 > \alpha_2$, there exists $\alpha_3 \in (0,1)$ such that $\alpha_1 \ast \alpha_3 \geq \alpha_2$.
2. For any $\alpha_4 \in (0,1)$, there exists $\alpha_5 \in (0,1)$ such that $\alpha_5 \ast \alpha_3 \geq \alpha_4$.

Example (2.4): [11] Let $(X, \| \|)$ be a normed space. $a * b = a \cdot b$ for all $a, b \in X$ and for all $x \in X, t > 0$

$$N(x, t) = \begin{cases} \frac{t}{x + |x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Then $(X, N, *)$ is fuzzy normed space.
**Definition (2.5)** [4]: Let $X$ be a real linear space, $\ast$ be a continuous $t$-norm on $I = [0,1]$. A function $H: X \times X \times \mathbb{R} \rightarrow [0,1]$ is called a fuzzy pre-Hilbert function if it satisfies the following axioms for every $x,y,z \in X$ and $s,t,r \in \mathbb{R}$:

Note: $h(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$

1. $H(x,x,0) = 0$ and $H(x,x,t) > 0$ for each $t > 0$
2. $H(x,x,t) \neq h(t)$ for some $t \in \mathbb{R}$ if and only if $x \neq 0$
3. $H(x,y,t) = H(y,x,t)$
4. For any real number $\alpha$:
   
   $$H(\alpha x,y,t) = \begin{cases} H(x,y,\frac{t}{\alpha}), & \alpha > 0 \\ h(t), & \alpha = 0 \\ 1 - H(x,y,\frac{t}{\alpha}), & \alpha < 0 \end{cases}$$

5. $H(x,x,t) \ast H(y,y,s) \leq H(x+y,x+y,t+s)$
6. $\sup_{s+r=1} (H(x,z,s) \ast H(y,z,r)) = H(x+y,z,t)$
7. $H(.,.,.) : \mathbb{R} \rightarrow [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$.
8. $\lim_{t \rightarrow +\infty} H(x,y,t) = 1$.

$(X,\ast,\cdot)$ is a fuzzy pre-Hilbert space.

**Example (2.6)** [4]: Let $(X,\langle \cdot \rangle)$ be an ordinary pre-Hilbert space. We define a function $H : X \times X \times \mathbb{R} \rightarrow [0,1]$ as follows:

$$H(\alpha x,y,t) = \begin{cases} \frac{t^2}{t^2 + ||(\alpha x,y)||^2}, & \alpha \geq 0, t > 0 \\ \frac{1}{t^2 + ||(\alpha x,y)||^2}, & \alpha < 0, t > 0 \\ 0, & t \leq 0 \end{cases}$$

Define $\alpha \ast \beta = \min \{ \alpha, \beta \}$ for all $\alpha, \beta \in X$. This is a fuzzy pre-Hilbert and called the standard fuzzy pre-Hilbert induced by the pre-Hilbert $(\cdot,\cdot)$.

**Definition (2.7)** [4]: Let $(X,\ast,\cdot)$ be a fuzzy pre-Hilbert space. $x,y \in X$ is said to be fuzzy orthogonal if $H(x,y,t) = h(t)(\forall t \in \mathbb{R})$ and it is denoted by $x \perp y$.

**Definition (2.8)** [4]: Let $(X,\ast,\cdot)$ be a fuzzy pre-Hilbert space. A subset $B$ of $X$ is called fuzzy orthogonal if $x \perp y$, for each $x,y \in B$.

**Lemma (2.9)** [4]: If $(X,\ast,\cdot)$ be a fuzzy pre-Hilbert space, then $(X,\ast,\cdot)$ is non decreasing with respect to $t$, for each $x,y \in X$.

**Definition (2.10)** [4]: If $B$ is a subset of the fuzzy pre-Hilbert space $(X,\ast,\cdot)$, then $B^\perp = \{ x \in X : x \perp y, \forall y \in B \}$.

**Definition (2.11)** [7]: A t-norm $\ast : [0,1] \times [0,1] \rightarrow [0,1]$ is called strong if it has the two following properties:

1. For all $a,b \in (0,1)$, $a \ast b > 0$.
2. For all $a,b,c,d \in [0,1]$ and $a > b, c > d$ we have $a \ast b > c \ast d$.

**Theorem (2.12)** [4]: Suppose that $(X,\ast,\cdot)$ be a fuzzy pre-Hilbert space, where $\ast$ is a strong $t$-norm and for each $x,y \in X$,

$$\sup \{ t \in \mathbb{R} : H(x,y,t) < 1 \} < \infty.$$ 

Define $(\cdot,\cdot) : X \times X \rightarrow \mathbb{R}$ by

$$\langle x,y \rangle = \sup \{ t \in \mathbb{R} : H(x,y,t) < 1 \}.$$ 

Then $(X,\langle \cdot,\cdot \rangle)$ is a pre-Hilbert space.

**Corollary (2.13)** [4]: Let $(X,\ast,\cdot)$ be a fuzzy pre-Hilbert space, where $\ast$ is a strong $t$-norm and for each $x,y \in X$, $\sup \{ t \in \mathbb{R} : H(x,y,t) < 1 \} < \infty$. If we define $||x|| = (\sup \{ t \in \mathbb{R} : H(x,x,t) < 1 \})^{\frac{1}{2}}$, then $(X,||\cdot||)$ is a normed space.

**Definition (2.14)** [4]: Let $(X,\ast,\cdot)$ be a fuzzy pre-Hilbert space, where $\ast$ is a strong $t$-norm and for each $x,y \in X$, $\sup \{ t \in \mathbb{R} : H(x,y,t) < 1 \} < \infty$ and $||x|| = (\sup \{ t \in \mathbb{R} : H(x,x,t) < 1 \})^{\frac{1}{2}}$. We say that $(X,\ast,\cdot)$ is a fuzzy Hilbert space if $(X,||\cdot||)$ is complete normed space.
Theorem (2.15) : [6] Let \((X, H,\ast)\) be a fuzzy pre-Hilbert space. And \(A \subseteq X;\)
(1) The relation of Orthogonality symmetric (i.e. if \(x \perp y \rightarrow y \perp x\))
(2) If \(x \perp y \rightarrow ax \perp y \forall t \in \mathbb{R}\)
(3) Let \(A \subseteq B \rightarrow B^{\perp} \subseteq A^{\perp}\)
(4) \(A \subseteq A^{\perp}\)
(5) \(A \subseteq B^{\perp} \rightarrow B \subseteq A^{\perp}\)
(6) If \(x \perp x \rightarrow x = 0 \forall t \in \mathbb{R}\)
(7) \(X^{\perp} = \{0\}\) for all \(t \in \mathbb{R}\)
(8) \(A \cap A^{\perp} = \{0\}\) for all \(t \in \mathbb{R}\)
(9) For every vector \(x \in X\), we have \(0 \perp x \forall x \in X\)
3. Eigenvalue and Eigenvector In Fuzzy Normed Spaces.
Definition (3.1) : [5] A function \(T : X \rightarrow Y\) is called an operator from \(X\) into \(Y\) if \(X\) and \(Y\) are linear spaces over the same field \(F\).
Definition (3.2) : [5] A linear operator \(T\) is an operator such that \(T(ax + \beta y) = \alpha T(x) + \beta T(y)\) for all \(x, y \in X\) and for all \(\alpha, \beta \in F\).
Definition (3.3) : Let \((X, N,\ast)\) be a fuzzy normed spaces over \(F\) and \(T \in L(X)\) then
(1) A scalar \(\lambda \in F\) is called an eigenvalue of \(T\), if there exists non zero \(x \in X\) such that \(T(x) = \lambda x\)
(2) Let \(x\) be an eigenvector of \(T\) corresponding to eigenvalue \(\lambda \Rightarrow T(x) = \lambda x\)
Example (3.4) : Let \(X = \mathbb{R}^2\) and \(T : (X, N,\ast) \rightarrow (X, N,\ast)\) Define
By \(T(x, y) = (\ast y, x)\) for all \((x, y) \in \mathbb{R}^2\) and \(N : \mathbb{R}^2 \times (0, \infty) \rightarrow [0, 1]\)
Define fuzzy norm in example(2.4) and \(T\) is linear operator has no eigenvalue
Example (3.5) : Let \(X = \mathbb{R}^2\) and \(T : (X, N,\ast) \rightarrow (X, N,\ast)\) Define
By \(T(x, y) = (x + 2y, 3x + 2y)\) for all \((x, y) \in \mathbb{R}^2\) and \(N : \mathbb{R}^2 \times (0, \infty) \rightarrow [0, 1]\)
Define fuzzy norm in example(2.4) and \(T\) is linear operator has eigenvalue \(\lambda = -1, \lambda = 4\)
Theorem (3.6) : Let \((X, N,\ast)\) be a fuzzy normed spaces over \(F\) and \(T \in L(X)\) if \(x\) one eigenvector of \(T\)
then \(\lambda x\) is also an eigenvector of \(T\) corresponding to the same eigenvalue \(\lambda\)
proof: since \(x\) is an eigenvector of \(T\) corresponding to the eigenvalue \(\lambda\) then \(x \neq 0\). \(T(x) = \lambda x\) since \(x \neq 0\) and \(\alpha \neq 0 \Rightarrow \alpha x \neq 0\)
\(T(ax) = \alpha T(x) = \alpha(\lambda x) = (\alpha\lambda)x = \lambda(\alpha x)\)
therefore \(\alpha x\) is an eigenvector of \(T\) corresponding to the eigenvalue \(\lambda\)
Remark(3.7) : Corresponding to an eigenvalue \(\lambda\) there may correspond more than one eigenvectors.
Theorem (3.8) : Let \((X, N,\ast)\) be a fuzzy normed spaces over \(F\) and \(T \in L(X)\) if \(x\) one eigenvector of \(T\), then \(x\) cannot correspond to more than one eigenvalues of \(T\)
proof: Let be an eigenvector of \(T\) corresponding to two distinct eigenvalues \(\lambda_1\) and \(\lambda_2\) of \(T\), \(T(x) = \lambda_1 x\) and also \(T(x) = \lambda_2 x\) therefore we have \(\lambda_1 x = \lambda_2 x\) \(\Rightarrow \lambda_1 x - \lambda_2 x = 0\) \(\Rightarrow (\lambda_1 - \lambda_2)x = 0\)
since \(x \neq 0\) \(\Rightarrow \lambda_1 - \lambda_2 = 0\) \(\Rightarrow \lambda_1 = \lambda_2\)
and \(\alpha\) is any non-zero scalar, then \(\alpha x\) is also an eigenvector of \(T\) corresponding to the same eigenvalue \(\lambda\)
Definition (3.9) : [10] Let \((X, N_1,\ast)\) and \((Y, N_2,\ast)\) be a fuzzy normed spaces. A linear operator
\(T : (X, N_1,\ast) \rightarrow (Y, N_2,\ast)\) is said to be fuzzy bounded if and only if there exists \(r > 0\), such that for each \(t > 0\)
\(N_2(T(x), t) \geq N_1 \left( x, \frac{1}{r} \right) , \forall x \in X\)
Remark (3.10) : [13] Let \((X, N_1,\ast)\) and \((Y, N_2,\ast)\) be a fuzzy normed spaces over \(F\). \(FB(X,Y)\) is the space of all fuzzy bounded linear operator from \(X\) to \(Y\).
Definition (3.11) : [13] Let \((X, H,\ast)\) and \((Y, H,\ast)\) be a fuzzy Hilbert spaces over \(F\), and let \(T \in FB(X,Y)\). A fuzzy Hilbert-adjoint operator \(T^*\) of \(T\) is the operator
\(T^* : (Y, H,\ast) \rightarrow (X, H,\ast)\) such that:
\(\sup \{ t \in \mathbb{R}, H(T(x), y, t) < 1 \} = \sup \{ t \in \mathbb{R}, H(x, T^*(y), t) < 1 \}\) for all \(x \in X\) and \(y \in Y\).
Remark (3.12) : [13] We denoted \(FB(X,Y)\) by \(FB(X)\).
Theorem (3.13) : [13] (Some Properties of fuzzy Hilbert-adjoint operator)
Let \((X, H,\ast)\) and \((Y, H,\ast)\) be a fuzzy Hilbert spaces over \(F\), and let \(S \in FB(X,Y)\). Then we have:
(a) \( \sup \{ t \in \mathbb{R}, H(T^*(y), x, t) < 1 \} = \sup \{ t \in \mathbb{R}, H(y, T(x), t) < 1 \} \) for all \( x \in X \) and \( y \in Y \)

(b) \((T + S)^* = T^* + S^*\)

(c) \((T^*)^* = T\)

**Definition (3.14)** : [13] Let \((X, H, \cdot)\) be a fuzzy Hilbert space over \(F\) and let \(T \in B(X)\). \(T\) is said to be Normal if \(T \circ T^* = T^* \circ T\).

**Theorem (3.15)** : [13] Let \((X, H, \cdot)\) be a fuzzy Hilbert space over \(F\), and \(T \in F B(X)\). Then \(T = 0\) if and only if \(\sup\{t \in \mathbb{R}, H(T(x), T(x), t) < 1\} = 0\) for all \(x \in X\).

**Theorem (3.16)** : Let \(T\) be a normal operator on a Fuzzy finite dimensional Hilbert space \(X\) over \(F\) then

1. \(T - \lambda I\) is normal
2. Every eigenvector of \(T\) is also eigenvector for \(T^*\)

**Proof:**

1. Since \(T\) is normal \(\Rightarrow T \circ T^* = T^* \circ T\)

\( (T - \lambda I) = T^* - \lambda I  \)

\( = T \circ T^* - \lambda T^* + \lambda I  \)

\( = T^* \circ T - \lambda T^* + \lambda I  \) \(\Rightarrow (T - \lambda I)^* = (T - \lambda I)\)

Therefore \(T - \lambda I\) is normal

\(\sup\{t \in \mathbb{R}, H(T(x), T(x), t) < 1\}

= \sup\{t \in \mathbb{R}, H(x, T^*(T(x)), t) < 1\}

= \sup\{t \in \mathbb{R}, H(x, T \circ T^*(x), t) < 1\}

= \sup\{t \in \mathbb{R}, H(x, T(T^*(x)), t) < 1\}

\(= \sup\{t \in \mathbb{R}, H(T^*(x), T^*(x), t) < 1\}

Since \(T - \lambda I\) is normal, therefore \(x \in X\) we have

\(\sup\{t \in \mathbb{R}, H(T^*(x), T^*(x), t) < 1\} \), \(\sup\{t \in \mathbb{R}, H((T - \lambda I)^*(x), (T - \lambda I)^*(x), t) < 1\}\)

Since \(T(x) = \lambda x \Rightarrow T(x) - \lambda I(x) = 0 \Rightarrow (T - \lambda I)(x) = 0 \Rightarrow (T - \lambda I)(x) = 0 \) then by theorem (3.15)

\(\sup\{t \in \mathbb{R}, H(T^*(y), x, t) < 1\} \Rightarrow \sup\{t \in \mathbb{R}, H((T - \lambda I)^*(x), (T - \lambda I)^*(x), t) < 1\} \Rightarrow (T - \lambda I)' = 0\) by theorem (3.15), for each \(x \in X\)

\(\Rightarrow T^*(x) = \lambda I(x) \Rightarrow T^*(x) = \lambda I(x) \Rightarrow T^*(x) = \lambda x\)

Therefore \(x\) is eigenvector of \(T^*\) and corresponding eigenvalue is \(\lambda\).

**Theorem (3.17)** : [6] Let \(B\) be a non-empty subset of a fuzzy pre-Hilbert space \((X, H, \cdot)\), then \(B^\perp\) is closed fuzzy subspace

**Proof:** Since \(H(0, y, t) = h(t)\), \(\forall y \in B\Rightarrow 0 \in B^\perp\) then \(B^\perp \neq \emptyset\)

Let \(x, y \in B^\perp\) and \(\alpha, \beta, r \in \mathbb{R}\)

\(H(x, z, t) = h(r)\) \(\forall z \in B\)

For every \(\forall z \in B\) we have:

If \(\alpha > 0, \beta > 0\)

\(H(\alpha x + \beta y, z, t) = \sup_{s + t = r}\left(H\left(x, z, \frac{1}{\alpha}\right) + H\left(y, z, \frac{1}{\beta}\right)\right)\)

= \(\frac{1}{\alpha}\) * \(\frac{1}{\beta}\) = \(h(r)\) \(\forall r \in \mathbb{R}\)

If \(\alpha < 0, \beta < 0\)

\(H(\alpha x + \beta y, z, t) = \sup_{s + t = r}\left(1 - H\left(x, z, \frac{1}{\alpha}\right) + 1 - H\left(y, z, \frac{1}{\beta}\right)\right)\)

= \(h(r)\) \(\forall r \in \mathbb{R}\)

Therefore \(B^\perp\) is a fuzzy subspace

Let \(x \in B^\perp\) \(\exists x_n\) in \(B^\perp\) such that \(x_n \to x\)

Let \(y \in B\Rightarrow H(x_n, y, t) = h(t)\) \(\forall n \in \mathbb{N}^+\)

And \(t \in \mathbb{R}\) \(x_n \in B^\perp \forall n \in \mathbb{N}^+\)

Since \(x_n \to x\Rightarrow H(x_n, y, t) = H(x, y, t)\)

\(\Rightarrow H(x, y, t) = h(t)\) for all \(y \in B\)

\(\Rightarrow x \in B^\perp\Rightarrow \overline{B^\perp} = B^\perp\)

\(B^\perp\) is closed fuzzy subspace

**Definition (3.18)** : Let \(M\) be a closed of a fuzzy Hilbert space \(X\) and \(x \in M\) said that projection of \(x \in X\) onto \(M\) if there is \(z \in M\)

\(N(x, z, t) = \sup \left\{ \frac{t}{||x - y||} : y \in M, t > 0 \right\}\), we write \(y = P_M(x)\)
Theorem (3.19): If M is subspace of a fuzzy Hilbert space X, for x ∈ X there exist a unique y ∈ M such that x = y ⊥ M and y = p_M(x)
Proof: Define (,.) : X × X → ᵃ by (x, y) = sup t ∈ ᵃ, H(x, y, t) < 1}. From theorem (2.12), we have (X, (,.) ) is a pre-Hilbert space. Also
‖x‖ = (sup t ∈ ᵃ, H(x, x, t) < 1)½ from corollary (2.13) we have (X, || ||) is a normed space since X is a fuzzy Hilbert space then (X, || ||) is complete normed space then X is Hilbert space then by using [12] for x ∈ X there exist a unique y ∈ M such that x = y ⊥ M and y = p_M(x)
Then (x, y, z) = 0 ∀x, y ∈ M then sup{t ∈ ᵃ, H(x, y, t) < 1} = 0 there for x = y ⊥ M ∀x, y ∈ M in X fuzzy Hilbert space, since y = p_M(x) then by [12] there is b ∈ M such that
‖x - b‖ = inf ||x - y||: y ∈ M then
‖x - b‖ ≤ ‖x - y‖, y ∈ M = t + ‖x - y‖ ≤ t + ‖x - y‖, t > 0 ⇒ ‖x - y‖ ≤ ‖x - b‖ ≤ t + ‖x - y‖, t > 0
there fore N(x - b, t) = sup{t ∈ ᵃ, ||x - y|| ≤ ‖x - b‖: y ∈ M, t > 0}

Theorem (3.20): If M is subspace of a fuzzy Hilbert space X then X = M ⊕ M⊥, that is each x ∈ X can be uniquely decomposed from x = y + z with y ∈ M, z ∈ M⊥
Proof: For all x ∈ X and M is subspace then exist y so that x = y ⊥ M with x = y ∈ M⊥ and y ∈ M such that y = p_M(x) and z = x - y ⇒ x = y + z ⇒ X = M + M⊥ also since M ∩ M⊥ = {0} by theorem (2.15), there fore X = M ⊕ M⊥

Theorem (3.21): If M is subspace of a fuzzy Hilbert space X, then M is fuzzy closed iff M = M⊥⊥
Proof: Since M ⊂ M⊥⊥ by theorem (2.15), we show that M ⊥⊥ ⊂ M
Let x ∈ M⊥⊥ then by theorem(3.20) x = y + z ,where y ∈ M, z ∈ M⊥ since
M ⊂ M⊥⊥ and M ⊥⊥ is subspace z = x - y M⊥⊥ but z ∈ M⊥ ⇒ z ∈ M⊥⊥ ∩ M⊥⊥ since M⊥⊥ ∩ M⊥⊥ = {0} then z = 0 , thus x = y ∈ M there fore M ⊥⊥ ⊂ M thus M = M⊥⊥
Conversely suppose M = M⊥⊥ since (M⊥⊥)⊥ = M⊥⊥ is close set then M is close set.

Theorem (3.22): Let M be a closed subspace of a fuzzy Hilbert space X over F, and let T ∈ FB(X). Then M is invariant under T iff M = M⊥⊥ is invariant under T⊥⊥

Proof: Suppose M is invariant under T
Let y ∈ M⊥⊥. To prove that T⊥⊥(y) ∈ M⊥⊥(i.e. T⊥⊥(y) ⊂ M)
Let x ∈ M, since M is invariant under T ⇒ T(x) ⊂ M
Since y ∈ M⊥⊥ ⇒ sup t ∈ ᵃ, H(T(x), y, t) < 1 = 0
sup t ∈ ᵃ, H(T(x), y, t) < 1 = 0 .Thus
T⊥⊥(y) ⊂ M

Conversely suppose M⊥⊥ is invariant under T⊥⊥.
Since M⊥⊥ is closed subspace of a fuzzy Hilbert space X by theorem (3.17) and since M⊥⊥ is invariant under T⊥⊥, therefore by first case (M⊥⊥)⊥⊥ is invariant under T⊥⊥ but (M⊥⊥)⊥⊥ = M⊥⊥ = M and (T⊥⊥)⊥⊥ = T⊥⊥ Therefore M⊥⊥ is invariant under T

Definition (3.23): Let M be a closed subspace of a fuzzy Hilbert space X over F. Then M reduces an operator T iff M is invariant under both T and T⊥⊥

Proof: Suppose M reduces .Then by the definition of reducibility both M and M⊥⊥ are invariant under T by theorem (3.22). If M⊥⊥ is invariant under T
Then (M⊥⊥)⊥⊥, i.e. M is invariant T⊥⊥ then M is invariant under both T and T⊥⊥
Conversely suppose that M is invariant under both T and T⊥⊥
Since M is invariant under T⊥⊥, therefore by theorem (3.22), M⊥⊥ is invariant under (T⊥⊥)⊥⊥, i.e. T⊥⊥, thus both M and M⊥⊥ are invariant under T⊥⊥ therefore M reduces T

Definition (3.25): Let X be a fuzzy normed space over F, T ∈ FB(X) and λ be eigenvalue of T then set consisting of all eigenvectors of T which correspond to eigenvalue λ together with the vector 0 is called eigenspace of T corresponding to the eigenvalue λ and is denoted by M_λ
(1)Since by definition an eigenvector is non zero vector, there fore the set M_λ necessary contains some non zero vector
(2)Since by definition of M_λ a non zero vector x is in M_λ iff T(x) = λx
Also it is given that the vector 0 is in M_λ the vector 0 definitely satisfies
The equation T(x) = λx there for
M_λ = {x ∈ X : T(x) = λx} = {x ∈ X : (T - λI)(x) = 0}
Thus M_λ is null space (or kernel) of linear operator T - λI on X. Hence M_λ is a subspace of X
Let $x \in X$ since $M_\lambda$ is a subspace of $X$ and $\lambda \in F$ 
\[ \Rightarrow \lambda x \in M_\lambda \] 
since $M_\lambda$ is a subspace of $X$ and $\lambda F$ is a scalar field.
\[ x \in M_\lambda \Rightarrow T(x) = \lambda x \Rightarrow T(x) \in M_\lambda \Rightarrow M_\lambda \] is an invariant under $T$.

from (1),(2) and (3) we have $M_\lambda$ is non zero subspace of $X$ invariant under of $T$.

(4)If $T \in FB(X)$ then $M_\lambda$ is closed subspace of $X$, $M_\lambda$ is called eigenspace of $T$ corresponding to the eigenvalue $\lambda$.

Theorem (3.26): If $T$ be a normal operator on $n$ dimensional fuzzy Hilbert Space $X$ over $F$, then each eigenspace reduces $T$.

Proof: Let $x_i$ belong to $M_i$ the eigenspace of $T$ and corresponding eigenvalue be $\lambda_i$, so that $T(x_i) = \lambda_i x_i$ since $T$ is normal then by theorem(3.16) eigenvalue for $T^*$ (i.e. $T^*(x_i) = \overline{\lambda_i} x_i$) since $M_i$ is a subspace 
\[ \overline{\lambda_i} x_i \in M_i \Rightarrow T^*(x_i) \in M_i \Rightarrow M_i \] is invariant under $T^*$ but $M_i$ is invariant under $T$ then by theorem(3.24) $M_i$ is reduces $T$.

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لوحة الفضاء الذاتي

المستخلص:
في هذا البحث قدمنا بعض التعريف المتعلقة بنظرية الطيفية للمؤثر الخطي T المعرف على الفضاء معيار الضبابي الذاتي. وكما سترهم بعض الحقائق المتعلقة بقيم الذاتية والمتجه الذاتي في فضاء هلبرت الضبابي T إذا كان T مؤثر خطي سوي. وكما نبرهن كل فضاء ذاتي في فضاء هلبرت الضبابي ذو البعد n يختزل المؤثر الخطي T excellently.