On approximation $f$ by $(\alpha, \beta, \gamma)$-Baskakov Operators

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Abstract:
In the present paper, we study some application properties of the approximation for the sequences $M_{n,\gamma}^{\alpha,\beta}(f; x)$ and $B_{n,\gamma}^{\alpha,\beta}(f; x)$. These sequences depend on the arbitrary (but fixed) parameters $\alpha, \beta$ and $\gamma$. Here, we study the effect of these parameters on tends speed of the two families of operators $M_{n,\gamma}^{\alpha,\beta}(f; x)$ and $B_{n,\gamma}^{\alpha,\beta}(f; x)$ and the CPU times which are occurring on the approximation by a choosing fixed $n$.

Key word: Korovkins’ conditions, $(\alpha, \beta, \gamma)$-Baskakov Operators, $(\alpha, \beta, \gamma)$- Baskakov Kantorovich operators.

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1- Introduction
The classical Baskakov operators $(L_n)$ of bounded continuous functions $f(x)$ on the interval $[0, \infty)$, which defined as: [3]

Suppose that

$$p_{n,k}(x) = (-1)^k \sum_{k=0}^{\infty} \frac{x^k}{k!} \varphi_n^{(k)}(x),$$

The $n$-th order of classical Baskakov is defined as:

$$(L_n f)(x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{x}{n}\right), \quad \left(1.1\right)$$

where $n \in \mathbb{N}, x \in [0, b], b > 0$.

The article proved the Korovkins’ conditions for the convergence of Baskakov operators. [4]

Berens and Suzuki were studied the classes for continuous functions with compact support and getting some results concerning bounded continuous functions. [8], [9]

Bernstein polynomials and Szasz-Mirakian operators are the especial cases of Baskakov operators considered by May. [7]

In recent years, some applications had been done for sequences of linear positive operators by use Maple programs.

Sharma was studied the rate of convergence of q-Durrmeyer operators and he used maple programming to describe the approximation for two sequences of operators. [5]

Mursaleen and Asif khan, they studied approximation properties of q-Bernstein–Shurer operators and they found the error estimate. In addition, they proved graphically the convergence for $f$ by these operators. [6]

Gupta introduced and studied a generalization of the Baskakov–Durrmeyer operators. This generalization are defined as:

For $x \in [0, \infty), \gamma =1$,

$$B_{n,\gamma}(f; x) = \sum_{k=0}^{\infty} p_{n,k,\gamma}(x) \int_{0}^{\infty} b_{n,k,\gamma}(t) f(t) dt + P_{n,0,\gamma}(x)f(0)$$

where $P_{n,k,\gamma}(x)$ and $b_{n,k,\gamma}(t)$ as defined as:

$$P_{n,k,\gamma}(x) = \frac{\gamma}{\gamma + (k+1) r \left(\frac{1}{r+1}\right)} \left(\frac{x^k}{\eta \left(1 + \eta x\right) \left(k+1\right)^{\eta x}}\right)$$

$$b_{n,k,\gamma}(t) = \frac{\gamma}{\gamma + (k+1) r \left(\frac{1}{r+1}\right)} \left(\frac{t^k}{\eta \left(1 + \eta \frac{t}{\eta}\right) \left(k+1\right)^{\eta t}}\right)$$

(1.2)

Then, he introduced modification of Baskakov operators using weight functions of Bate base functions depend of parameter $\gamma$, and getting some results concerning Baskakov operators from them approximation theorem, rate of convergence, weighted approximation theorem. [1], [2]
We define \((\alpha, \beta, \gamma)\)- Baskakov operators
\(M_{n,y}^{\alpha,\beta}(f; x)\) in this research, we prove the Korovkin conditions for the operators
\(M_{n,y}^{\alpha,\beta}(f; x)\) and
\(B_{n,y}^{\alpha,\beta}(f; x)\).

In this paper is an application study to the sequences \(M_{n,y}^{\alpha,\beta}(\cdot; x)\), \(B_{n,y}^{\alpha,\beta}(\cdot; x)\) and \(L_n(f, x)\) on the two test function \(f(x) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{16} x\), \(f(t) = \sin(10t) \exp(-3t) + 0.3\) to show that the effect of the parameters \((\alpha, \beta, \gamma)\) in the sequences \(M_{n,y}^{\alpha,\beta}(\cdot; x)\), \(B_{n,y}^{\alpha,\beta}(\cdot; x)\) on the tends speed of approximation. The results which are done are describe by the graphs of the test function and the approximations of the sequences \(M_{n,y}^{\alpha,\beta}(\cdot; x)\), \(B_{n,y}^{\alpha,\beta}(\cdot; x)\) and \(L_n(f, x)\). In addition, we give some tables of the CPU time which are occurring on the approximation of the test function by a choosing fixed \(n\).

2- Construction of the Operators \(\{M_{n,y}^{\alpha,\beta}(f, x)\}\)

In this part, we introduce the operators \(M_{n,y}^{\alpha,\beta}(f, x)\) and state some of their properties.

Definition 2-1

Let \(f \in [0,1]\), \(x \in [0, \infty)\), \(n \in N = \{0, 1, 2, \ldots\}\) for some \(0 \leq \alpha \leq \beta\) and \(n \in N = \{1, 2, \ldots\}\). The \((\alpha, \beta, \gamma)\)-Baskakov Operators in special case i.e. \(\gamma = 1\), \(\alpha = \beta = 0\) is reduce to the operators (1.1).

The will known \((\alpha, \beta, \gamma)\)- Baskakov operators \(M_{n,y}^{\alpha,\beta}, (\alpha, \beta, \gamma)\)- Baskakov Kantorovich operators \(B_{n,y}^{\alpha,\beta}\) with two parameters \(\alpha\) and \(\beta\) with \(0 \leq \alpha \leq \beta\) on two test function \(f(x)\) and investigated convergence and approximation properties of these operators, such as defined:

\[M_{n,y}^{\alpha,\beta}(f(t), x) = \sum_{k=0}^{n} P_{n,k,y}(x) (k+\alpha) f\left(\frac{\alpha}{n+\beta}\right)\]  
\[
B_{n,y}^{\alpha,\beta}(f(t); x) = n \sum_{k=0}^{n} P_{n,k,y} \frac{k+1}{n} f(t) dt
\]

Where

\[P_{n,k,y}(x) = \frac{\binom{n+k}{k} y^k}{\binom{n}{k} (1+y)^{n+k}}, \quad \int_{0}^{x} f(t) e^{-(n+\beta) t} dt = \left[\frac{f(x)}{(n+\beta)}\right]^{(n+\beta)}
\]

\[f(x) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{16} x \]  
\[f(t) = \sin(10t) \exp(-3t) + 0.3
\]

The following theorem help us to study the Korovkin conditions for convergence for two operators \(M_{n,y}^{\alpha,\beta}, B_{n,y}^{\alpha,\beta}\).

Theorem (2-1) (Korovkin Theorem):

For \(x \in [0, \infty), f \in [0,1]\) and by applying Korovkin Theorem on the operator \(M_{n,y}^{\alpha,\beta}(f, x)\), we have:

1. \(M_{n,y}^{\alpha,\beta}(1; x) = 1\)
2. \(M_{n,y}^{\alpha,\beta}(t; x) = \frac{nx}{n+\beta} + \frac{\alpha}{n+\beta}\)
3. \(M_{n,y}^{\alpha,\beta}(t^2; x) = \frac{n^2x^2}{(n+\beta)^2} + \frac{1+2\alpha}{(n+\beta)^2} (nx) + \frac{\alpha^2}{(n+\beta)^2} \)
4. \(M_{n,y}^{\alpha,\beta}(t^m; x) = \frac{n^m x^m}{(n+\beta)^m} + \frac{m(m-1)+2m}{2(n+\beta)^m} \{n^{m-1}x^{m-1}\} + \frac{m^m x^m}{(n+\beta)^m}

Proof:

The operators \(M_{n,y}^{\alpha,\beta}\) are well define on the function \(1, t, t^2, t^m\) we obtain.

1. \(M_{n,y}^{\alpha,\beta}(1; x) = \sum_{k=0}^{n} P_{n,k,y}(x) = 1\)
2. \(B_{n,y}^{\alpha,\beta}(t; x) = \sum_{k=0}^{\infty} P_{n,k,y}(x) \frac{k+\alpha}{n+\beta}\)
3. \(M_{n,y}^{\alpha,\beta}(t^2; x) = \sum_{k=0}^{\infty} P_{n,k,y}(x) f\left(\frac{k+\alpha}{n+\beta}\right)^2
\]
4. \(M_{n,y}^{\alpha,\beta}(t^m; x) = \sum_{k=0}^{\infty} P_{n,k,y}(x) f\left(\frac{k+\alpha}{n+\beta}\right)^m
\]
\[ = \frac{1}{(n+\beta)^m} \left( \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} \right) k^m + \frac{\alpha m}{(n+\beta)^m} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} k^{m-1} + \frac{T. L. P(x)}{n+\beta} \]
\[ = \frac{2nx + \frac{1}{2n} + 1}{2n} \rightarrow x \text{ as } n \rightarrow \infty \]

3. \[ B_{n,y}^{\alpha,\beta} (t^2, x) = \frac{n}{n+\beta} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} \left( \frac{k+1}{n} \right)^2 t^2 dt \]
\[ = \frac{n}{3n^2} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} \{ (k+1)^2 - k^2 \} \]
\[ = \frac{n}{3n^2} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} \{ 3k^2 + 3k + 1 \} \]
\[ = \frac{1}{n^2} \left( n^2 x^2 + y^2 + x + \frac{1}{n^2} \right) \rightarrow x^2 \text{ as } n \rightarrow \infty \]

4. \[ B_{n,y}^{\alpha,\beta} (t^m, x) = \frac{n}{n+\beta} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} \left( \frac{k+1}{n} \right)^m t^m dt \]
\[ = \frac{n}{m+1} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} \{ (k+1)^m - k^m \} \]
\[ = \frac{1}{m+1} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} \{ m^m + (m+1)k^m + \ldots + (m+1)k + 1 - k^m \} \]
\[ = \frac{1}{m+1} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} k^m + \frac{m}{2n^m} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} k^{m-1} + \ldots + \frac{1}{m+1} \sum_{k=0}^{\infty} P_{n,k,y}^{(x)} k + \frac{1}{n^m} \]
\[ B_{n,y}^{\alpha,\beta} (t^m, x) = x^m + \frac{m^2}{2n} x^{m-1} + T. L. P(x) + \frac{1}{(m+1)n^m} \]

3- Numerical Example

Here, we give a numerical example for the approximation of operators \( M_{n,y}^{\alpha,\beta}(f, x) \) for different values of the parameters \( \alpha, \beta, \gamma \) by take the two test functions on \([0, 1]\).

\[ f(x) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{16} x \]

(2.3)

\[ f(t) = \sin(10t) \exp(-3t) + 0.3 \]

(2.4)
Figure (3.1)
Approximation test function $f(x)$ by $M_{n,y}^{\alpha,\beta}(f, x)$ for $n = 50$
Figure 3.1, explains the tends speed of the operators \( M_{n,y}^{\alpha,\beta}(f,x) \) by first test function (2.3), when the values \( n=50, \gamma = 1 \) fixed, such as if \( n \) increases tends speed of \( M_{n,y}^{\alpha,\beta}(f,x) \) will fail in application, and take variance values of the \( \alpha,\beta \), such that \( 0 \leq \alpha \leq \beta \) we get the best tends speed by \( M_{n,y}^{\alpha,\beta}(f,x) \) to approximating the test function when \( \alpha = 0.5, \beta = 1 \) and \( \gamma = 1 \). In addition, the \( M_{n,y}^{\alpha,\beta}(f,x) \) operators is returns to the classical operators \( L_{n}(f,x) \) when \( \gamma = 1, \alpha = 0, \beta = 0 \).

### The CPU time

The following table is explain the CPU time for the operators \( M_{n,y}^{\alpha,\beta}(f,x), L_{n}(f,x) \) by test function (2.3), where \( n=50 \). We found the best CPU time introduced by \( L_{n}(f,x) \) by using the same test function \( f \).

<table>
<thead>
<tr>
<th>The sequence</th>
<th>( \gamma )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{n,y}^{\alpha,\beta}(f,x) )</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>12.12s</td>
</tr>
<tr>
<td>( L_{n}(f,x) )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>11.07s</td>
</tr>
</tbody>
</table>

Table (3.1) 
Explains the CPU time for \( n = 50 \)
Figure 3.2 Approximation test function by \( B_{n,\gamma}^{\alpha,\beta}(f, x) \) and \( M_{n,\gamma}^{\alpha,\beta}(f, x) \) for Hanadi A.

- \( n = 100, \gamma = 1, \alpha = 0.25, \beta = 0.50 \)
- \( n = 100, \gamma = 1, \alpha = 0.50, \beta = 0.75 \)
- \( n = 100, \gamma = 1, \alpha = 0.80, \beta = 1 \)
- \( n = 100, \gamma = 1, \alpha = 1, \beta = 1 \)
Figure 3.2 explains the tends speed of \((\alpha,\beta,\gamma)\)-Baskakov operators \(M_{n,\gamma}^{\alpha,\beta}\) with \((\alpha,\beta,\gamma)\)-Baskakov Kantorovich operators \(B_{n,\gamma}^{\alpha,\beta}\) by first test function (2.3), when take the values \(n = 100, \gamma = 1\) and take variance values of the \(\alpha,\beta\), such that \(0 \leq \alpha \leq \beta\) we get the best case is \(\alpha = 1\) and \(\beta = 1\).

3-2 The CPU time

The following table is explain the CPU time for the operators \(M_{n,\gamma}^{\alpha,\beta}(f, x)\), \(B_{n,\gamma}^{\alpha,\beta}(f, x)\) where \(n = 100\). We found the best CPU time introduced by \(B_{n,\gamma}^{\alpha,\beta}(f, x)\) by using the same test function \(f\).

Table (3.2)
Explain the CPU time for \(n = 100\)

<table>
<thead>
<tr>
<th>The sequence</th>
<th>(\gamma)</th>
<th>A</th>
<th>B</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_{n,\gamma}^{\alpha,\beta}(f, x))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>31.26S</td>
</tr>
<tr>
<td>(B_{n,\gamma}^{\alpha,\beta}(f, x))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>28.48S</td>
</tr>
</tbody>
</table>
Now we will test the second function (2.4) on the same two sequence of operators with the same steps as above.

Figure (3.3)
Approximation $f(x)$ by $M_{n,\gamma}^{a,\beta}(f, x)$ for $n = 50$
3.3 The CPU time: The following table is explain the CPU time for the operators $M_{n,y}^{\alpha,\beta}(f,x)$, $L_n(f,x)$ by test function (2.4), where $n=50$.

Table (3.3)
Explains the CPU time for $n = 50$

<table>
<thead>
<tr>
<th>The sequence</th>
<th>$\gamma$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{n,y}^{\alpha,\beta}(f,x)$</td>
<td>1</td>
<td>0.5</td>
<td>1</td>
<td>4.71s</td>
</tr>
<tr>
<td>$L_n(f,x)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4.78s</td>
</tr>
</tbody>
</table>
Figure 3.4
Approximation test function $f(x)$ by $M_{n,y}^{\alpha,\beta}(f, x)$ and $B_{n,y}^{\alpha,\beta}(f, x)$ for $n = 100$
3-4 The CPU time

The following table is explain the CPU time for the operators $M_{n,y}^{a,b}(f,x), B_{n,y}^{a,b}(f,x)$ by test function(2.4), where $n = 100$. We found the best CPU time introduced by $M_{n,y}^{a,b}(f,x)$ by using the same test function $f$.

<table>
<thead>
<tr>
<th>The sequence</th>
<th>$\gamma$</th>
<th>$\alpha$</th>
<th>$B$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{n,y}^{a,b}(f,x)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4.45S</td>
</tr>
<tr>
<td>$B_{n,y}^{a,b}(f,x)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>19.01S</td>
</tr>
</tbody>
</table>

Table (3.4)
Explains the CPU time for $n = 100$

4- Comparing Between Test Functions

<table>
<thead>
<tr>
<th>Test function</th>
<th>The operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test function (2.3)</td>
<td>$M_{n,y}^{a,b}(f(t),x) = \sum_{k=0}^{\infty} P_{n,k,y}(x) f\left(\frac{k + \alpha}{n + \beta}\right)$</td>
</tr>
<tr>
<td>Test function (2.4)</td>
<td>$M_{n,y}^{a,b}(f(t),x) = \sum_{k=0}^{900} P_{n,k,y}(x) f\left(\frac{k + \alpha}{n + \beta}\right)$</td>
</tr>
<tr>
<td>Test function (2.3)</td>
<td>$B_{n,y}^{a,b}(f(t);x) = n \sum_{k=0}^{\infty} P_{n,k,y} \int_{k/n}^{(k+1)/n} f(t) dt$</td>
</tr>
<tr>
<td>Test function (2.4)</td>
<td>$B_{n,y}^{a,b}(f(t);x) = n \sum_{k=0}^{900} P_{n,k,y} \int_{k/n}^{(k+1)/n} f(t) dt$</td>
</tr>
<tr>
<td>Test function (2.4)</td>
<td>The best tends speed of $M_{n,y}^{a,b}(f(t),x)$</td>
</tr>
<tr>
<td>Test function (2.4)</td>
<td>The best CPU time for $M_{n,y}^{a,b}(f(t),x)$, where $n = 100$</td>
</tr>
</tbody>
</table>
5- Conclusions

In this paper, we defined the sequence of a linear positive operators \( M_{n,y}^{a,b}(f,x) \) depends on the parameters \( a,\beta,\gamma \) and give some of its properties. In addition, we made an application of the sequences \( M_{n,y}^{a,b}(f,x) \), \( B_{n,y}^{a,b}(f,x) \) to show the effect of these parameters \( a,\beta,\gamma \) on tends speed occurs by these operators are betters than all tends speed of the sequence \( L_n(f,x) \), where \( f \) is the test function. We also find a better effect of the parameters when \( 0 \leq a \leq \beta \) betters than previous cases of parameters \( a,\beta,\gamma \). Finally, by the applying the two operators \( M_{n,y}^{a,b}(f,x) \), \( B_{n,y}^{a,b}(f,x) \) we get the best CPU time introduced by \( M_{n,y}^{a,b}(f,x) \) by using the second test function.

References