T-semimaximal submodules

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Abstract
In this paper, we define and study the notions of t-semimaximal submodule as a generalization of semimaximal submodule. We provided many properties and characterizations of this concept are provided.

Key words: maximal submodule, semimaximal submodule, t-semimaximal submodule and t-semisimple modules.

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1. Introduction

Throughout this paper $R$ is a ring with unity and $M$ unitary a right $R$-module. The second singular (or Goldi torsion) of $M$ is denoted by $Z_2(M)$ and defined as $Z_2(M) = Z(M)/Z_2(M)$ where $Z(M)$ is the singular submodule of $M$[5]. A module $M$ is called $Z_2$-torsion if $Z_2(M) = M$. A submodule $A$ of an $R$-module $M$ is said to be essential in $M$ (denoted by $A \leq_{ess} M$), if $A \cap W \neq (0)$ for every non-zero submodule $W$ of $M$[7].

The concept of t-essential submodules is introduced as a generalizations of essential submodules [2]. A submodule $N$ of $M$ is said to be t-essential in $M$ (denoted by $(N \leq_{tes} M)$ if for every submodule $B$ of $M$, $N \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$. A submodule $N$ of a module $M$ is called small in $M$ and denoted by $N \ll M$ if for every $K \leq M$ the equality $M = N + K$ implies $M = K$. A module $M$ is called hollow if every proper submodule of $M$ is small in $M$ "[10].

Asgari and Haghany in [3] introduced the concept of t-semisimple modules and t-semisimple rings; A module $M$ is called t-semisimple if every submodule $N$ of $M$ contains a direct summand $K$ of $M$ such that $K$ is t-essential in $N$. A submodule $N$ of a module $M$ is called semimaximal if $M/N$ is a semisimple module [9].

In this paper, we introduce a generalization of semimaximal submodule, namely t-semimaximal. A submodule $N$ of a module $M$ is called t-semimaximal if $M/N$ is t-semisimple module. This paper consists of two sections, in section two of this paper, we define and study the concept of t-maximal submodules and give some properties and charerizations of it.

Proposition (1.1)[2]:" The following statements are equivalent for a submodule $A$ of an $R$-module $M$:

(1) $A$ is t-essential in $M$.
(2) $(A + Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$.
(3) $A + Z_2(M)$ is essential in $M$;
(4) $M/A$ is $Z_2$-torsion [3]."
Corollary (1.2) [3]:" Let M be a t-semisimple module. Then:
(1) Every submodule of M is t-semisimple.
(2) Every homomorphic image of M is t-semisimple".

Corollary (1.3) [3]: "A module M is t-semisimple if and only if M has no proper t-essential submodule which contains $Z_2(M)$".

Corollary (1.4) [3]: Every direct sum of t-semisimple modules is t-semisimple.

2. t-semimaximal submodules

In this section, we will introduce and study the concept of t-semimaximal submodule

Definition (2.1): A submodule $N$ of module $M$ is called t-semimaximal if $M/N$ is a t-semisimple module.

Proposition (2.2): Let $M$ be an $R$-module. $Z_2(M)$ is semimaximal submodule of $M$ if and only if $Z_2(M)$ is t-semimaximal submodule of $M$.

Proof: $\Rightarrow$ It is clear.
$\Leftarrow$ Since $Z_2(M)$ is t-semimaximal submodule of $M$, then $M/Z_2(M)$ is a t-semisimple module. Hence $M/Z_2(M)$ is semisimple by [3, Theorem 2.3], but $Z_2(M/Z_2(M)) = (0)$. Hence $M/Z_2(M)$ is semisimple module. Thus $Z_2(M)$ is semimaximal submodule of $M$.

Remarks and Examples (2.3):

(1) It is clear that every semimaximal submodule of a right $R$-module is t-semimaximal submodule but not conversely, for example: $4Z$ is a t-semimaximal submodule of $Z$ as $Z$-module (because $Z/4Z$ is t-semisimple $Z$-module [3])

(2) Every $t$-essential (or essential) submodule $N$ of $M$ is t-semimaximal (by [3, Example 2.2(i)])

(3) Let $N \leq M$ and $W$ be the complement of $N$, then $N \oplus W$ is t-semimaximal of $M$.

(4) Let $N \leq W \leq M$ and $N$ be a t-semimaximal submodule, then $W$ is a t-semimaximal submodule of $M$.

Proof: Let $f: M/N \rightarrow M/W$ defined by $f(m+N) = m+W$, for all $m \in M$.

It is clear that $f$ is a well-defined and epimorphism. Since $M/N$ is t-semisimple it follows from Hence $M/W$ is t-semisimple by Corollary 1.2(2) that $M/W$ is t-semisimple and hence $W$ is a t-semimaximal submodule of $M$.

If $N$ is t-semimaximal of $M$ and $N \leq K \leq M$, then $N$ is t-semimaximal of $K$.

Proof: Since $N$ is t-semimaximal of $M$ it follows that $M/N$ is t-semisimple. But $K/N \leq M/N$, hence by Corollary 1.2(1) $K/N$ is t-semisimple. Thus $N$ is t-semimaximal of $K$.

(6) Let $\{M_i, i \in I\}$ be a family of $R$-modules and let $M = \bigoplus_{i \in I} M_i$. If $A_i$ is t-semimaximal of $M_i$, then $\bigoplus_{i \in I} A_i$ is t-semimaximal of $\bigoplus_{i \in I} M_i$.

Proof: Since $A_i$ is t-semimaximal of $M_i$, it follows that $M_i/A_i$ is t-semisimple and hence $\bigoplus_{i \in I} M_i/A_i$ is t-semisimple by Corollary 1.4. Thus $\bigoplus_{i \in I} A_i$ is t-semimaximal of $\bigoplus_{i \in I} M_i$.

(7) Let $N \leq K \leq M$ . Then $K$ is t-semimaximal submodule if and only if $M/K_B$ is a t-semimaximal submodule of $M/B$.

Proof: $\Rightarrow$Since $K$ is a t-semimaximal submodule of $M$, it follows that $M/K$ t-semisimple. But $N/K \leq M/K$, it follows that $N/K$ is a t-semisimple module and hence $K_B$ is a t-semimaximal submodule of $M/B$.

$\Leftarrow$ By similarly way of first direction.

(8) Rad $(M)$ is t-semimaximal submodule of $M$ if and only if $M = M_1 \oplus M_2$ such that $M_1$ is semisimple and $\text{Rad } M \leq \text{rad } M_2$ [3, Proposition 2.10].

(9) If $M$ is a t-semimaximal submodule of a module $M$ then $N$ is t-semimaximal submodule, for each non-zero submodule $N$ of $M$.
Proof: suppose that (0) is t-semimaxmal submodule of a module \( M \), thus \( M/(0) \approx \frac{M}{(0)} \). Hence \( M \) is t-semisimple. Hence \( M/N \) is t-semisimple by \([3, \text{Corollary 2.4(2)}]\). Thus \( N \) is t-semisimal.

(10) If \( N \) is a nonzero t-semimaxmal submodule (0) need not be t-semimaxmal, for example: \( 6\mathbb{Z} \) in \( \mathbb{Z} \)-module is t-semimaxmal. But (0) is not t-semimaxmal since \( \mathbb{Z}/(0) \approx \mathbb{Z} \) is not t-semisimple

(11) \( M \) is t-semisimple \( R \)-module if and only if \( M \) is t-semisimple \( R/\text{ann}(M)(0) \)-module.

Proof: Since every submodule of \( M \) \( R \)-module if and only if every submodule of \( M \)

\[ R/\text{ann}(M)(0) \]

\[ \text{module} \] [10].

**Proposition (2.4):** Every submodule of \( M \) \( R \)-module is t-semimaxmal submodule.

Proof: Let \( U \leq M \) and \( \pi: M \rightarrow M/U \) be the natural epimorphism. Hence \( M/U \) is t-semisimple by Corollary 1.2(2). Thus \( U \) is t-semimaxmal.

**Proposition (2.5):** The intersection of any two t-semimaxmal submodules of an \( R \)-module is t-semimaxmal submodule.

Proof: Let \( U_1, U_2 \) be two t-semimaxmal submodules of \( M \). Thus \( M/U_1 \) and \( M/U_2 \) are t-semimaxmal modules and hence \( M/U_1 \oplus M/U_2 \) is t-semisimple by Corollary 1.4. Since \( M/U_1 \cap U_2 \) is an isomorphism to a submodule of \( M/U_1 \oplus M/U_2 \) it follows that \( M/U_1 \cap U_2 \) is t-semisimple. Thus \( U_1 \cap U_2 \) is a t-semimaxmal submodule of \( M \).

**Proposition (2.6):** Let \( U_1 \) be a t-semimaxmal submodule of an \( R \)-module \( M_1 \) and \( U_2 \) be a t-semimaxmal submodule of an \( R \)-module \( M_2 \). Then \( U_1 \oplus U_2 \) is a t-semimaxmal submodule of \( M_1 \oplus M_2 \).

Proof: By hypothesis, \( M_1/U_1 \) and \( M_2/U_2 \) are t-semisimple \( R \)-module and hence from Corollary 1.4 we have that \( M_1/U_1 \oplus M_2/U_2 \) is t-semisimple. Since \( M_1 \oplus M_2 \cong M_1/U_1 \oplus M_2/U_2 \) it follows that \( M_1 \oplus M_2 \cong M_1/U_1 \oplus M_2/U_2 \) is t-semisimple and hence \( U_1 \oplus U_2 \) is t-semisimple in \( M_1 \oplus M_2 \).

**Proposition (2.7):** Let \( M \) be an \( R \)-module and \( N \leq M \). Then \( N \) is a t-semimaxmal if and only if \( M/N \) is semisimple, for each t-closed submodule \( W \) of \( M \) and \( W \supseteq N \).

Proof: \( \Rightarrow \) Let \( W \) be a t-closed submodule of \( M \) with \( W \supseteq N \). Hence \( M/W \) is a t-closed submodule of \( M/N \) by \([4, \text{Lemma 2.5}]\). But \( N \) is a t-semimaxmal by hypothesis, so \( M/N \) is t-semisimple. Then by \([3, \text{Corollary 2.17}]\), \( M/N/W \) is semisimple and hence \( M/W \) is semisimple.

\( \Leftarrow \) To prove \( N \) is a t-semimaxmal submodule of \( M \). Let \( C/N \) be a t-closed in \( M/N \); hence \( C \) is a t-closed of \( M \), and \( C \supseteq N \). So that \( M/C \) is semisimple by hypothesis, but \( M/C \cong M/N/C/N \) so that \( M/N/C/N \) is semisimple for each t-closed submodule \( C/N \) of \( M/N \), which implies \( M/N \) is t-semisimple by \([3, \text{Corollary 2.17}]\). Thus \( N \) is t-semimaxmal submodule of \( M \).

**Proposition (2.8):** If \( Rad(M) \) is a t-semimaxmal and \( M \) is hollow then \( M/Rad(M) \) is \( Z_2 \)-torsion.

Proof: Since \( Rad(M) \) is t-semimaxmal, \( M/Rad(M) \) is t-semisimple. By \([3, \text{Proposition 2.10}]\) we have that \( M = M_1 \oplus M_2 \), where \( M_1 \) is semisimple and \( Rad(M) \leq \text{tes} M_2 \). Let \( A \ll M \), then \( A \leq Rad(M) \leq \text{tes} M_2 \), so if \( M \) is hollow, every submodule of \( M \) contain in \( M_2 \). Hence \( M = M_2 \) and thus \( M/Rad(M) \) is \( Z_2 \)-torsion.

**Remark (2.10):** If \( R \) is a t-semi simple ring and \( M \) is an \( R \)-module, then every submodule of \( M \) is t-semi maximal.
Proof: Since $R$ is a t-semisimple, every $R$-module $M$ is t-semisimple [3,Theorem 3.2]. Hence by Proposition 2.3 every submodule of $M$ is t-semimaximal.

Proposition (2.11): Let $N \leq M$. Then $N$ is a t-semimaximal submodule in $M$ if and only if for each submodule $A$ of $M$ with $A \supseteq N$, there exist $K, K' \leq N$ such that $A = K + K'$ and $M = K + L$ for some $L \leq M$ and $N \leq_{tes} K', K \cap L = N, K \cap K' = N$.

Proof: $\Rightarrow$ Let $N$ be a t-semimaximal submodule in $M$, then $M/N$ is t-semisimple.

For each $A \supseteq N$, $A/N \leq M/N$. Hence by [3, Proposition 2.13(3)] $A/N = K'/N \oplus K'/N$ for each $K, K' \leq M$ with $K/N \leq M/N$ and $K'/N$ is $Z_2$-torsion. Hence $N \leq_{tes} K'$ by Proposition 1.1. $K/N \leq M/N$, then $K/N \oplus L/N = M/N$ for some $L \leq M$ with $N \leq L$, then $K + L = M$ with $K \cap L = N$.

$\Leftarrow$ For any $A/N \leq M/N$. As $A = K + K', K \cap K' = N$, then $A/N = K/N \oplus K'/N$. Also, $K + L = M, K \cap L = N$, then $K/N \oplus L/N = M/N$, so $K/N \leq M/N$. But $N \leq_{tes} K'$ implies $K'/N$ is $Z_2$-torsion. Hence $A/N = K/N \oplus K'/N$ with $K/N \leq M/N$ and $K'/N$ is $Z_2$-torsion implies $M/N$ is t-semisimple by [3, Proposition 2.13(3)]. Thus $N$ is t-semimaximal submodule in $M$.

Proposition (2.12): An $R$-module $M$ is t-semisimple if and only if $\forall N \leq M, N + Z_2(M)$ is semisimple.

Proof: $\Rightarrow$ Suppose that $M$ is t-semisimple, then $N + Z_2(M) = M$ is closed in $M$, $\forall N \leq M$ by [3,Corollary 2.8]. But $N + Z_2(M)$ contains $Z_2(M)$, so $N + Z_2(M)$ is t-closed. Hence by [3, Corollary 2.17], $M/N + Z_2(M)$ is semisimple.

$\Leftarrow$ Since $\forall N \leq M, N + Z_2(M)$ is semisimple, so that $M/N + Z_2(M)$ is semisimple. Hence $M/N + Z_2(M)$ is semisimple (if $N = 0$). This implies $M$ is t-semisimple [3, Theorem 2.3].

References


المقاسات الجزئية العظمى من النمط $T$

فرحان داخل شياع
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المستخلص:

في هذا البحث عرفنا و درسنا مفهوم المقاسات الجزئية العظمى من النمط $T$ كتعميم المقاسات الجزئية العظمى العديد من الخواص والمميزات لهذا المفهوم برهنت.