Close-to-convex Function Generates Remarkable Solution of 2\textsuperscript{nd} order Complex Nonlinear Differential Equations

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Abstract.
Consider the complex nonlinear differential equation
\[ f''(z) - \frac{2}{z} f'(z) - \frac{2}{z^2} f^3(z) = H(z), \]
where \( P(z) = -\frac{2}{z}, Q(z) = -\frac{2}{z^2} \) are complex coefficients, and \( H(z) \) be a complex function performs non-homogeneous term of given equation.
In this paper, we investigated that \( w(z) = \frac{zf'}{f} \) is a remarkable solution of given equation and belongs to hardy space \( H^2 \); with studying the growth of that solution by two ways; through the maximum modulus and Brennan’s Conjecture and another by finding the supremum function of a volume of the surface area \( K_\theta \). Furthermore, we discussed the solution behaviour with meromorphic coefficients properties for given equation.

Keywords: Univalent Function, Positive Harmonic Functions, Growth of Solution.

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Introduction
Consider a second order of complex nonlinear differential equation
\[ f'' + P(z) f' + Q(z) f^3 = H(z), \]
where \( P(z), Q(z) \) are complex coefficients, and \( H(z) \) be a complex function performs non-homogeneous term of given equation.
Let \( f \) be a close-to-convex function defined on the simply connected domain \( \Omega \) onto unit disk \( D \) conformally, so there exist starlike function \( \varphi(z) \) satisfies the condition \( \Re\left(\frac{\varphi'}{\varphi}\right) > 0 \), for more convenience we can suppose that \( f(0) = 0 \), that is to say; \( f \) is a starlike function satisfies the condition \( \Re\left(\frac{zf'}{f}\right) > 0 \), instead of \( \varphi \).

In this paper, we consider a proper solution \( w(z) = \frac{zf'}{f} \), which generated through the close-to-convex function \( f \) itself such that,
\[
\begin{align*}
\frac{w''}{w} &= \frac{zf'' - 3zf' + zf'' - 2f^2}{f^2} \\
&\quad + \frac{zf^3}{2f^2}
\end{align*}
\]
\[
\begin{align*}
\frac{w'}{w} &= \frac{2\left(\frac{zf'f - zf^2}{f} + \frac{f'}{f}ight) + \frac{2zf^3}{z^2}}{zf^2 - \frac{zf^2}{f^2} + \frac{zf^2}{f}} \\
&\quad + \frac{2f'zf'' - 3zf' + zf'}{zf^2 - \frac{zf^2}{f^2} + \frac{zf^2}{f}}
\end{align*}
\]
\[
\frac{w''}{w} - \frac{2}{z} w'(z) - \frac{2}{z^2} w^3(z) = H(z).
\]
it is easy to see that \( w(z) \) satisfies the general equation (1) above

\[
\frac{2f''}{zf} - \frac{3zf'}{f^2} + \frac{x}{f}.
\]

At this point, we have got 2\(^{nd}\) order complex differential equation of type (non-homogeneous) with coefficients of meromorphic function as follows:

\[
f''(z) - \frac{2}{z} f'(z) - \frac{2}{z^2} f^3(z) = H(z) \ldots \ldots (2)
\]

As a result, has been reached to the proper solution for given equation to mark this type of coefficients.

Some research papers cover this type of topics which depending on how the solution of second order complex differential equations (linear - nonlinear) satisfying such condition to be in.

In [1] studied the same kind of complex differential equation with same conditions but when the equation’s coefficients are polynomials, while the author in [4] considers such kind of complex differential equation and proved if the solution satisfies the condition

\[
\frac{x}{f}.
\]

By changing the variables we will be able to rewrite (3) in terms of

\[
\int_D |f|^2 p \, dx dy < \infty.
\]

Definition (Maximum Modulus). [6]

The modulus of a function \( f \) analytic in a domain \( D \), does not have weak local maximum in \( D \) unless \( f \) is constant. If \( f \) is an analytic in a bounded domain and continuous in the closure, then \( |f(z)| \) must have maximum value on the boundary \( \partial D \).

Theorem (Distortion Theorem). [6]

For each \( f \) in the class of univalent function \( S \) defined on unit disk \( D \),

\[
1 - r \leq |f'(z)| \leq 1 + r,
\]

where \( |z| = r < 1 \).

For each \( z \in D, z \neq 0 \), equality works if and only if \( f \) satisfies rotation of the koebe function \( \frac{z}{1-z} \).

Theorem (Prawitz’s theorem). [6],[8]

If \( f(z) \) belongs to the class of univalent functions \( S \), so that \( f:D \rightarrow \Omega \), then for

\[
0 < p < \infty, M_p^\Omega(r,f) \leq p \int_0^r \frac{1}{\rho} M_p^\Omega(\rho,f) d \rho.
\]

1. Problem Statement

In this section, we study the solution behaviour \( w(z) = \frac{z}{f} \) for a second order complex non-linear differential equation (2) by letting

\[
w(z) = w_1(z) + iw_2(z)
\]

as follows:

1. Examine the mean square solution \( w(z) \) of given equation (2) belongs to Hardy space \( H^2(\Omega) \) through the harmonicity property for the real part of the solution \( \Re(w(z)) = w_1(z) \).
2. Examine the role of each of the maximum modulus and Brennan’s Conjecture in the growth of solution \( w(z) \) for given equation (2) depending on a fact that the theory of conformal maps considers as an identically relation between the boundaries of the image and of the pre-image.

3. Show that, the solution \( w(z) \) of given equation (2) can be bounded if its coefficients are bounded.

2. Main Theorems

**Theorem (2.1).** If the solution \( w(z) = \frac{zf'}{f} \) of equation (2) whose a positive harmonic real part, then its solution \( w(z) \) belongs to Hardy space \( H^2(\Omega) \).

**Proof.**
Given conformally, that is \( f \) whose inverse function \( f^{-1}: D \rightarrow \Omega \).

Let \( w(z) = w_1(z) + iw_2(z) \), with
\[
w(z) = \frac{zf'}{f} \tag{4}
\]
Then
\[
f^{-1}(w) = \frac{f(f^{-1})}{f^{-1}} \tag{5}
\]
One can rewrite equation (5) as the form
\[
\frac{1}{w(z)} = \frac{zf^{-1}(f)}{f^{-1}}.
\]
As known that, \( f(z) \) is a starlike function, so its having a positive real part, that is :
\[\Re\left(\frac{zf}{f}\right) > 0, \quad \frac{1}{w(z)} = \frac{zf^{-1}(f)}{f^{-1}} > 0.
\]
Hence, we obtain
\[w_1(z) = \frac{f^{-1}}{z(f^{-1})} > 0.
\]
Obviously, \( f^{-1} \) is an analytic function on \( D \) with \( \Re(f^{-1}) > 0 \).
As a result, \( w_1(z) \) is a positive harmonic function that can be formulated by Poisson integral with an unique positive unit measure \( d\mu(\theta) \) as follows
\[
w_1(r e^{i\theta}) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta);
\]
where \( \mu(\theta) \geq 0 \) and \( \int d\mu(\theta) = 1 \).
Now, would be take modulus for both of sides with suppose that \( z = re^{i\theta} \) to obtain
\[
|w_1(re^{i\theta})| = \left|\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right| = \frac{1}{1-r} \left|\int_0^{2\pi} \frac{d\mu(\theta)}{1-r}\right| = \frac{2\mu(2\pi)}{2\pi} < \frac{C}{2\pi} ; \quad C \text{ is a constant.}
\]

Normally we need to do some calculations for the integral mean of the positive harmonic real function \( w_1(re^{i\theta}) \) in order to check the possibility of the solution existence and its finite,
\[
\frac{1}{2\pi} \int_0^{2\pi} \left|w_1(re^{i\theta})\right|^2 \, r \, dr \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} |w_1(re^{i\theta})||w_1(re^{i\theta})| \, r \, dr \, d\theta,
\]
\[
< \frac{1}{2\pi} \int_0^{2\pi} C \left|w_1(re^{i\theta})\right| \, r \, dr \, d\theta = \int_0^1 (1-r)^{-1} \, dr \cdot \frac{2\pi}{2\pi} \left|w_1(re^{i\theta})\right| \, ds
\]
where, \( s = r \theta \Rightarrow ds = r \, d\theta; \int_0^{2\pi} \left|w_1(re^{i\theta})\right| \, ds < K, \) since \( w_1(re^{i\theta}) \in H^2(D) \), which implies to
\[
\frac{1}{2\pi} \int_0^{2\pi} \left|w_1(re^{i\theta})\right|^2 \, r \, dr \, d\theta < K \int_0^1 (1-r)^{-1} \, dr
\]
Consequently, for all values of \( p \) ; we obtain
\[
\int_0^1 \int_0^{2\pi} \left|w_1(re^{i\theta})\right|^2 \, r \, dr \, d\theta < \infty .
\]
Now, by doing a short simplifications which including Brennan’s conjecture, we obtain
\[
\int_0^\Omega |w(z)|^2 \, dx \, dy < \infty .
\]
Finally, we conclude that \( w(z) \in H^2(\Omega) \).

**Growing of Solution [8].**

In fact, a growth of solution, it was and still an important objective for researchers in this aspect and for along years. It’s obvious that the growth of any solution belongs to given domain, it is controlled by a growing the coefficients of given equation, so there is a match between the solution and those certain coefficients of given equation.

Here, we shall concentrate in terms of the maximum modulus as a main tool to study the growth of solution especially for a harmonic solution which is a positive real part of holomorphic function \( f \).

**Theorem (2.2).** If \( w(z) \) is a solution of equation (2), with
\[
\int_{|w|<1} |w'(w)|^2 \, dx \, dy = \int_{\Omega} \mu(\Omega).
\]
then
Proof. For more convenience, let \( f^{-1} = \psi \).

Applying Second Green Identity as follow:

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial |\psi|^2}{\partial r} r \, d\theta = \frac{1}{2\pi} \int_{|\omega|<1} \Delta (|\psi|^2) \, dx \, dy,
\]

We notice that Laplacian operator in \( L^p \)-space defined as \( \Delta (|\psi|^p) = p^2 |\psi|^{p-2} |\psi|^2 \), such that when \( p = -2 \) implies to

\[
\frac{1}{2\pi} \int_0^{2\pi} |\psi|^{-2} \, d\theta \in (|\omega|<1)
\]

where

\[
= \frac{2}{\pi r} \int |w|<1 \ (|\psi|^{-4}) \ |\psi|^2 \, dx \, dy,
\]

where

\[
\int |w|<1 \ |\psi|^2 \, dx \, dy = \int \Omega \ |\psi|^2 \, dx \, dy.
\]

As a result, \( \psi \) has a maximum modulus \( \mathcal{M}_r(r, \psi) = \max_{r<1} |\psi(z)| \),

so that

\[
|\psi(z)| \leq \max_{r<1} |\psi(z)|.
\]

Applying Second Green Identity as follow:

\[
\int_0^{2\pi} |\psi|^{-2} \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |\psi|^{-2} \, d\theta \in (|\omega|<1)
\]

where

\[
= \frac{2}{\pi r} \int |w|<1 \ (|\psi|^{-4}) \ |\psi|^2 \, dx \, dy.
\]

The integral (1) over \( \Omega \) could be shrink over unit disk where \( \omega \) and \( \rho \) that means \( f \) is defined conformally from \( \Omega \) onto unit disk \( D \), we apply to obtain

\[
J(r^{e^{i\theta}}) = \max_{0<\rho<1} C_i(\rho^{e^{i\theta}}) \quad \text{on} \quad [0, \frac{1}{2}]
\]

which presents the largest value of the function \( C_i(\rho^{e^{i\theta}}) \), where \( i = 0, 1, 2 \) along the edge of given domain \( D \in \Omega \) as well as we shall divide the interval into \( n \) parts between of \( r = 0 \) and \( \rho = \frac{1}{2} \) as follows:

\[
\mathcal{H} = \{u_0, u_1, ..., u_{n-1}, u_n = \rho\} \in [0, \frac{1}{2}]
\]

which represents the interior area of \( D \in \Omega \), such that \( J(r^{e^{i\theta}}) \) and \( \mathcal{H} \) both of them show the volume of the surface area \( D \in \Omega \) that is why, one can define a relation \( V((r^{e^{i\theta}}), \mathcal{H}) \) such that

\[
V((r^{e^{i\theta}}), \mathcal{H}) - \int_0^{\frac{1}{2}} \max_{0<\rho<1} C_i(\rho^{e^{i\theta}}) \, dr < \delta_0,
\]

Proof. Let \( r \in (0,1) \) in \( D \), hence we have to suppose \( \rho^{1+r} \); that is, \( \rho \in \left[ \frac{1}{2}, 1 \right] \).

Set,

\[
J(r^{e^{i\theta}}) = \max_{0<\rho<1} C_i(\rho^{e^{i\theta}}) \quad \text{on} \quad [0, \frac{1}{2}]
\]

which presents the largest value of the function \( C_i(\rho^{e^{i\theta}}) \); \( i = 0, 1, 2 \) along the edge of given domain \( D \in \Omega \) as well as we shall divide the interval into \( n \) parts between of \( r = 0 \) and \( \rho = \frac{1}{2} \) as follows:

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\[
V((r^{e^{i\theta}}), \mathcal{H}) - \int_0^{\frac{1}{2}} \max_{0<\rho<1} C_i(\rho^{e^{i\theta}}) \, dr < \delta_0,
\]
where $\delta_0$ is a constant.

Define such kind of function

$$K_\theta = 3 \sup \{ |r e^{i\theta}| \} \quad \text{for all } r \in [u_0, \rho].$$

Hence, we already have

$$K_\theta \geq 3 \sup \{ |r e^{i\theta}| \} \quad \text{for all } r \in [u_0, \rho].$$

So when an inequality (7) satisfies the equality with taking the integral on the interval $[u_0, \rho]$

$$\int_{u_0}^{\rho} K_\theta \, dr = V\{(r e^{i\theta}), \mathcal{H}\}$$

Consequently,

$$\int_{u_0}^{\rho} K_\theta \, dr < \left(\int_{u_0}^{\rho} \max_{0 \leq r \leq 2\pi} |C_i(\rho e^{i\theta})| \right)_{i=0,1,2} \, dr + \delta_0,$$

then

$$\exp \left(\int_{u_0}^{\rho} K_\theta \, dr \right) < \exp \left(\int_{u_0}^{\rho} \max_{0 \leq r \leq 2\pi} |C_i(\rho e^{i\theta})| \right)_{i=0,1,2} \, dr + \delta_0,$$

Look at the left – hand side of the statement (8) which shows a growth estimate the solution of equation (2), that is

$$w(z) = \exp \left(\int_{u_0}^{\rho} K_\theta \, dr \right)$$

depending on the volume of the surface area $D \subset \Omega$, that generated by coefficients $C_i(z) = \{P(z) = \frac{z}{z^2}; Q(z) = \frac{z^2}{z^2}\}$.

Finally,

$$|w(re^{i\theta})| \leq \exp \left(\int_{u_0}^{\rho} \max_{0 \leq r \leq 2\pi} |C_i(\rho e^{i\theta})| \right)_{i=0,1,2} \, dr + \delta_0,$$

The proof is complete.

Solution Behaviour with Meromorphic Coefficients. [3]

One can study the behaviour of solution which depending on the kind of given equation such (linear- nonlinear) or its coefficients.[2]

We back here into a second order of complex non-linear differential equation (non-homogenous $f''(z) - \frac{2}{z} f'(z) - \frac{2}{z^2} f(z) = H(z)$; where it’s necessary to refer to the role of the coefficients $P(z) = \frac{z}{z^2}; Q(z) = \frac{z^2}{z^2}$; which possess the irregular singularity point at origin. As known in theory of meromorphic differential equations; one can define a special sector around a singularity point 0 of meromorphic coefficient $Q(z) = \frac{z^2}{z^2}$ must be bounded by stokes ray, which presents through the argument of $z$ that takes arbitrary real values connected with solution to be in $\Omega$ rather than in $C_i(0)$.

Let $Q(z) = \frac{z^2}{z^2}$ be meromorphic function in $D$ and let the values which $Q(z)$ assumes in $D$ lie in a domain $\Omega$ (simply connected domain) whose boundary $\gamma$ has positive logarithmic energy $T(r, Q)$ is bounded, then $Q(z)$ is of bounded in $D \subset \Omega$ by coefficients properties.

Theorem (2.4). If the coefficients of equation (2) be meromorphic functions with pole at origin have a property of bounded, then $w(z)$ already is bounded solution.

Proof.

Given $f''(z) - \frac{2}{z} f'(z) - \frac{2}{z^2} f(z) = H(z)$ be a second order non-homogeneous complex differential equation with $P(z), Q(z)$ are a meromorphic functions in $\Omega$, having a pole in origin point with some properties as follows:

Let $P(z) = \frac{z^2}{z^2} \implies P'(z) = \frac{z^2}{z^2}$; which implies to $P'(z) = Q(z)$.

On the other word, consider $P(z) = 2(\log z)^2$ such that, $P(z) = 2(\log z)^2 = Q(z)$.

Set,

$$|r e^{i\theta}| = \max_{0 \leq r \leq 1} |Q(\rho e^{i\theta})|,$$

By maximal inequality we obtain

$$\int_0^{2\pi} \sqrt{|(r e^{i\theta})|} \, d\theta \leq C \int_0^{2\pi} \sqrt{|Q(\rho e^{i\theta})|} \, d\theta,$$

(cf. [7]) the solution $w(z)$ of equation (2) has been able to grow in as shown in previous theorem (2.2), one can rewrite $w(z)$ in more detailed as follows $w(z) \to w(z e^{i\theta})$; where $z = re^{i\theta}$, for any periodic solution to obtain

$$\int_0^{2\pi} \log |w(re^{i\theta})| \, d\theta \leq K$$

by inequality (9) we obtain

$$\int_0^{2\pi} \log |w(re^{i\theta})| \, d\theta \leq K + C \int_0^{2\pi} \sqrt{|Q(\rho e^{i\theta})|} \, d\theta.$$

Now, we already have

$$|Q(\rho e^{i\theta})| = |2(\log z)^2| = \frac{2}{\rho^2}$$

Obviously.
\[ \int_0^{2\pi} \sqrt{|Q(z)|} \, d\theta = 2\sqrt{2}\pi \rho. \]

It is so clear to notice that, as \( \rho \to 1 - 0, \)
\[ \int_0^1 \int_0^{2\pi} \sqrt{|Q(z)|} \, d\theta d\rho \]
\[ = \sqrt{2}\pi \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10) \]

In sense \( w(z) \) already is a bounded solution by properties of the equation’s coefficients because the function \( Q(z) \) is meromorphic function at the origin point and the integral (9) is a bounded.

Conclusions.
In this research paper, we concluded that: The harmonicity property of real part of the solution \( \Re \left( \frac{w(z)}{f} \right) \) for a second order complex non-linear differential equation (2) had a role in:
1. Determine the space which contains such kind of solution.
2. Examine the growth of solution \( w(z) \) for given equation (2).
3. Examine that the equation (2) must have a bounded solution \( w(z) \) if its coefficients already are bounded.

References


الدالة القريبة إلى التحذب تولد حلاً ملحوظاً للمعادلات اللاخطية اللاخطية التفاضلية ذات المرتبة الثانية

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المستخلص:
في هذه الورقة البحثية نتناول معادلات اللاخطية التفاضلية العقلية ذات المرتبة الثانية من النوع الغير متجانس
\[ f''(z) - \frac{2}{z} f'(z) - \frac{2}{z^2} f^2(z) = H(z), \]
حيث ثوابت عقدية، و (hardy space \( H^2 \)) \( H^2 \) هو حل ملحوظ للمعادلة المعطاة. يتم التحقق من أن
\[ w(z) = \frac{2f''}{f} \]
هو حل ملحوظ للمعادلة المعطاة ويتم إلى فضاء هاردي، مع دراسة نمو الحل الأولي من خلال معامل الحد الأقصى
 وكل يعملي BRENNAN’S CONJECTURE
وتحمين برينان ويخترى عن طريق إيجاد الحد الأعلى لحجم المساحة السطحية

علاوة على ذلك، نناقش سلوك الحل مع خصائص معاملات المعادلة المعطاة (من نوع دوال ميرومورفيك).