On $\mathbb{P} –$ Hausdorff Topological Random Systems

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Abstract
We are taking a view of topological random systems which introduce considered as a mixing between two fundamental branches of mathematics "topology" and "probability Theory". The concept of $\mathbb{P} –$ Hausdorff topological random system is studied and some properties of such system are given and proved.

Key words. Random set, Topological random system, $\mathbb{P} –$ neighborhood, $\mathbb{P} –$ limit point, $\mathbb{P} –$ closed set and $\mathbb{P} –$ Hausdorff system.
Introduction. The "topology" and "probability Theory" (specially random sets) are the important tools in the study of pure and applied mathematics. Therefore we mix here these two theories by define the topological random system. As first step of our study we are taking avie of topologicall random system and define the concept of Hausdorff topological random system. Throughout our paper we state and prove some properties of Hausdorff topological random system. This work consist of three sections. In section 2 we state the definition of random set and some concepts related with probability theory.

In section 3, our new concept "the topological random system" is introduced and some concept in terms of probability concepts such as neighborhood, limit point and closed set are given. In Section 4 the concept of Hausdorff system is introduced and some essential properties are proved. Throughout this paper all probability space are complete (A" probability space is said to be complete "[1,2,3] if for ever y with ,then for all subsets ).

2. Random Sets. The origin of the recent concept of a random set energies as far back as the inspiring book by A.N. Kolmogorov [5](first published in 1933) where he arranged out the foundations of probability theory. In this section the definition of random set is given and some properties of such sets. The set of random closed set if for every opensets the set is measurable. The complement of random closed set is called random open set.

Let be a function whose values are subsets of . A function is individually determined by its graph . Conversely, each subset defines such a function via .

Definition 2.1[1]: "Let be a metric space which is considered a measurable space with Borel algebra and be a measurable space. The set-valued function , is said to be random set if for every the function is measurable. If is closed (compact) for all , it is called a random closed(compact) set".

Proposition 2.2[1]: Let the set-valued function take values in the subspace of closed subsets of a Polish space . Then:

(i) A is a random closed set if and only if for all open sets the set is measurable.
(ii) If A is a random closed set then . The property of being a random closed set is thus slightly stronger than being measurable and being closed.

For convenient, throughout this paper we adopt the following definition of random set" .

Definition 2.3 [1] Agreement be a set-valued function where is a topological space. Then A is said to be a random closed set if for every opensets the set is measurable. Then the complement of random closed set is called random open set.
Examples [1]

(a) "The set \( A = \{ \zeta \} \) is an RCS where \( \zeta \) is a random point in".

(b) "The set \( A = (-\infty, \zeta] \) is RCS on \( X = \mathbb{R}^1 \) if \( \zeta \) is RV. Also the set \( A = (-\infty, \zeta_1] \times (-\infty, \zeta_2] \times \cdots \times (-\infty, \zeta_n] \) is RCS in \( \mathbb{R}^n \) if \( \zeta_1, \ldots, \zeta_n \) is \( n \) -dimensional random vector".

Theorem 2.4[1]

(i) the closure of the complement of any closed random set is closed random set.

(ii) The closure of any random open set is closed random set.

(iii) The interior of any closed random set is open random set

(iv) the intersection of any two random set is random set.

For more detail about random set see[1] and [2].

3. Topological Random System. In this section the new concept of topological random system is introduced. Also the concepts of, \( \mathbb{P} \) -limit point and \( \mathbb{P} \) -closed set are introduced.

Definition 3.1 (Topological Random System)

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space and \( (X, \tau) \) be a topological space (which is considered as measurable space with Borel \( \sigma \) - algebra \( \mathcal{B}(X) \)). The triple \( (\Omega, X, \mathcal{R}) \) is called topological random system (shortly, TRS), where \( \mathcal{R} \) is the collection of random sets in \( X \).

Example 3.2 Agreement \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space, where \( \Omega := \{ H, T \} \), \( \mathcal{F} := 2^\Omega \) and \( \mathbb{P}(\{ H \}) = \mathbb{P}(\{ T \}) = \frac{1}{2} \) and let \( \mathbb{R} \) be the set of all real numbers endowed with the usual topology (in this case \( \mathbb{R} \) is polish space).

Define the collection \( \mathcal{R} := \{ A, B, C, D \} \) of sub sets of \( \mathbb{R} \), where

\[
\begin{align*}
A &: \Omega \to \mathcal{B}(\mathbb{R}) \ , \ A(\omega) = \emptyset \ , \ \forall \omega \in \Omega \\
B &: \Omega \to \mathcal{B}(\mathbb{R}) \ , \ B(\omega) = [0, \infty) \ , \ \forall \omega \in \Omega \\
C &: \Omega \to \mathcal{B}(\mathbb{R}) \ , \ C(\omega) = (-\infty, 0] \ , \ \forall \omega \in \Omega \\
D &: \Omega \to \mathcal{B}(\mathbb{R}) \ , \ D(\omega) = \mathbb{R} \ , \ \forall \omega \in \Omega.
\end{align*}
\]

Thus \( \mathcal{R} := \{ A, B, C, D \} \) be the collection of random sets in \( \mathbb{R} \). (In fact is the collection of closed random set in \( \mathbb{R} \)). Hence the triple \( (\Omega, \mathbb{R}, \mathcal{R}) \) is TRS.

Definition 3.3 (sub-topological random system).

The triple \( (\Omega, \mathcal{Y}, \mathcal{R}) \) is said to be sub-topological random system of \( (\Omega, X, \mathcal{R}) \) if \( Y \) is a subspace (as a topological space) of \( X \) and the intersection of each open random set in \( X \) with \( Y \) is random open set in \( Y \).

Definition 3.4 (Random Neighborhood).

Let \( (\Omega, X, \mathcal{R}) \) be a TRS and \( x \in X \). A random neighborhood (shortly, RN) of \( x \) is a random set \( N \) such that there exists random open set \( U \) with the property that

\[
\mathbb{P}(\{ \omega : x \in U(\omega) \subseteq N(\omega) \}) = 1.
\]

The collection \( \mathcal{R}_x \) denoted to all RNhd of \( x \) and is called random neighborhood system (RNS) at \( x \).
Example 3.5 Consider $\mathbb{R}$ endowed with the usual topology, and $(\Omega, \mathcal{F}, \mathbb{P})$ be any complete probability space. Let $\omega \in \Omega$, be a real-valued random process on $\Omega$ with continuous sample paths. Then $A = \{\omega: x > 0\}$ is RN of each element of itself.

Theorem 3.6 The RN $X$ has the following properties.

[RN1] If $N \in \mathcal{R}_x$, then $\mathbb{P}\{\omega: x \in N(\omega)\} = 1$.

[RN2] If $N, M \in \mathcal{R}_x$, then $N \cap M \in \mathcal{R}_x$.

[RN3] If $N \in \mathcal{R}_x$, then for each $y \in M$, $N \in \mathcal{R}_y$.

[RN4] If $N \in \mathcal{R}_x$ and $\mathbb{P}\{\omega: N(\omega) \subseteq M(\omega)\} = 1$, then $N \in \mathcal{R}_x$.

[RN5] $G$ is random open set if and only if $G$ contains an RN of each of its points.

Proof.

[RN1]: Suppose that $N \in \mathcal{R}_x$, then $\mathbb{P}\{\omega: x \in U(\omega) \subseteq N(\omega)\} = 1$. Let $\omega \in \{\omega: x \in U(\omega) \subseteq N(\omega)\}$, then $x \in U(\omega) \subseteq N(\omega)$, i.e., $x \in N(\omega)$. Thus $\omega \in \{\omega: x \in N(\omega)\}$. Therefore $\{\omega: x \in U(\omega) \subseteq N(\omega)\} \subseteq \{\omega: x \in N(\omega)\}$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete "probability space", then $\{\omega: x \in N(\omega)\} \in \mathcal{F}$.

Now by properties of $\mathbb{P}$ we have $\mathbb{P}\{\omega: x \in U(\omega) \subseteq N(\omega)\} \subseteq \mathbb{P}\{\omega: x \in N(\omega)\}$ or, equivalently $1 \leq \mathbb{P}\{\omega: x \in N(\omega)\}$. Hence $\mathbb{P}\{\omega: x \in N(\omega)\} = 1$.

[RN2]: Suppose that $N, M \in \mathcal{R}_x$, then there exists two random opensets $U$ and $V$ such that

$$\mathbb{P}\{\omega: x \in U(\omega) \subseteq N(\omega)\} = 1 = \mathbb{P}\{\omega: x \in V(\omega) \subseteq M(\omega)\}.$$

Clearly that

$$\{\omega: x \in U(\omega) \subseteq N(\omega)\} \cap \{\omega: x \in V(\omega) \subseteq M(\omega)\} \subseteq \{\omega: x \in U(\omega) \cap V(\omega) \subseteq N(\omega) \cap M(\omega)\}.$$ 

Then

$$\mathbb{P}\{(\omega: x \in U(\omega) \subseteq N(\omega)) \cap (\omega: x \in V(\omega) \subseteq M(\omega))\} \subseteq \mathbb{P}\{(\omega: x \in U(\omega) \cap V(\omega) \subseteq N(\omega) \cap M(\omega))\}.$$ 

Thus

$$\mathbb{P}\{(\omega: x \in U(\omega) \subseteq N(\omega)) \cap (\omega: x \in V(\omega) \subseteq M(\omega))\} = \mathbb{P}\{(\omega: x \in U(\omega) \subseteq N(\omega)) \cap (\omega: x \in V(\omega) \subseteq M(\omega))\}.$$ 

Therefore

$$\{\omega: x \in U(\omega) \subseteq N(\omega)\} \cap \{\omega: x \in V(\omega) \subseteq M(\omega)\} \subseteq \{\omega: x \in U(\omega) \cap V(\omega) \subseteq N(\omega) \cap M(\omega)\} \in \mathcal{F}.$$ 

Hence

$$\{\omega: x \in U(\omega) \cap V(\omega) \subseteq N(\omega) \cap M(\omega)\} = 1.$$ 

By definition of RN we get $N \cap M \in \mathcal{R}_x$.

[RN3]: Suppose that $N \in \mathcal{R}_x$, and take $M = \text{Int} (N)$. Then for each $y \in M$, $y \in \text{Int} (N)$. Then $\mathbb{P}\{\omega: y \in \text{Int} (N) \subseteq N\} = 1$. So $N \in \mathcal{R}_y$. 

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Suppose that \( N \in \mathcal{R}_X \). Then there exists a random open set \( U \) with the property that \( \mathbb{P}(\omega : x \in U(\omega) \subseteq N(\omega)) = 1 \). Then \( \mathbb{P}(\omega : x \in \text{Int}(N(\omega)) \subseteq N(\omega)) = 1 \). If \( \mathbb{P}(\omega : N(\omega) \subseteq M(\omega)) = 1 \), then \( \mathbb{P}(\omega : \text{Int}(N(\omega)) \subseteq \text{Int}(M(\omega))) = 1 \). So \( \mathbb{P}(\omega : x \in \text{Int}(M(\omega))) = 1 \) and hence \( \mathbb{P}(\omega : x \in \text{Int}(M(\omega)) \subseteq M(\omega)) = 1 \). Therefore \( M \in \mathcal{R}_X \).

If \( G \) is a random open set, and \( x \in G \). Since \( G = \text{Int}(G) \). Then \( \mathbb{P}(\omega : x \in \text{Int}(G(\omega)) \subseteq G(\omega)) = 1 \). Hence \( G \in \mathcal{R}_X \). Conversely, if \( G \in \mathcal{R}_X \) for every \( x \in G \). Then there exists a random open set \( V_x \) such that \( \mathbb{P}(\omega : x \in V_x(\omega) \subseteq G(\omega)) = 1 \). (In fact \( \{ \omega : x \in V_x(\omega) \subseteq G(\omega) \} = \Omega \)). Hence \( \bigcup_{\omega \in G} \text{Int}(V_x(\omega)) = G(\omega) \). Therefore \( G \) is a random open set by Theorem (2.4).

**Definition 3.7** (**\( \mathbb{P} \)-limit point**). Let \((\Omega, X, \mathcal{R})\) be a TRS and \( A \) be a random set in \((\Omega, X, \mathcal{R})\). A point \( x \in X \) is said to be a **\( \mathbb{P} \)-limit point** of \( A \) if \( \mathbb{P}(\omega : [N(\omega) - \{x\}] \cap A(\omega) \neq \emptyset) = 1 \) for every \( \Omega \)-null set \( x \).

The set of all \( \mathbb{P} \)-limit points of \( A \) is called the \( \mathbb{P} \)-derived set and is denoted by \( \mathbb{P} - D(A) \).

**Example 3.8** Consider the TRS \((\Omega, \mathcal{R}, \mathcal{R})\) given in Example 3.2 with \( = (-1, \infty) \). Then \(-1 \) is \( \mathbb{P} \)-limit point of \( A \).

**Definition 3.9** (**\( \mathbb{P} \)-closed set**). The (deterministic) subset \( A \) of \( X \) is said to be a **\( \mathbb{P} \)-closed set** if it has all of its \( \mathbb{P} \)-limit points. That is, \( A \) is a **\( \mathbb{P} \)-closed set** if and only if \( \mathbb{P} - D(A) \subseteq A \). The complement of \( \mathbb{P} \)-closed set is called **\( \mathbb{P} \)-open set**.

**Example 3.10** Consider Example 3.8. Then all intervals of the form \([a, \infty) \subseteq \mathbb{R} \) are **\( \mathbb{P} \)-closed**.

**Note 3.11**. The set \( G \) is \( \mathbb{P} \)-open set if and only if \( \mathbb{P} - D(G) \cap G = \emptyset \).

**Lemma 3.12**. The finite union of all \( \mathbb{P} \)-closed sets is \( \mathbb{P} \)-closed set.

**Proof.** Assume that \( \{A_i : i = 1, 2, ..., n\} \) be a collection of \( \mathbb{P} \)-closed sets. To show that \( A = \bigcup_{i=1}^{n} A_i \) is an \( \mathbb{P} \)-closed set. Let \( x \in X \) be a \( \mathbb{P} \)-limit point of \( A \). Then for every \( \mathbb{P} \)-null set \( N \) of \( x \) we have \( \mathbb{P}(\omega : [N(\omega) - \{x\}] \cap (A(\omega) \neq \emptyset) = 1 \). Hence \( \bigcup_{\omega \in A} \text{Int}(V_x(\omega)) = A(\omega) \). Therefore \( A \) is a random open set by Theorem (2.4).

4. **\( \mathbb{P} \)-Hausdorff system.** In this final section the concept of \( \mathbb{P} \)-Hausdorff system is introduced and studied. The "Hausdorff property" is one of the important properties in the study of the topology and its applications. Therefore we focus our study to study this concept in terms of probability theory and random set.
Definition 4.1 A topological random system $(\Omega, X, \mathcal{R})$ is said to be $\mathbb{P} -$ Hausdorff (or $\mathbb{P} - T_2$) if for every there exist two distinct points $x, y \in X$, two random opensets $A$ and $B$ in $X$ such that $x \in A$, $y \in B$ and $\mathbb{P}(\omega : A(\omega) \cap B(\omega) \neq \emptyset) = 0$ or equivalently $\mathbb{P}(\omega : A(\omega) \cap B(\omega) = \emptyset) = 1$.

Example 4.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where $\Omega := \{H, T\}$, $\mathcal{F} := 2^\Omega$ and $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}$ and let $\mathbb{R}$ be the set of all real numbers endowed with the usual topology. Define the collection $\mathcal{R} := \{A_x : x \in \mathbb{R}\}$ of sub sets of $\mathbb{R}$, where $A_x : \Omega \to \mathcal{B}(\mathbb{R})$, $A_x(\omega) = \{x\}$, $\forall \omega \in \Omega$.

Thus $\mathcal{R} := \{A_x : x \in \mathbb{R}\}$ be the collection of random sets in $\mathbb{R}$. Hence the triple $(\Omega, \mathbb{R}, \mathcal{R})$ is a TRS. Since $\mathbb{P}(\omega : A(\omega) \cap B(\omega) = \emptyset) = \mathbb{P}(\Omega) = 1$. Then $(\Omega, \mathbb{R}, \mathcal{R})$ is $\mathbb{P} - T_2$ space.

Theorem 4.3 The subspace of a $\mathbb{P} - T_2$ space is $\mathbb{P} - T_2$.

Proof: Let $(\Omega, X, \mathcal{R})$ be a topological random system and $(Y, \tau_y)$ be a subspace of $(X, \tau)$. Let $x, y \in Y$ with $x \neq y$. Then $x, y \in X$. By hypothesis, there exist two open random sets $A$ and $B$ in $X$ such that $x \in A$, $y \in B$ and $\mathbb{P}(\omega : A(\omega) \cap B(\omega) \neq \emptyset) = 0$. Define $C(\omega) := A(\omega) \cap Y$ and $D(\omega) := B(\omega) \cap Y$ are two random sets in $Y$ with $x \in C(\omega)$ and $y \in D(\omega)$. Let $F = \{\omega : A(\omega) \cap B(\omega) \neq \emptyset\}$, such that $\mathbb{P}(F) = 0$. Then $\{\omega : C(\omega) \cap D(\omega) \neq \emptyset\} \subseteq F$. By completeness of $(\Omega, \mathcal{F}, \mathbb{P})$ we have $\omega : C(\omega) \cap D(\omega) \neq \emptyset \in \mathcal{F}$. Thus $\mathbb{P}(\omega : C(\omega) \cap D(\omega) \neq \emptyset) \leq \mathbb{P}(\omega : A(\omega) \cap B(\omega) \neq \emptyset) = 0$ . It follows from completeness of the probability space that $\mathbb{P}(\omega : C(\omega) \cap D(\omega) \neq \emptyset) = 0$. Thus the subsystem $(\Omega, Y, \mathcal{R})$ is $\mathbb{P} - T_2$.

Theorem 4.4 Every singleton set in a $\mathbb{P} - T_2$ space is $\mathbb{P} - T_2$ closed set.

Proof. Let $(\Omega, X, \mathcal{R})$ be a $\mathbb{P} - T_2$ and let $x \in X$. To show that $\{x\}$ is $\mathbb{P} - T_2$ closed set. Let $y \in X$, with $x \neq y$. To prove that $y$ is not $\mathbb{P} - T_2$ limit point of $\{x\}$ i.e., there exists $N_y$ of $y$ such that $\mathbb{P}(\omega : [N(\omega) - \{y\}] \cap \{x\} = \emptyset) = 1$. Since $(\Omega, X, \mathcal{R})$ is $\mathbb{P} - T_2$, there exists two random opensets $A$ and $B$ in $X$ such that $x \in A$, $y \in B$ and $\mathbb{P}(\omega : A(\omega) \cap B(\omega) \neq \emptyset) = 0$. Set $F = \{\omega : M(\omega) \cap N(\omega) = \emptyset\}$, with $\mathbb{P}(F) = 1$. Then $M(\omega) \cap N(\omega) = \emptyset$, $\forall \omega \in F$. Then $x \in N(\omega)$, $\forall \omega \in F$. That is $\mathbb{P}(\omega : x \in N(\omega)) = 1$ or $\mathbb{P}(\omega : (x) \cap N(\omega) = \emptyset) = 1$. Hence $\mathbb{P}(\omega : (x) \cap N(\omega) - \{y\} = \emptyset) = 1$. Consequently $\{x\}$ is $\mathbb{P} - T_2$ closed set.

Corollary 4.5 A finite (deterministic) sub set of a $\mathbb{P} - T_2$ space is $\mathbb{P} - T_2$ closed set.

Proof. This follows from Theorem 4.4 and Lemma 3.12.

Theorem 4.6 The RS $(\Omega, X, \mathcal{R})$ is $\mathbb{P} - T_2$ for each pair $x, y \in X$, there exists a $\mathbb{P} - \text{nhd}$ $N_y$ of $y$ such that $\mathbb{P}(\omega : x \in \overline{N_y}(\omega)) = 1$.

Proof. Supposing $(\Omega, X, \mathcal{R})$ is $\mathbb{P} - T_2$ and let $x, y \in X$ with $x \neq y$. Then $x, y \in X$. By hypothesis, there exist two random opensets $G$ and $H$ in $X$ such that $x \in G$, $y \in H$ and $\mathbb{P}(\omega : G(\omega) \cap H(\omega) = \emptyset) = 1$. Then $\mathbb{P}(\omega : y \in H(\omega) \subseteq X - G(\omega)) = 1$. Then $X - G(\omega)$ is closed $\mathbb{P} - \text{nhd}$ of $y$ and $\mathbb{P}(\omega : x \notin X - G(\omega)) = 1$. Set $N_y = X - G(\omega)$, then $\overline{N_y} = N_y$, so $N_y$ is $\mathbb{P} - \text{nhd}$ and $\mathbb{P}(\omega : x \notin \overline{N_y}(\omega)) = 1$. 

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Conversely, suppose that for each pair $x, y \in X$, there exists a $P$-nhd $N_y$ of $y$ such that $P\{\omega : x \in \overline{N}_y(\omega)\} = 1$. Since $N_y \supseteq N_y$, then by Theorem (3.6) $RN_4 \overline{N}_y$ is $P$-nhd of $y$. Since $N_y$ is closed random set, then $X - \overline{N}_y$ is open random set with $x \in X - \overline{N}_y$ and $y \notin X - \overline{N}_y$. Put $-\overline{N}_y = N_x$, we see that there is a $P$-nhd $N_x$ of $x$ and a $P$-nhd $N_y$ of $y$ such that $P\{\omega : N_x(\omega) \cap N_y(\omega) = \emptyset\} = 1$. Consequently, $P\{\omega : N_x(\omega) \cap N_y(\omega) = \emptyset\} = 1$. Therefore $(\Omega, X, \mathcal{R})$ is $P - T_2$.

**Theorem 4.7** The RS $(\Omega, X, \mathcal{R})$ is $P - T_2$ if and only if for every collection $\{F_\lambda : \lambda \in \Lambda\}$ of closed $P$-nhd of each $x \in X$, we have $P\{\omega : \cap_{\lambda \in \Lambda} F_\lambda(\omega) = \{x\}\} = 1$.

**Proof.** Suppose that $(\Omega, X, \mathcal{R})$ is a $P - T_2$ RS. Let $x, y \in X$ with $x \neq y$. Then there exist $G, H \in ROS$ such that $x \in G, y \in H$ and $P\{\omega : G(\omega) \cap H(\omega) = \emptyset\} = 1$. Thus $P\{\omega : G(\omega) \subseteq X - H(\omega)\} = 1$. Hence $X - H(\omega)$ is a closed $P$-nhd of $x$, and by completeness of $(\Omega, \mathcal{F}, P)$, we have $P\{\omega : y \in X - H(\omega)\} = 0$. If $\{F_\lambda : \lambda \in \Lambda\}$ is a collection of closed $P$-nhd of each $x \in X$, then $P\{\omega : y \in \cap_{\lambda \in \Lambda} F_\lambda(\omega)\} = 0$. Since $y$ is an arbitrary, then $P\{\omega : \cap_{\lambda \in \Lambda} F_\lambda(\omega) = \{x\}\} = 1$. Consequently, suppose that $P\{\omega : \cap_{\lambda \in \Lambda} F_\lambda(\omega) = \{x\}\} = 1$ for every collection $\{F_\lambda : \lambda \in \Lambda\}$ of closed $P$-nhd of each $x \in X$. Let $y \in X$ with $x \neq y$. Since $P\{\omega : y \in \cap_{\lambda \in \Lambda} F_\lambda(\omega)\} = 0$, then there exists closed $P$-nhd $N$ of each $x$. $P\{\omega : y \notin N(\omega)\} = 1$. Therefore, exists $G \in ROS$ such that $P\{\omega : x \in G(\omega) \subseteq N(\omega)\} = 1$.

Therefore $G, N^c \in ROS$ such that $x \in G$ and $y \in N^c$. Finally, we must show that $F = \{\omega : G(\omega) \cap N^c(\omega) = \emptyset\} \in \mathcal{F}$, and $P(F) = 1$.

We have $F = \{\omega : G(\omega) \cap N^c(\omega) = \emptyset\} = \{\omega : G(\omega) \subseteq N(\omega)\}$,

$$
\supseteq \{\omega : x \in G(\omega) \subseteq N(\omega)\}.
$$

By completeness of $(\Omega, \mathcal{F}, P)$, $F \in \mathcal{F}$ and $P(F) = 1$. This means that $(\Omega, X, \mathcal{R})$ is a $P - T_2$ RS.

**Corollary 4.8** The RS $(\Omega, X, \mathcal{R})$ is $P - T_2$ if and only if for every collection $\{N_\lambda : \lambda \in \Lambda\}$ of closed $P$-nhd of each $x \in X$, we have $P\{\omega : \cap_{\lambda \in \Lambda} N_\lambda(\omega) = \{x\}\} = 1$.

**Proof** Since the closure of random set is closed random set (by Theorem 2.4) then the result followed directly from Theorem 4.6.

**Theorem 4.9** The RS $(\Omega, X, \mathcal{R})$ is $P - T_2$ if and only if for every finite (deterministic) subset $\{x_i : i = 1, 2, \ldots, n\}$ of $X$ there exists RN $N_i$ of $x_i$ for every $i = 1, 2, \ldots, n$ such that for every $i, j = 1, 2, \ldots, n$ with $i \neq j$, we have $P\{\omega : N_i(\omega) \cap N_j(\omega) = \emptyset\} = 1$.

**Proof.** Supposing that $(\Omega, X, \mathcal{R})$ be a $P - T_2$ and $(x_i : i = 1, 2, \ldots, n) \subseteq X$ with $x_i \neq x_j$, $\forall i, j = 1, 2, \ldots, n$ with $i \neq j$. By hypothesis there exist $N_{ij}, N_{ij} \in ROS$ such that $x_i \in N_{ij}, x_j \in N_{ij}$ and $P\{\omega : N_{ij}(\omega) \cap N_{ij}(\omega) = \emptyset\} = 1$. Let $N_i(\omega) = \cap \{N_{ij}(\omega) : j = 1, 2, \ldots, n, i \neq j\}$. 
Then by Theorem 2.4 $N_i(\omega) \in ROS$, for every $i = 1, 2, \ldots, n$. To show that $\mathbb{P}\{\omega: N_i(\omega) \cap N_j(\omega) = \emptyset\} = 1$, for every $i, j = 1, 2, \ldots, n$ with $i \neq j$. Set $F = \{\omega: N_i(\omega) \cap N_j(\omega) = \emptyset\}$, for every $i, j = 1, 2, \ldots, n$ with $i \neq j$. Then $\forall \omega \in F$ we have $N_i(\omega) \cap N_j(\omega) = (\bigcap_{i \neq j} N_i(\omega)) \cap (\bigcap_{i \neq j} N_j(\omega)) = \emptyset$.

Then $F = \{\omega: N_i(\omega) \cap N_j(\omega) = \emptyset\}$ and hence $\mathbb{P}(F) = 1$.

Conversely, suppose that for every finite (deterministic) subset $\{x_i: i = 1, 2, \ldots, n\}$ of $X$ there exists RN $N_i$ of $x_i$ for every $i = 1, 2, \ldots, n$ such that for every $i, j = 1, 2, \ldots, n$ with $i \neq j$ we have $\mathbb{P}\{\omega: N_i(\omega) \cap N_j(\omega) = \emptyset\} = 1$. It follows in particular that for any two distinct points $x, y$ there exist $M, N \in ROS$ such that $x \in N, y \in M$ and $\mathbb{P}\{\omega: M(\omega) \cap N(\omega) = \emptyset\} = 1$. Thus $(\Omega, X, \mathcal{R})$ be a $\tau$-closed $\mathbb{P}$-nhd of $x$ by hypothesis we have $\mathbb{P}\{\omega: \bigcap_{\lambda \in \Lambda} F_j(\omega) = \{x\}\} = 1$. It follows that $(F_j: \lambda \in \Lambda)$ be a collection of $\tau$-closed $\mathbb{P}$-nhd of $x$. Consequently, $(\Omega, X, \mathcal{R})$ is $\mathbb{P} - T_2$.

**Theorem 4.10** Let $(\Omega, X, \mathcal{R})$ be an RS. If each point of $X$ admits a $\tau$-closed $\mathbb{P}$-nhd of $x$ which is a $\mathbb{P} - T_2$ sub-system of $(\Omega, X, \mathcal{R})$, then $(\Omega, X, \mathcal{R})$ is $\mathbb{P} - T_2$.

**Proof.** Let $x \in X$ and let $Y$ be a $\tau$-closed $\mathbb{P}$-nhd of $x$ in $X$. Such that $(\Omega, Y, \mathcal{R})$ is $\mathbb{P} - T_2$ sub-system of $(\Omega, X, \mathcal{R})$. First we need to show that every $\tau_Y$-closed $\mathbb{P}$-nhd of $x$ is a $\tau$-closed $\mathbb{P}$-nhd of $x$. $N^*$ be $\mathbb{P}$-closed $\mathbb{P}$-nhd of $x$. Then there is a $\tau$-closed $\mathbb{P}$-nhd of $x$ such that $N^* = N \cap Y$. Since $N$ and $Y$ are random $\tau$-closed sets then by Theorem 2.4 $N^* = N \cap Y$ is random $\tau$-closed set and so $N^*$ is $\tau$-closed $\mathbb{P}$-nhd of $x$. Now, let $\{F_j: \lambda \in \Lambda\}$ be a collection of $\tau_Y$-closed $\mathbb{P}$-nhd of $x$ by hypothesis we have $\mathbb{P}\{\omega: \bigcap_{\lambda \in \Lambda} F_j(\omega) = \{x\}\} = 1$. It follows that $(F_j: \lambda \in \Lambda)$ be a collection of $\tau$-closed $\mathbb{P}$-nhd of $x$. Consequently, $(\Omega, X, \mathcal{R})$ is $\mathbb{P} - T_2$.

**References:**


حول النظم التبولوجية العشوائية الهاوزدورفية من النمط – 

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المستخلص:
قدمنا في هذا البحث مفهوم النظام التبولوجي العشوائي الذي يعتبر كخليط بين فروعين أساسيين من فروع الرياضيات " التبولوجيا" و "نظرية الاحتمال". تم دراسة النظم التبولوجية العشوائية الهاوزدورفية من النمط – 

وتم تقديم بعض خواص هذا النظام مع برهانها.