On a Completion of Fuzzy Measure

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Abstract: In this paper, we introduce some properties in completeness of fuzzy measure and we get some relations between them.

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1. Introduction

The fuzzy measure, defined on a classical $\sigma$-field, was introduced by Sugeno [7]. Ralescu and Adams [1] generalized the concepts of fuzzy measure and fuzzy integral to the case that the value of a fuzzy measure can be infinite, and to realize an approach from subjective.


The notion of fuzzy measure was extended by Avallone and Barbieri, Jiang and Suzuki [9], Narukawa and Murofushi [10], Ralscu and Adams [1] as a set function which was defined on $\sigma$-field with valus in $[0, \infty]$. After that, many authors studied the fuzzy measure and proved some results about it as Guo and Zhang [10], Kui [6], Li and Yasuda [3], Lushu and Zhaohu [5], Minghu [2].

In this paper, we mention the definition of completion of fuzzy measure with some properties, and prove some new relations deal with completeness of fuzzy measure.

Definition (1):[13]
Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mu: \mathcal{F} \rightarrow [0, \infty)$ is called a fuzzy measure if

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \mu(B)$, where $A \subseteq B$

Definition (2):
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space, $A \in \mathcal{F}$ is said to be $\mu - \text{null set}$ if $\mu(A) = 0$. The fuzzy measure $\mu$ is said to be complete on $\mathcal{F}$ if $\mathcal{F}$ contains the subset of every $\mu - \text{null}$ sets.

Definition (3):[12]
$\mu$ is called countably weakly null-additive, if for any $\{A_n\} \subset \mathcal{F}$,

$$\mu(A_n) = 0, \text{ for all } n \geq 1 \Rightarrow \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0$$

Definition (4):[12]
$\mu$ is said to be additive, if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$. 

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2. Main results

**Theorem (1):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measurable space and \(\mu\) is countably weakly null-additive and \(\delta_\mu = \{E: E \subset A \in \mathcal{F} \text{ and } \mu(A) = 0\}\). Then \(\delta_\mu\) is \(\sigma - \text{ring}\).

**Proof:**

1. Clearly \(\emptyset \in \delta_\mu\).
2. Let \(E_1, E_2 \in \delta_\mu \Rightarrow\) there exists \(A_1, A_2 \in \mathcal{F}\) such that \(E_1 \subset A_1, E_2 \subset A_2\) and \(\mu(A_1) = 0, \mu(A_2) = 0\).

\(E_1 / E_2 \subset E_1 \subset A_1 \in \mathcal{F}\) So \(E_1 / E_2 \in \delta_\mu\).
3. Let \(\{E_n\}\) be a sequence of sets in \(\delta_\mu\) \(n=1,2,\ldots\) \(\Rightarrow\) there exist a sequence \(\{A_n\}\) \(n=1,2,\ldots\) of sets in \(\mathcal{F}\) such that \(E_n / A_n\) and \(\mu(A_n) = 0\).

\[\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} A_n\]

Since \(\mathcal{F}\) is \(\sigma - field\)

\[\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}\]

Since \(\mu\) is countably weakly null-additive

\[\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0\]

So

\[\bigcup_{n=1}^{\infty} E_n \in \delta_\mu\]

Therefore

\[\delta_\mu\) is \(\sigma - \text{ring}\]

**Theorem (2):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measurable space and \(\mu\) is additive, define \(\mathcal{F} = \{(E \cup E_1) / E_2 : E \in \mathcal{F}, E_1, E_2 \in \delta_\mu\}\). Then \(A \in \mathcal{F}\) \(\iff\) there exists \(M, N \in \mathcal{F}\) such that \(M \subset A \subset N\) and \(\mu(M / N) = 0\).

**Proof:**

Let \(M, N \in \mathcal{F}\) and \(M \subset A \subset N\) and \(\mu(N / M) = 0\).

So

\[A = (N \cup \emptyset) / (N / A)\]

Since

\[N / A \subset N / M \in \mathcal{F}\) and \(\mu(N / M) = 0 \Rightarrow N / A \in \delta_\mu\).

Therefore

\[A \in \mathcal{F}\].

Suppose that \(A \in \mathcal{F}\), then \((E \cup E_1) / E_2, E \in \mathcal{F}, E_1, E_2 \in \delta_\mu\).

\[\Rightarrow\) there exist \(A_1, A_2 \in \mathcal{F}\) such that \(\mu(A_1) = 0, \mu(A_2) = 0\)

and \(E_1 \subset A_1, E_2 \subset A_2\).

\[E \cup A_1, E / A_2 \in \mathcal{F}\) and \(\mu((E \cup A_1) / (E / A_2)) = \mu((A_1 / E) \cup (A_2 \cap E)) = \mu(A_1 / E) + \mu(A_2 \cap E)\)

Since

\[A_1 / E \subset A_1 \text{ and } A_2 \cap E \subset A_2 \Rightarrow \mu(A_1 / E) = 0\) and \(\mu(A_2 \cap E) = 0\)

So

\[\mu((E \cup A_1) / (E / A_2)) = 0.\]

**Corollary (1):**

Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measurable space and \(\mu\) is additive. Then \(A \in \mathcal{F}\) \(\iff\) \(A = E \cup M, E \in \mathcal{F}\) and \(M \in \delta_\mu\).

**Proof:**

Suppose that \(A \in \mathcal{F}\). By theorem (2) there exist \(M, N \in \mathcal{F}\) such that \(N \subset A \subset M\) and \(\mu(M / N) = 0\)

\[A = N \cup (A / N), N \in \mathcal{F}\]
Since
\[ A/N \subseteq M/N \in \mathcal{F} \text{ and } \mu(M/N) = 0 \implies A/N \in \delta_{\mu} \]

Conversely
Suppose \( A = E \cup M, E \in \mathcal{F} \) and \( M \in \delta_{\mu} \)
\[ A = (E \cup M)/\emptyset, \emptyset \in \delta_{\mu} \implies A \in \bar{\mathcal{F}} \]

**Corollary (2):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measurable space and \(\mu\) is additive. Then \(A \in \bar{\mathcal{F}}\) iff \(A = E / D\) with \(E \in \mathcal{F}\) and \(D \in \delta_{\mu}\).

**Proof:**
Suppose that \(A \in \bar{\mathcal{F}}\)
\[ \implies \text{there exist } M, N \in \mathcal{F} \text{ such that} \]
\[ N \subseteq A \subseteq M \text{ and } \mu(M/N) = 0 \]
\[ A = M/(M/A), M \in \mathcal{F} \]

Since
\[ M/A \subseteq M/N \in \mathcal{F} \text{ and } \mu(M/N) = 0 \]
So
\[ M/A \in \delta_{\mu} \]

Conversely
Suppose that \(A = E/D\) where \(E \in \mathcal{F}\) and \(D \in \delta_{\mu}\)
\[ \implies A = (E \cup \emptyset)/D \]
\[ D, \emptyset \in \delta_{\mu} \]
\[ \implies A \in \bar{\mathcal{F}} \]

**Theorem (3):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measurable space and \(\mu\) is additive. Then \(\bar{\mathcal{F}}\) is \(\sigma - \text{ring}\).

**Proof:**
1. Clearly \(\emptyset \in \bar{\mathcal{F}}\).
2. Let \(\{A_n\}_{n=1}^{\infty} \text{ where } A_n \in \bar{\mathcal{F}}\) be a sequence of sets such that \(A_n \in \bar{\mathcal{F}}\)
\[ \implies A_n = M_n \cup N_n \text{ where } M_n \in \mathcal{F} \text{ and } N_n \in \delta_{\mu} \]
\[ \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (M_n \cup N_n) = \left( \bigcup_{n=1}^{\infty} M_n \right) \cup \left( \bigcup_{n=1}^{\infty} N_n \right) \]

Since
\[ \bar{\mathcal{F}} \text{ is } \sigma - \text{field and } \delta_{\mu} \text{ is } \sigma - \text{ring} \]
\[ \implies \bigcup_{n=1}^{\infty} M_n \in \mathcal{F}, \bigcup_{n=1}^{\infty} N_n \in \delta_{\mu} \]

So
\[ \bigcup_{n=1}^{\infty} A_n \in \bar{\mathcal{F}} \]

3. Let \(A, B \in \bar{\mathcal{F}}\) from Corollary(1) we obtain
\[ A = M_1 \cup N_1 \]
\[ B = M_2 \cup N_2 \]
\[ A/B = (M_1 \cup N_1)/(M_2 \cup N_2) \]
\[ = (M_1/M_2) \cup (N_1/M_2) \]
\[ = [(M_1/M_2)/E_2] \cup ((E_2/N_2) \cup (M_1/M_2)/N_2) \]
\[ = [(M_1/M_2)/E_2] \cup (E_2/N_2) \cup (M_1/M_2/N_2) \]
\[ N_2 \subseteq E_2 \subseteq \mathcal{F}, \mu(E_2) = 0 \]
\[ A/B \in \bar{\mathcal{F}} \]

Therefore
\[ \bar{\mathcal{F}} \text{ is } \sigma - \text{ring.} \]
References


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الملخص: في هذا البحث، قمنا بعض الخصائص في كمالية القياس الضبابي وحصولنا على بعض العلاقات بينها.