On Differential Sandwich Theorems of Meromorphic Univalent Functions

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Recived : 15/4/2018           Revised : 26/4/2018          Accepted : 22/5/2018
Available online : 5/8/2018
DOI: 10.29304/jqcm.2018.10.3.399

Abstract

By using of linear operator, we obtain some Subordinations and superordinations results for certain normalized meromorphic univalent analytic functions in the punctured open unit disk $U^*$. Also we derive some sandwich theorems.

Keywords: Analytic Function, Differential Subordination, Hadamard Product, Meromorphic Univalent Function.

Mathematics Subject Classification : 30C45
1. Introduction

Let \( \mathcal{H} \) be the Linear space of all analytic functions in \( U \). For a positive integer number \( n \) and \( a \in \mathbb{C} \), we let
\[
\mathcal{H}[a,n] = \{ f \in \mathcal{H} : f(z) = a + a_1 z + a_2 z^2 + \cdots \}.
\]
For two functions \( f \) and \( g \) analytic in \( U \), we say that the function \( g \) is subordinate to \( f \) in \( U \) and write \( g(z) < f(z) \), if there exists a Schwarz function \( \omega \), which is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) (\( z \in U \)), such that \( g(z) = f(\omega(z)) \), \( z \in U \).

If the function \( f(z) \) is if the function \( f \) is univalent in \( U \), then we have
\[
g(z) < f(z) \iff g(0) = f(0) \quad \text{and} \quad g(U) \subset f(U),
\]
which are analytic and meromorphic univalent function in the punctured open unit disk \( U^* = \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} \).

Let \( p, h \in \mathcal{H} \), and \( \Phi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C} \).
If \( p \) and \( \Phi(p(z),zp'(z),z^2p''(z);z) \) are univalent functions in \( U \) and if \( p \) satisfies the second- order superordination
\[
h(z) \prec \Phi(p(z),zp'(z),z^2p''(z);z), \quad (z \in U),
\]
then \( p \) is called a solution of the differential superordination (1.2), (if \( f \) subordinate to \( g \), then \( g \) is superordinate to \( f \)).

An analytic function \( q \) is called a subordinate of the differential superordination if \( q < p \) for all \( p \) satisfying (1.2). A univalent subordinate \( \tilde{q} \) that satisfies \( q < \tilde{q} \) for all subordinates \( q \) of (1.2) is said to be the best subordinate. Recently Miller and Mocnu [3] obtained sufficient conditions on the functions \( h, p \) and \( \Phi \) for which the following implication holds:
\[
h(z) \prec \Phi(p(z),zp'(z),z^2p''(z);z) \Rightarrow q(z) < p(z), \quad (z \in U).
\]

If \( f \in W \) is given by (1.1) and \( g \in W \) given by
\[
g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k.
\]

The Hadamard product (or convolution) of \( f \) and \( g \) is defined by
\[
(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z).
\]

Using the results, Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination preserving integral operator [1]. Ali et al. [5], have used the results of Bulboacă [4] to obtain sufficient conditions for normalized analytic functions to satisfy:
\[
q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),
\]
where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \). Also, Tunesci [6] obtained a sufficient conditions for starlikeness of \( f \) in terms of the quantity
\[
\frac{f''(z)f(z)}{(f'(z))^2}.
\]

Recently, Shanmugam et al. [7,8] and Goyal et al. [9] also obtained sandwich results for certain classes of analytic functions.

Ali et al. [10] introduced and investigated the linear operator
\[
I_1(n,\lambda) : W \to W
\]
which is defined as follows:
\[
I_1(n,\lambda)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(k+\lambda)^n}{\lambda-1} a_k z^k,
\]
(\( z \in U^*, \lambda > 1 \)).

The general Hurwitz- lerch zeta function
\[
\Phi(z,s,r) = \sum_{k=0}^{\infty} \frac{z^k}{(r+k)^s}, \quad r \in \mathbb{C} \setminus \mathbb{Z}_0, s \in \mathbb{C}
\]
when \( 0 < |z| < 1 \).
Definition 1.1. Let \( f \in W, z \in U^*, r \in \mathbb{C} \setminus \mathbb{Z}_0^+ \), \( s \in \mathbb{C} \) and \( \lambda > 1 \), we define the operator \( J_{s,r,1}(\lambda, f(z)) : \mathbb{W} \rightarrow \mathbb{W} \), where
\[
J_{s,r,1}(\lambda, f(z)) = \frac{\Phi(z, s, r)}{z^{r-s}} \ast I_1(\lambda, f(z))
\]
\[
= \frac{1}{z} \sum_{k=0}^{\infty} \frac{r}{1+k+r} \left( \frac{r+k+\lambda}{\lambda-1} \right)^n a_k z^k
\]
We note from (1.5) that, we have
\[
\lambda J_{s,r,1}(\lambda, f(z)) = z \left( \frac{J_{s,r,1}(\lambda, f(z))}{\lambda - 1} \right) - J_{s,r,1}(\lambda, f(z))
\]
\[
(\lambda - 1) J_{0,r,1}(\lambda, f(z)) = I_1(\lambda, f(z))
\]
and
\[
J_{0,r,1}(0, f(z)) = f(z).
\]
The main object of this idea is to find sufficient conditions for certain normalized analytic functions \( f \) to satisfy:
\[
q_1(z) < \left( \frac{(1-\beta)z J_{s,r,1}(n, \lambda, f(z)) + \beta z J_{s,r,1}(n+1, \lambda, f(z))}{\beta + 1} \right) < q_2(z)
\]
and
\[
q_1(z) < \left( z J_{s,r,1}(n, \lambda, f(z)) \right)^{\delta} < q_2(z)
\]
where \( q_1(z) \) and \( q_2(z) \) are given univalent functions in \( U \) with \( q_1(0) = q_2(0) = 1 \).

2. Preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas.

Definition 2.1. [2]: Denote by \( Q \) the set of all functions \( q \) that are analytic and injective on \( \mathbb{U} \setminus E(q) \), where \( \mathbb{U} = U \cup \{ z \in \partial U \} \), and
\[
E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\}
\]
and are such that \( q'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(q) \). Further, let the subclass of \( Q \) for which \( q(0) = \alpha \) be denoted by \( Q(\alpha) \). \( Q(0) \equiv Q_0 \) and \( Q(1) \equiv Q_1 \).

Lemma 2.1. [5] Let \( q(z) \) be convex univalent function in \( U \), let \( \alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\} \) and suppose that
\[
\Re \left( 1 + \frac{z q''(z)}{q'(z)} \right) > \max \left\{ 0, -\Re \left( \frac{\alpha}{\beta} \right) \right\}
\]
If \( p(z) \) is analytic in \( U \) and
\[
\alpha p(z) + \beta z p'(z) < \alpha q(z) + \beta z q'(z)
\]
then \( p(z) < q(z) \) and \( q \) is the best dominant.

Lemma 2.2. [1] Let \( q \) be univalent in \( U \) and let \( \emptyset \) and \( \Theta \) be analytic in the domain \( D \) containing \( q(U) \) with \( \emptyset(\omega) \neq 0 \), when \( \omega \in q(U) \). Set
\[
Q(z) = z q'(z) \emptyset(q(z)) \text{ and } h(z) = \Theta(q(z)) + Q(z),
\]
suppose that
\[
1 - Q \text{ is starlike univalent in } U,
\]
\[
2 - \Re \left( \frac{z h'(z)}{Q(z)} \right) > 0, z \in U.
\]
If \( p \) is analytic in \( U \) with \( p(0) = q(0) \), \( p(U) \subseteq D \) and
\[
\emptyset(p(z)) + z p'(z) \emptyset(p(z)) < \emptyset(q(z)) + z q'(z) \emptyset(q(z)),
\]
then \( p < q \), and \( q \) is the best dominant.

Lemma 2.3. [3] Let \( q(z) \) be convex univalent in the unit disk \( U \) and let \( \Theta \) and \( \Phi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that
\[
1 - \Re \left( \frac{\Theta'(q(z))}{\Phi(q(z))} \right) > 0 \text{ for } z \in U,
\]
\[
2 - z q'(z) \Phi(q(z)) \text{ is starlike univalent in } z \in U.
\]
If \( p \in \mathcal{H}[q(0), 1] \cap Q \), with \( p(U) \subseteq D \), and \( \Theta(p(z)) + z p'(z) \Phi(p(z)) \) is univalent in \( U \), and
\[
\Theta(q(z)) + z q'(z) \Phi(q(z)) < \Theta(p(z)) + z p'(z) \Phi(p(z))
\]
then \( q < p \), and \( q \) is the best subordinant.

Lemma 2.4. [3]: Let \( q(z) \) be convex univalent in \( U \) and \( q(0) = 1 \). Let \( \beta \in \mathbb{C} \), that \( \Re(\beta) > 0 \). If \( p(z) \in \mathcal{H}[q(0), 1] \cap Q \) and \( p(z) + \beta z p'(z) \) is univalent in \( U \), then
which implies that \( q(z) < p(z) \) and \( q(z) \) is the best subordinant.

### 3. Subordination Results

**Theorem 3.1.** Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1, \eta, \delta \in \mathbb{C}\setminus\{0\} \). Suppose that

\[
\text{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, \text{Re}\left(\frac{\delta}{\eta}\right)\right\}.
\]

(3.1)

If \( f \in W \) is satisfies the Subordination

\[
G(z) < q(z) + \frac{\eta}{\delta} zq'(z),
\]

(3.2)

where

\[
G(z) = \left(\frac{(1-\beta)z\partial_{n+1}(n\lambda)f(z) + \beta z\partial_{n+1}(n+1,1)f(z)}{\beta + 1}\right)^{\delta}(1 + \eta \times \left(\frac{(-2z-1-\beta)\partial_{n+1}(n\lambda)f(z) + (2z + \beta)\partial_{n+1}(n+1,1)f(z)}{1-\beta\partial_{n+1}(n\lambda)f(z) + \beta\partial_{n+1}(n+1,1)f(z)}\right)).
\]

(3.3)

then

\[
\left(\frac{(1-\beta)z\partial_{n+1}(n,\lambda)f(z) + \beta z\partial_{n+1}(n+1,1)f(z)}{\beta + 1}\right)^{\delta} < q(z),
\]

(3.4)

and \( q(z) \) is the best dominant.

**Proof.** Define a function \( g(z) \) by

\[
g(z) = \left(\frac{(1-\beta)z\partial_{n+1}(n\lambda)f(z) + \beta z\partial_{n+1}(n+1,1)f(z)}{\beta + 1}\right)^{\delta},
\]

(3.5)

then the function \( g(z) \) is analytic in \( U \) and \( q(0) = 1 \), therefore, differentiating (3.5) logarithmically with respect to \( z \) and using the identity (1.6) in the resulting equation,

\[
G(z) = \left(\frac{(1-\beta)z\partial_{n+1}(n\lambda)f(z) + \beta z\partial_{n+1}(n+1,1)f(z)}{\beta + 1}\right)^{\delta}(1 + \eta \times \left(\frac{(-2z-1-\beta)\partial_{n+1}(n\lambda)f(z) + (2z + \beta)\partial_{n+1}(n+1,1)f(z)}{1-\beta\partial_{n+1}(n\lambda)f(z) + \beta\partial_{n+1}(n+1,1)f(z)}\right)).
\]

\[
g(z) + \frac{\eta}{\delta} zg'(z).
\]

Thus the subordination (3.2) is equivalent to

\[
g(z) + \frac{\eta}{\delta} zg'(z) < q(z) + \frac{\eta}{\delta} zq'(z).
\]

An application of Lemma (2.1) with \( \beta = \frac{n}{\delta} \) and \( \alpha = 1 \), we obtain (3.4).

Taking \( q(z) = \frac{1+Az}{1+Bz} (-1 \leq B < A \leq 1) \), in Theorem (3.1), we obtain the following Corollary.

**Corollary 3.2.** Let \( \eta, \delta \in \mathbb{C}\setminus\{0\} \) and \( -1 \leq B < A \leq 1 \). Suppose that

\[
\text{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\text{Re}\left(\frac{\delta}{\eta}\right)\right\}.
\]

If \( f \in W \) is satisfy the following Subordination condition:

\[
G(z) < \frac{1+Az}{1+Bz} + \frac{\eta}{\delta} \frac{(A-B)z}{(1+Bz)^2},
\]

where \( G(z) \) given by (3.3), then

\[
\left(\frac{(1-\beta)z\partial_{n+1}(n\lambda)f(z) + \beta z\partial_{n+1}(n+1,1)f(z)}{\beta + 1}\right)^{\delta} < \frac{1+Az}{1+Bz},
\]

and \( \frac{1+Az}{1+Bz} \) is best dominant.

Taking \( A = 1 \) and \( B = -1 \) in Corollary (3.2), we get following result.

**Corollary 3.3.** Let \( \eta, \delta \in \mathbb{C}\setminus\{0\} \) and suppose that

\[
\text{Re}\left(1+\frac{z}{1-z}\right) > \max\left\{0, -\text{Re}\left(\frac{\delta}{\eta}\right)\right\}.
\]

If \( f \in W \) is satisfy the following Subordination

\[
G(z) < \frac{1+z}{1-z} \frac{2z}{(1-z)^2},
\]

where \( G(z) \) given by (3.3), then

\[
\left(\frac{(1-\beta)z\partial_{n+1}(n\lambda)f(z) + \beta z\partial_{n+1}(n+1,1)f(z)}{\beta + 1}\right)^{\delta} < \frac{1+z}{1-z},
\]

and \( \frac{1+z}{1-z} \) is best dominant.

**Theorem 3.4.** Let \( q(z) \) be convex univalent in unit disk \( U \) with \( q(0) = 1, c, \eta, \delta \in \mathbb{C}\setminus\{0\}, \alpha, \beta, t, \mu, \tau \in \mathbb{C}, f \in W \) and suppose that \( f \) and \( q \) satisfy the following conditions

\[
\text{Re}\left(\frac{\mu q(z) + \tau q^2(z)}{\mu t + q^2(z)}\right) > 0,
\]

(3.6)
and 
\[ zJ_{s,r,1}(n,\lambda)f(z) \neq 0. \quad (3.7) \]
If
\[ r(z) < t + \mu q(z) + \tau a q^2(z) + \zeta \frac{q'(z)}{q(z)}, \quad (3.8) \]
where
\[ r(z) = (zJ_{s,r,1}(n,\lambda)f(z))^\delta \times \left( (\mu + \tau a \left( zJ_{s,r,1}(n,\lambda)f(z) \right)^\delta + t + \phi(\lambda - 1) \right) \left( \frac{J_{s,r,1}(n+1,\lambda)f(z)}{J_{s,r,1}(n,\lambda)f(z)} - 1 \right), \quad (3.9) \]
then
\[ (zJ_{s,r,1}(n,\lambda)f(z))^\delta < q(z), \quad \text{and } q(z) \text{ is best dominant.} \]

**Proof.** Define analytic function \( g(z) \) by
\[ g(z) = \left( zJ_{s,r,1}(n,\lambda)f(z) \right)^\delta. \quad (3.10) \]
Then the function \( g(z) \) is analytic in \( U \) and \( g(0) = 1 \), differentiating (3.10) logarithmically with respect to \( z \), we get
\[ \frac{zq'(z)}{g(z)} = \delta(\lambda - 1) \left( \frac{J_{s,r,1}(n+1,\lambda)f(z)}{J_{s,r,1}(n,\lambda)f(z)} - 1 \right). \quad (3.11) \]
By setting \( \theta(w) = t + \mu w + \tau a w^2 \) and \( \phi(w) = \frac{\zeta}{w} \), it can be easily observed that \( \theta(w) \) is analytic in \( C \), \( \phi(w) \) is analytic in \( C \setminus \{0\} \) and that \( \phi(w) \neq 0, w \in C \setminus \{0\} \).
Also, if we let
\[ Q(z) = zq'(z)\phi(z) = \zeta \frac{zq'(z)}{q(z)} \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z) \]
\[ = t + \mu q(z) + \tau a q^2(z) + \zeta \frac{q'(z)}{q(z)}, \]
we find that \( Q(z) \) is starlike univalent in \( U \), we have
\[ h'(z) = \mu q'(z) + 2\tau a q(z)q'(z) + \zeta \frac{q'(z)}{q(z)} + \zeta^2 \frac{q''(z)}{q(z)}, \]
and
\[ zq'(z) = q(z) + \frac{2\tau a}{\zeta} q^2(z) + 1 + \frac{q''(z)}{q(z)} - \frac{q'(z)}{q(z)}, \]
hence that
\[ \text{Re} \left( \frac{zq'(z)}{q(z)} \right) = \text{Re} \left( \frac{\mu q(z) + \frac{2\tau a}{\zeta} q^2(z) + 1 + \frac{q''(z)}{q(z)} - \frac{q'(z)}{q(z)}}{q(z)} \right) > 0. \]
By using (3.11), we obtain
\[ \mu q(z) + \tau a q^2(z) + \zeta \frac{q'(z)}{q(z)} < \mu q(z) + \tau a q^2(z) + \zeta \frac{q'(z)}{q(z)} \]
and by using Lemma (2.2), we deduce that subordination (3.8) implies that \( g(z) < q(z) \) and the function \( q(z) \) is the best dominant.
Taking the function \( q(z) = \frac{1+Az}{1+Bz} \) \(( -1 \leq B < A \leq 1 ) \), in Theorem (3.4), the condition (3.6) becomes
\[ \text{Re} \left( \frac{\mu + \frac{2\tau a}{\zeta} \left( \frac{1+Az}{1+Bz} \right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Az)} - \frac{2Bz}{1+Bz} }{(1+Bz)(1+Az)} \right) > 0 \quad (s \in C \setminus \{0\}), \quad (3.12) \]
hence, we have the following Corollary.

**Corollary. 3.5.** Let \((-1 \leq B < A \leq 1), s, \delta \in C \setminus \{0\}, \alpha, t, \mu \in C . \) Assume that (3.12) holds. If \( f \in W \) and
\[ r(z) < t + \mu \frac{1+Az}{1+Bz} + \tau a \left( \frac{1+Az}{1+Bz} \right)^2 + \zeta \frac{(A-B)z}{(1+Bz)(1+Az)} \]
where \( r(z) \) is defined in (3.9), then
\[ (zJ_{s,r,1}(n,\lambda)f(z))^\delta < \frac{1+Az}{1+Bz} \text{ and } \frac{1+Az}{1+Bz} \text{ is best dominant.} \]
Taking the function \( q(z) = \left( \frac{1+Az}{1+Bz} \right)^\rho \) \((0 < \rho \leq 1) \), in Theorem (3.4), the condition (3.6) becomes
\[ \text{Re} \left( \frac{\mu}{\zeta} \left( \frac{1+Az}{1+Bz} \right)^\rho + \frac{2\tau a}{\zeta} \left( \frac{1+Az}{1+Bz} \right)^{2\rho} + \frac{2Bz}{1+Bz} \right) > 0 \]
\[ (\zeta \in C \setminus \{0\}), \quad (3.13) \]
hence, we have the following Corollary.
Corollary 3.6. Let

\[ 0 < \rho \leq 1, \zeta, \delta \in \mathbb{C}\setminus\{0\}, \alpha, t, \tau, \mu \in \mathbb{C} \text{. Assume that (3.13) holds.} \]

If \( f \in W \) and

\[ r(z) < t + \mu \left( \frac{1 + z}{1 - z} \right)^{\rho} + \tau \left( \frac{1 + z}{1 - z} \right)^{\rho} + \zeta \frac{2\rho z}{1 - z} \]

where \( r(z) \) is defined in (3.9), then

\[ \left( z J_{s,r,1}(n, \lambda) f(z) \right)^{\delta} < \left( \frac{1 + z}{1 - z} \right)^{\rho} \text{, and } \left( \frac{1 + z}{1 - z} \right)^{\rho} \text{ is best dominant.} \]

4. Superordination Results

Theorem 4.1. Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1, \delta \in \mathbb{C}\setminus\{0\}, \text{Re}(\eta) > 0 \), if \( f \in W \), such that

\[ (1 - \beta) z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z) \neq 0 \]

and

\[ \left( \frac{1 - \beta}{\beta + 1} z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z) \right)^{\delta} \]

\[ \in \mathcal{H}(q(0), 1) \cap Q \text{.} \]

If the function \( G(z) \) defined by (3.3) is univalent and the following superordination condition:

\[ q(z) + \frac{\eta}{\delta} z q'(z) < G(z) \text{,} \]

holds, then

\[ q(z) < \left( \frac{(1 - \beta) z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z)}{\beta + 1} \right)^{\delta} \]

(4.3)

and \( q(z) \) is the best subordinant.

Proof. Define a function \( g(z) \) by

\[ g(z) = \left( \frac{(1 - \beta) z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z)}{\beta + 1} \right)^{\delta} \]

(4.4)

Differentiating (4.4) with respect to \( z \) logarithmically, we get

\[ \frac{zg'(z)}{g(z)} = \delta \left( \frac{(1 - \beta) z J_{s,r,1}(n, \lambda) f(z)' + \beta z J_{s,r,1}(n + 1, \lambda) f(z)'}{(1 - \beta) z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z)} \right) \]

(4.5)

A simple computation and using (1.6), from (4.5), we get

\[ G(z) = \left( \frac{(1 - \beta) z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z)}{\beta + 1} \right)^{\delta} \times (1 + \eta) \]

\[ = g(z) + \eta + \zeta g'(z) \]

now, by using Lemma (2.4), we get the desired result.

Taking \( q(z) = \frac{1 + Az}{1 + Bz} (-1 < B < A \leq 1) \), in Theorem (4.1), we get the following Corollary.

Corollary 4.2. Let \( \text{Re}(\eta) > 0, \delta \in \mathbb{C}\setminus\{0\} \text{ and } -1 < B < A \leq 1 \), such that

\[ \left( \frac{(1 - \beta) z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z)}{\beta + 1} \right)^{\delta} \]

\[ \in \mathcal{H}(q(0), 1) \cap Q \text{.} \]

If the function \( G(z) \) given by (3.3) is univalent in \( U \) and \( f \in W \) satisfies the following superordination condition:

\[ 1 + Az + \eta \frac{(A - B)z}{(1 + Bz) \delta} < G(z) \text{,} \]

then

\[ 1 + Az < \frac{(1 - \beta) z J_{s,r,1}(n, \lambda) f(z) + \beta z J_{s,r,1}(n + 1, \lambda) f(z)}{\beta + 1} \]

and the function \( \frac{1 + Az}{1 + Bz} \) is the best subordinant.

Theorem 4.3. Let \( q(z) \) be convex univalent in unit disk \( U \), let \( \zeta, \delta \in \mathbb{C}\setminus\{0\}, \alpha, t, \tau, \mu \in \mathbb{C}, q(z) \neq 0 \), and \( f \in W \). Suppose that \( \text{Re} \left\{ \frac{q(z)}{\delta} (2\tau a q(z) + \mu) \right\} q'(z) > 0 \)

and satisfies the next conditions

\[ z J_{s,r,1}(n, \lambda) f(z) \in \mathcal{H}(q(0), 1) \cap Q \text{,} \]

and

\[ z J_{s,r,1}(n, \lambda) f(z) \neq 0 \text{.} \]

If the function \( r(z) \) is given by (3.9) is univalent in \( U \),

\[ t + \mu q(z) + \tau a q^2(z) + \zeta \frac{q'(z)}{q(z)} < r(z) \text{,} \]

(4.7)
implies
\[ q(z) < \left( \frac{zJ_{s,r,t}(n,\lambda)f(z)}{\delta} \right)^{\frac{1}{\delta}}, \] and \( q(z) \) is the best subordinant.

\textbf{Proof}. Let the function \( g(z) \) defined on \( U \) by (3.14). Then a computation show that
\[ \frac{zg'(z)}{g(z)} = \theta(\lambda - 1) \left( \frac{J_{s,r,t}(n+1,\lambda)f(z)}{J_{s,r,t}(n,\lambda)f(z)} - 1 \right). \] (4.8)
By setting \( \theta(w) = t + \mu w + \tau w^2 \) and \( \mathcal{O}(w) = \frac{w}{w} \), it can be easily observed that \( \theta(w) \) is analytic in \( \mathbb{C} \), \( \mathcal{O}(w) \) is analytic in \( \mathbb{C}\setminus\{0\} \) and that \( \mathcal{O}(w) \neq 0 \) \( (w \in \mathbb{C}\setminus\{0\}) \). Also, we get
\[ Q(z) = zq'(z)\mathcal{O}(q(z)) = \zeta \frac{zg'(z)}{g(z)}, \]
It observed that \( Q(z) \) is starlike univalent in \( U \).
Since \( q(z) \) is convex, it follows that
\[ \text{Re} \left( \frac{z\theta'(q(z))}{\mathcal{O}(q(z))} \right) = \text{Re} \left( \frac{q(z)}{\zeta} \left( 2t\alpha q(z) + \mu \right) q'(z) \right) > 0. \]
By making use of (4.8) the hypothesis (4.7) can be equivalently written as
\[ \theta(q(z) + zq'(z)\mathcal{O}(q(z))) = \theta(q(z) + zg'(z)\mathcal{O}(g(z))), \]
thus, by applying Lemma (2.3), the proof is completed.

5. Sandwich Results
Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich theorem.

\textbf{Theorem 5.1.} Let \( q_1 \) and \( q_2 \) be two convex univalent functions in \( U \) such that \( q_1(0) = 1 \), \( q_2(0) \neq 0 \) \( (i = 1,2) \). Suppose that \( q_1 \) and \( q_2 \) satisfies (3.8), respectively.\( F \) \( f \) satisfies the next conditions:
\[ (1-\beta)zJ_{s,r,t}(n,\lambda)f(z) + \beta zJ_{s,r,t}(n+1,\lambda)f(z) \]
and the function \( G(z) \) defined by (3.3) is univalent and satisfies
\[ q_1(z) + \frac{\eta}{\delta} z q_1(z) < G(z) \]
\[ q_2(z) + \frac{\eta}{\delta} z q_2(z), \] (5.1)
then
\[ q_1(z) < \left( \frac{(1-\beta)zJ_{s,r,t}(n,\lambda)f(z) + \beta zJ_{s,r,t}(n+1,\lambda)f(z)}{\beta + 1} \right) ^{\frac{1}{\delta}} q_2(z), \]
where \( q_1 \) and \( q_2 \) are, respectively, the subordinant and the best dominant of (5.1).
Combining Theorem (3.4) with Theorem (4.3), we obtain the following sandwich theorem.

\textbf{Theorem 5.2.} Let \( q_i \) be two convex univalent functions in \( U \) such that \( q_i(0) = 1 \), \( q_i(0) \neq 0 \) \( (i = 1,2) \). Suppose that \( q_1 \) and \( q_2 \) satisfies (4.8) and (3.8), respectively.\( F \) \( f \) satisfies the next conditions:
\[ (1-\beta)zJ_{s,r,t}(n,\lambda)f(z) \]
and the function \( G(z) \) defined by (3.3) is univalent and satisfies
\[ q_1(z) + \frac{\eta}{\delta} z q_1(z) < G(z) \]
\[ q_2(z) + \frac{\eta}{\delta} z q_2(z), \]
\[ \text{implies} \]
\[ q_1(z) < \left( \frac{(1-\beta)zJ_{s,r,t}(n,\lambda)f(z) + \beta zJ_{s,r,t}(n+1,\lambda)f(z)}{\beta + 1} \right) ^{\frac{1}{\delta}} q_2(z), \]
where \( q_1 \) and \( q_2 \) are the best subordinant and the best dominant respectively and \( r(z) \) is given by (3.9).

\textbf{References}


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