On Differential Sandwich Results For Analytic Functions

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Abstract: In this paper, we obtain some subordination and superordination results involving
the integral operator $F_{\alpha}$. Also, we get Differential sandwich results for classes of univalent functions in the unit disk.

Keywords: Analytic function, univalent function, differential subordination, superordination.

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1-Introduction:

Let $H=H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $n$ a positive number additionally $a \in \mathbb{C}$. Let $H[a,n]$ be the subclass of $H$ entailing of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots = \sum_{k=0}^{\infty} a_k z^k \quad (a \in \mathbb{C}). \quad (1.1)$$

Let $A$ be the subclass of $H$ entailing of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.2)$$

The function $f$ is said to be subordinate to $g$, or $g$ is said to be subordinate to $f$, if there exists a Schwarz function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), to such an extent that $f(z) = g(w(z))$. In such a case we compose $f < g$ or $f(z) < g(z)(z \in U)$. If $g$ is univalent function in $U$, then $f < g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p,h \in H$ and $\psi(r,\delta,t,z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If $p$ and $\psi(p(z),zp'(z),z^2p''(z))$ are univalent functions in $U$ and if $p$ fulfills the second-order differential superordination:

$$h(z) < \psi(p(z),zp'(z),z^2p''(z)); z), \quad (1.3)$$

then $p$ is called a result of the differential superordination (1.3). (If $f$ is subordinate to $g$, then $g$ is subordinate to $f$) . An analytic function $q$ is called a subordinate of (1.3), if $q < p$ for very the functions $p$ filling (1.3).

An univalent subordinate $q$ that fulfills $q < q$ for all the subordinants $q$ of (1.3) is called the best subordinate. Miller and Mocanu [5] have gotten adequate conditions for certain standardized systematic capacities $f$ to fulfill:

$$q_1(z) < \left(\frac{p_1^{\beta_1 + 1}(z)}{z}\right)^{\alpha_1} < q_2(z),$$

and

$$q_1(z) < \left(\frac{\sum_{k=2}^{\infty} a_k z^k}{z}\right)^{\alpha_1} < q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$. Additionally, Tuneski [9] acquired adequate conditions for starlikeness of $f$ in relations of the amount $\frac{f''(z)}{f'(z)}$. Recently, Shammugam et al.[7,8], Goyal et al.[4] also gotten sandwich consequences for certain classes of analytic functions.

The principle question of the present paper is to discover adequate conditions for certain standardized systematic capacities $f$ to fulfill:

$$q_1(z) < \left(\frac{f^{\beta_1 + 1}(z)}{z}\right)^{\alpha_1} < q_2(z),$$

and

$$q_1(z) < \left(\frac{\sum_{k=2}^{\infty} a_k z^k}{z}\right)^{\alpha_1} < q_2(z),$$

wherever $q_1$ and $q_2$ are known univalent functions in $U$ with $q_1(0)=q_2(0)=1$.

2-Preliminaries:

With the end goal to demonstrate our subordination and superordination result , we require the accompanying definition and lemmas.

Definition 2.1 [5]: Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\xi \in \partial U : \lim_{z \to \xi} f(z) = \infty\}$.

(2.1)

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 2.1 [5]: Let $q$ be univalent in the unit disk $U$ and let $\theta$ and $\partial \theta$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = q(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(i) $Q(z)$ is starlike univalent in $U$,

(ii) $Re\left(\frac{\partial h(z)}{Q(z)}\right) > 0$ for $z \in U$.

If $p$ is analytic in $U$ with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)) \quad (2.2)$$

then $p < q$ and $q$ is the best dominant of (2.2).

Lemma 2.2 [6]: Let $q$ be convex univalent in function in $U$ and let $\alpha \in \mathbb{C}, \beta \in \mathbb{C}/\{0\}$ with

$$Re\left(1 + \frac{zq''(z)}{q(z)}\right) > \max(0, -Re\left(\frac{\alpha}{\beta}\right)).$$

If $p$ is analytic in $U$ and $\alpha p(z) + \beta zp'(z) < \alpha q(z) + \beta zq'(z), \quad (2.3)$

then $p < q$ and $q$ is the best dominant of (2.3).
**Lemma 2.3** [6]: Let q be convex univalent in U and let $\beta \in \mathbb{C}$, further assume that $\text{Re} (\beta) > 0$. If $P \in H[q(0)] \cap Q$ and $P(z) + \beta z p'(z)$ is univalent in U, then
\[
q(z) + \beta z q'(z) < p(z) + \beta z p'(z),
\]
which implies that $q < p$ and q is the best subordinant of (2.4).

**Lemma 2.4** [3]: Let q be convex univalent in the unit disk U and let $\theta$ and $\emptyset$ be analytic in domain D containing q (U). Suppose that
\[
\text{Re} \left( \frac{\theta(q(z))}{\emptyset(q(z))} \right) > 0 \quad \text{for} \quad z \in U,
\]
(i) $Q(z) = z q'(z) \emptyset(q(z))$ is starlike univalent in U.
If $p \in H[q(0), 1] \cap Q$, with $p(U)$
\[
\subset D, \theta(p(z)) + z p'(z) \emptyset(p(z))
\]
is univalent in U and
\[
\text{Re} \left( \frac{\theta(q(z))}{\emptyset(q(z))} \right) > 0 \quad \text{for} \quad z \in U,
\]
\[
\theta(q(z)) + z q'(z) \emptyset(q(z)) < \theta(p(z)) + z p'(z) \emptyset(p(z)),
\]
then $q < p$ and q is the best subordinant of (2.5).

3-**Subordination Consequences**

**Theorem 3.1**: Let q be convex univalent function in U with $q(0) = 1$, $0 < \Psi \in \mathbb{C}$, $\lambda > 0$ also, assume that q satisfies
\[
\text{Re} \left( \frac{\Psi(q(z))}{q'(z)} \right) > \max(0, \text{Re} \left( \frac{\Psi}{q'(z)} \right)).
\]
If $f \in A$ satisfies the subordination
\[
\left( 1 - \Psi^2 + 1 \right) \left( \frac{p^{\delta + 1}(z)}{z} \right)^{\lambda} + 
\Psi^2 \left( \frac{p^{\delta + 1}(z)}{z} \right)^{\lambda} \left( \frac{p^{\delta}(z)}{p^{\delta + 1}(z)} \right) < q(z) + \frac{\Psi}{\lambda} z q'(z),
\]
(3.2)
then
\[
\left( \frac{p^{\delta + 1}(z)}{z} \right)^{\lambda} < q(z),
\]
and q is the best dominant of (3.2).

**Proof**: Characterize the capacity p by
\[
p(z) = \left( \frac{p^{\delta + 1}(z)}{z} \right)^{\lambda}.
\]
Differentiating (3.4) with admiration to $z$ logarithmically, we get
\[
z p'(z) \frac{p(z)}{p(z)} = \lambda \left( \frac{z p^{\delta + 1}(z)}{z p^{\delta + 1}(z)} - 1 \right).
\]
(3.5)
Presently, in perspective of (1.7), we get the accompanying subordination
\[
\left( \frac{z p'(z)}{p(z)} \right) = \lambda \left( c \left( \frac{p^{\delta}(z)}{p^{\delta + 1}(z)} - 1 \right) + \left( \frac{p^{\delta}(z)}{p^{\delta + 1}(z)} - 1 \right) \right),
\]
therefore
\[
\left( \frac{z p'(z)}{p(z)} \right) = \lambda \left( c \left( \frac{p^{\delta}(z)}{p^{\delta + 1}(z)} - 1 \right) + \left( \frac{p^{\delta}(z)}{p^{\delta + 1}(z)} - 1 \right) \right).
\]
The subordination (3.2) from the speculation moves toward becoming
\[
p(z) + \frac{\Psi}{\lambda} z p'(z) < q(z) + \frac{\Psi}{\lambda} z q'(z).
\]

**Corollary 3.1**: Let $0 < \Psi \in \mathbb{C}$, $\lambda > 0$ also
\[
\text{Re} \left( 1 + \frac{\Psi}{\lambda} \right) > \max(0, \text{Re} \left( \frac{\Psi}{\lambda} \right)).
\]
If $f \in A$ satisfies the subordination
\[
\left( 1 - \Psi + 1 \right) \left( \frac{p^{\delta + 1}(z)}{z} \right)^{\lambda} + \Psi^2 \left( \frac{p^{\delta + 1}(z)}{z} \right)^{\lambda} \left( \frac{p^{\delta}(z)}{p^{\delta + 1}(z)} \right) < (1 - z\lambda^2 + 2 \frac{\Psi}{\lambda} z) \left( \frac{1}{(1 - z\lambda^2) \lambda} \right),
\]
then
\[
\left( \frac{p^{\delta + 1}(z)}{z} \right)^{\lambda} < \left( \frac{1 + \Psi}{z} \right),
\]
and q(z) = $\left( \frac{1 + \Psi}{z} \right)$ is the best dominant.

**Theorem 3.2**: Let q be convex univalent function in U with $q(0) = 1$, $q(z) \neq 0(z \in U)$ furthermore, accept that q fulfills
\[
\text{Re} \left( 1 + \frac{\lambda}{\Psi} \right) > 0,
\]
(3.6)
where $\Psi \in \mathbb{C}/\{0\}$, $\lambda > 0$ and z $\in U$.
Supposing that $-\Psi z q'(z)$ is starlike univalent function in U, if $f \in A$ fulfills:
\[
\emptyset(\lambda, \delta, c, \Psi, z) < \lambda q(z) - \Psi z q'(z),
\]
(3.7)
where
\[
\emptyset(\lambda, \delta, c, \Psi, z) = \lambda \left( \frac{p^{\delta + 1}(z) + (1 - \lambda) p^{\delta}(z)}{z} \right)^{\lambda} - \lambda \Psi \left( \frac{p^{\delta + 1}(z) + (1 - \lambda) p^{\delta}(z)}{z} \right)^{\lambda}
\]
(3.8)
then
\[
\left( \frac{p^{\delta + 1}(z) + (1 - \lambda) p^{\delta}(z)}{z} \right)^{\lambda} < q(z),
\]
(3.9)
and q(z) is the best dominant of (3.7).

**Proof**: Express the function p by
\[
p(z) = \left( \frac{p^{\delta + 1}(z) + (1 - \lambda) p^{\delta}(z)}{z} \right)^{\lambda},
\]
by setting:
\[
\theta(w) = \lambda w \quad \text{and} \quad \emptyset(w) = -\Psi, w \neq 0.
\]
We see that \( \theta(w) \) is analytic in \( \mathbb{C} \), \( \varphi(w) \) is analytic in \( \mathbb{C}/\{0\} \) and so on \( \varphi(w) \neq 0 \), \( w \in \mathbb{C}^* \).

Too, we get
\[
Q(z) = zq'(z)\varphi q(z) = -\Psi zq'(z),
\]
and
\[
h(z) = \varphi q(z) = \lambda q(z) - \Psi zq'(z).
\]
It is clear that \( Q(z) \) is starlike univalent in \( U \),
\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) = \text{Re} \left( 1 - \frac{\lambda}{\Psi} + \frac{zq''(z)}{q'(z)} \right) > 0.
\]

By a straightforward computation, we obtain \( \lambda \varphi(z) - \Psi z p'(z) = \varphi(\lambda, \delta, c, \Psi; z) \), \( (3.11) \)
where \( \varphi(\lambda, \delta, c, \Psi; z) \) is given by \( (3.8) \).

From \( (3.7) \) and \( (3.11) \), we have
\[
\left( \frac{\varphi z}{z} \right) < \lambda q(z) - \Psi q'(z).
\]
\( (3.12) \)

So, by Lemma 2.1, we become \( p(z) < q(z) \). By using \( (3.10) \), we get the result.

Putting \( q(z) = \frac{1+Az}{1+Bz} \) (-1 \( \leq B < A \leq 1 \) in Theorem 3.2, we obtain the next corollary:

**Corollary 3.2**: Let -1 \( \leq B < A \leq 1 \) while \( \lambda \)

\[
\text{Re} \left( \frac{1 - \lambda}{\Psi} + \frac{z}{(1+Bz)} \right) > 0,
\]

where \( \Psi \in \mathbb{C}/\{0\} \) and \( z \in U \), if \( f \in A \) contains
\[
\varphi(\lambda, \delta, c, \Psi; z) < \left( \frac{1+Az}{1+Bz} \right) - \frac{\lambda}{\Psi}(1-Bz),
\]
and \( \varphi(\lambda, \delta, c, \Psi; z) \) is given by \( (3.8) \).

\[
\left( \frac{\varphi z}{z} \right) < 1 +Az
\]

while \( q(z) = \frac{1+Az}{1+Bz} \) is the best dominant.

**4-Superordination Consequences**:

**Theorem 4.1**: Let \( q \) be convex univalent function in \( U \) with \( q(0) = 1, \lambda > 0 \) and \( \text{Re} \{\Psi\} > 0 \). Let \( f \in A \) satisfies
\[
\left( \frac{\varphi z}{z} \right) \lambda \in \mathbb{H} [q(0), 1] \cap Q,
\]
and
\[
(1 - \Psi(c + 1)) \left( \frac{\varphi z}{z} \right) \lambda + \Psi(c + 1) \left( \frac{\varphi z}{z} \right) \lambda \left( \frac{\varphi z}{z} \right) \lambda + \Psi(c + 1) \left( \frac{\varphi z}{z} \right) \lambda \left( \frac{\varphi z}{z} \right) \lambda,
\]
exist univalent in \( U \). If
\[
q(z) + \frac{\psi z}{z} q'(z) < (1 - \Psi(c + 1)) \left( \frac{\varphi z}{z} \right) \lambda + \Psi(c + 1) \left( \frac{\varphi z}{z} \right) \lambda \left( \frac{\varphi z}{z} \right) \lambda,
\]
then
\[
q(z) < \left( \frac{\varphi z}{z} \right) \lambda,
\]
and \( q(z) \) is the best subordinant of \( (4.1) \).

**Proof**: Express the function \( p \) by
\[
p(z) = \left( \frac{\varphi z}{z} \right) \lambda.
\]
Differentiating \( (4.3) \) with respect to \( z \) logarithmically, we get
\[
zp'(z) = \lambda \left( \frac{\varphi z}{z} \right) \lambda \left( \frac{\varphi z}{z} \right) \lambda.
\]
(4.4)

After some computations and using \( (1.7) \), from \( (4.4) \), we obtain
\[
(1 - \Psi(c + 1)) \left( \frac{\varphi z}{z} \right) \lambda + \Psi(c + 1) \left( \frac{\varphi z}{z} \right) \lambda \left( \frac{\varphi z}{z} \right) \lambda,
\]
and now, by using Lemma 2.3, we get the desired result.

Putting \( q(z) = \frac{1+Az}{1+Bz} \) in Theorem 4.1, we acquire the accompanying corollary:

**Corollary 4.1**: Let \( \lambda > 0 \) and \( \text{Re} \{\Psi\} > 0 \). If \( f \in A \) satisfies:
\[
\left( \frac{\varphi z}{z} \right) \lambda \in \mathbb{H} [q(0), 1] \cap Q,
\]
and
\[
(1 - \Psi(c + 1)) \left( \frac{\varphi z}{z} \right) \lambda + \Psi(c + 1) \left( \frac{\varphi z}{z} \right) \lambda \left( \frac{\varphi z}{z} \right) \lambda,
\]
be univalent in \( U \). If
\[
1 - \frac{z^2 + 2z}{(1-z)^2},
\]
then
\[
\left( \frac{1 + z}{1 - z} \right) < \left( \frac{\varphi z}{z} \right) \lambda,
\]
and \( q(z) = \frac{1+Az}{1+Bz} \) is the best subordinant.

**Theorem 4.2**: Let \( q \) be convex univalent function in \( U \) with \( q(0) = 1 \), also, accept that \( q \) fulfills
\[
\text{Re} \{-\lambda q'(\eta)\} > 0,
\]
where \( \eta \in \mathbb{C}/\{0\} \) and \( z \in U \).

Assume that \( -\Psi z q'(z) \) is starlike univalent function in \( U \), let \( f \in A \) satisfies
\[
\left( \frac{\varphi z}{z} \right) \lambda \in \mathbb{H} [q(0), 1] \cap Q,
\]
and \( \varphi(\lambda, \delta, c, \Psi; z) \) is univalent function in \( U \), where \( \varphi(\lambda, \delta, c, \Psi; z) \) is given by \( (3.8) \). If
\[
\lambda q(z) - \Psi z q'(z) < \varphi(\lambda, \delta, c, \Psi; z),
\]
then
\[
q(z) < \left( \frac{\varphi z}{z} \right) \lambda.
\]
(4.7)
and q is the best subordinant of (4.6).

**Proof:** Express the function p by
\[ p(z) = \left( \frac{f^{(l+1)}(z) + (1-t)F_c^l(f(z))}{z} \right)^\lambda, \]
(4.8)
by setting
\[ \theta(w) = \lambda w \text{ and } \omega(w) = -\Psi, w \neq 0, \]
we see that \( \theta(w) \) is analytic in \( \mathbb{C} \), \( \omega(w) \) is analytic in \( \mathbb{C}^* \) and that \( \omega(w) \neq 0, w \in \mathbb{C}^* \). Too, we get
\[ Q(z) = zq'(z)\mathcal{Q}(z) = -\Psi q'(z). \]
It is clear that \( Q(z) \) is starlike univalent function in \( U \).
\[
\text{Re}\left\{ \frac{\theta'(q(z))}{\Psi(q(z))} \right\} = \text{Re}\left\{ \frac{-\lambda q'(z)}{\Psi} \right\} > 0.
\]
By a straightforward computation, we obtain
\[
\mathcal{Q}(\lambda, \delta, c, \Psi; z) = \lambda p(z) - \Psi z p'(z),
\]
(4.9)
where \( \mathcal{Q}(\lambda, \delta, c, \Psi; z) \) is given by (3.8).
From (4.6) and (4.9), we have
\[
\lambda q(z) - \Psi q'(z) < \lambda p(z) - \Psi p'(z).
\]
(4.10)
So, by Lemma 2.4, we become \( q(z) < p(z) \). By using (4.8), we get the outcome.

**5-Sandwich Consequences:**

Concluding the consequences of differential subordination and superordination we arrive at the next "sandwich consequence".

**Theorem 5.1:** Let \( q_1 \) be convex univalent function in \( U \) with \( q_1(0)=1, \text{Re} \{\Psi\}>0 \) and let \( q_2 \) be univalent in \( U \), \( q_2(0)=1 \) and fulfills (3.1), let \( f \in \mathbb{A} \) satisfies:
\[
\left( \frac{f^{(l+1)}(z)}{z} \right)^\lambda \in H[1,1] \cap \mathbb{Q}, \]
and
\[
(1 - \Psi(c + 1)) \left( \frac{F_c^{(l+1)}(z)}{z} \right)^\lambda
+ \Psi(c + 1) \left( \frac{F_c^{(l+1)}(z)}{z} \right)^\lambda
+ (1 - \Psi(c + 1)) \left( \frac{F_c^{(l+1)}(z)}{z} \right)^\lambda.
\]
be univalent in \( U \). If
\[
q_1(z) + \frac{\Psi}{z} q_1'(z) < (1 - \Psi(c + 1)) \left( \frac{F_c^{(l+1)}(z)}{z} \right)^\lambda + \Psi(c + 1) \left( \frac{F_c^{(l+1)}(z)}{z} \right)^\lambda
\]
\[
< q_2(z) + \frac{\Psi}{z} q_2'(z), \text{then}
\]
\[
q_1(z) < \left( \frac{F_c^{(l+1)}(z)}{z} \right)^\lambda < q_2(z),
\]
and \( q_1 \) and \( q_2 \) are correspondingly the best subordinant and the best dominant.

**Theorem 5.2:** Let \( q_1 \) be convex univalent function in \( U \) with \( q_1(0)=1, \) fulfills (4.5), let \( q_2 \) be univalent function in \( U \), \( q_2(0)=1, \) fulfills (3.6), let \( f \in \mathbb{A} \) satisfies
\[
\left( \frac{tF_c^{(l+1)}(z) + (1-t)F_c^l(f(z))}{z} \right)^\lambda \in H[1,1] \cap \mathbb{Q},
\]
and \( \mathcal{Q}(\lambda, \delta, c, \Psi; z) \) is univalent in \( U \). Where \( \mathcal{Q}(\lambda, \delta, c, \Psi; z) \) is given by (3.8). If \( \lambda q_1(z) - \Psi q_1'(z) < \lambda q_2(z) - \Psi q_2'(z) \)
then
\[
q_1(z) < \left( \frac{tF_c^{(l+1)}(z) + (1-t)F_c^l(f(z))}{z} \right)^\lambda < q_2(z).
\]
In addition \( q_1 \) and \( q_2 \) are correspondingly the best subordinant and the best dominant.

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نتائج الساندوج التفاضلية للدوال التحليلية

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الملخص:
في هذا البحث، نحصل على بعض نتائج التبعية والتبعية العليا باستخدام المشغل التكامل 
أيضا، وحصلنا على نتائج الساندوج التفاضلية لصنف من الدوال احادية التكافؤ في قرص الوحدة.