Approximately Quasi-Prime Submodules and Some Related Concepts

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Abstract:
“Let $R$ be a commutative ring with identity and $B$ is a left unitary $R$-module. A proper submodule $E$ of $B$ is called a quasi-prime submodule, if whenever $r sb \in E$, where $r, s \in R$, $b \in B$ implies that either $rb \in E$ or $sb \in E$”. As a generalization of a quasi-prime submodules, in this paper we introduce the concept of approximately quasi-prime submodules, where a proper submodule $E$ of $B$ is an approximately quasi-prime submodule, if whenever $r sb \in E$, where $r, s \in R$, $b \in B$ implies that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$, where $soc(B)$ is the intersection of all essential submodules of $B$. Many basic properties, characterization and examples of this concept are given. Furthermore, we study the behavior of approximately quasi-prime submodules under $R$-homomorphisms. Finally, we introduced characterizations of approximately quasi-prime submodule in class of multiplication modules.

Keywords. Prime submodules, Quasi-prime submodules, Approximately prime submodules, Socle of submodules, Multiplication Modules, Approximately quasi-prime submodules.
1. Introduction

A quasi-prime submodule was introduced and studied in 1999 by [1] as a generalization of a prime submodule, where a proper submodule \( E \) of an \( R \)-module \( B \) is called a prime, if whenever \( rb \in E \), where \( r \in R, b \in B \) implies that either \( b \in E \) or \( r \in [E; R] B \) where \([E; R] B = \{ r \in R : rB \subseteq E \} \) [3]. Recently several generalizations of quasi-prime submodules were introduced such as “Weakly quasi-prime, Nearly quasi-prime, WE-quasi-prime, Weakly quasi 2-absorbing, Nearly quasi 2-absorbing, and Pseudo quasi 2-absorbing submodules see [14,12,6,7,8,9]”. In this paper, we give another generalization of a quasi-prime submodule, where a proper submodule \( E \) of \( B \) is an approximately quasi-prime submodule, if whenever \( rsb \in E \), where \( r, s \in R, b \in B \) implies that either \( rb \in E + soc(B) \) or \( sb \in E + soc(B) \). The concept of approximately quasi-prime submodule is also, generalization of the concept of approximately prime submodules which appear in [10], also generalization of prime submodules. Recall that an \( R \)-module \( B \) is multiplication if every submodule \( E \) of \( B \) is of the form \( E = IB \) for some ideal \( I \) of \( R \), in particular \( E = [E; R] B \) [4]. Let \( E \) and \( D \) be a submodule of a multiplication \( R \)-module \( B \) with \( E = IB \) and \( D = JB \) for some ideals \( I \) and \( J \) of \( R \), then \( ED = IJB \) that is \( ED = ID \). In particular \( EB = IBB = IB = E \). Also for any \( b \in B \), we define \( Eb = E(b) = Ib \) [15].

2. Basic properties of Approximately Quasi-Prime Submodules

In this part of the paper we introduce the definition of approximately quasi-prime submodule, and give some basic properties, examples and characterizations of this concept.

**Definition 2.1** A proper submodule \( E \) of \( B \) is said to be an approximately quasi-prime submodule(for short app-quasi-prime), if whenever \( rsb \in E \), where \( r, s \in R, b \in B \) implies that either \( rb \in E + soc(B) \) or \( sb \in E + soc(B) \). An ideal \( I \) of a ring \( R \) is an app-

quasi-prime ideal of \( R \) if and only if \( I \) is an app-
quasi-prime submodule of an \( R \)-module \( R \).

**Remark 2.2** It is clear that every quasi-prime submodule is an app-quasi-prime, but the convers is not true in general, the following example explain that:

**Example 2.3** Let \( B = Z_{12} \), \( R = Z \) and \( E = (\bar{0}) \), and \( soc(Z_{12}) = (\bar{2}) \). \( E \) is an app-quasi-prime submodule of \( B \) since if \( rsb \in E \), where \( r, s \in Z, b \in Z_{12} \), implies that either \( rb \in E + (\bar{2}) = (\bar{2}) \) or \( sb \in E + (\bar{2}) = (\bar{2}) \). But \( E \) is not a quasi-prime submodule of \( B \), since \( 3.4.2 \in E \), but neither \( 3.2 \in E \) nor \( 4.2 \in E \).

The following proposition gives a characterization of app-quasi-prime submodules.

**Proposition 2.4** Let \( B \) be an \( R \)-module, and \( E \) be a proper submodule of \( B \). Then \( E \) is an app-quasi-prime submodule of \( B \) if and only if whenever \( I / J \subseteq E \), where \( I, J \) are ideals in \( R \) and \( D \) is a submodule of \( B \), implies that either \( ID \subseteq E + soc(B) \) or \( JD \subseteq E + soc(B) \).

**Proof** (\( \Rightarrow \)) Suppose that \( I / J \subseteq E \), where \( I, J \) are ideals in \( R \) and \( D \) is a submodule of \( B \), and with \( ID \subseteq E + soc(B) \) and \( JD \subseteq E + soc(B) \). So there exists \( d_1, d_2 \in D \) and \( r \in I, s \in J \) such that \( rd_1 \notin E + soc(B) \) and \( sd_2 \notin E + soc(B) \). Since \( E \) is an app-quasi-prime submodule of \( B \) and \( rsd_1 \in E \) and \( rd_1 \notin E + soc(B) \) implies that \( sd_1 \in E + soc(B) \). Also \( rsd_2 \in E \) and \( sd_2 \notin E + soc(B) \) implies that \( rd_2 \in E + soc(B) \). It follows that either \( ID \subseteq E + soc(B) \) or \( JD \subseteq E + soc(B) \).

(\( \Leftarrow \)) Assume that \( rsb \in E \), where \( r, s \in R, b \in B \) implies that \( (r)(s)(b) \subseteq E \), so by hypothesis either \( (r)(b) \subseteq E + soc(B) \) or \( (s)(b) \subseteq E + soc(B) \). Thus either \( rb \in E + soc(B) \) or \( sb \in E + soc(B) \). Hence \( E \) is an app-quasi-prime submodule of \( B \).
As direct application of proposition 2.4, we get the following corollaries.

**Corollary 2.5** Let $B$ be an $R$-module, and $E$ be a proper submodule of $B$. Then $E$ is an app-quasi-prime submodule of $B$ if and only if whenever $rsD \subseteq E$, where $r, s \in R$ and $D$ is a submodule of $B$, implies that either $rD \subseteq E + soc(B)$ or $sD \subseteq E + soc(B)$.

**Corollary 2.6** Let $B$ be an $R$-module, and $E$ be a proper submodule of $B$. Then $E$ is an app-quasi-prime submodule of $B$ if and only if whenever $rb \subseteq E$, where $r \in R, b \in B$ and $I$ is an ideal of $R$, implies that either $rb \subseteq E + soc(B)$ or $lb \subseteq E + soc(B)$.

**Proposition 2.8** Let $B$ be an $R$-module, and $E$ be a proper submodule of $B$ with $soc(B) \subseteq E$. Then $E$ is an app-quasi-prime submodule of $B$ if and only if $[E + soc(B): b]$ is a prime ideal of $R$ for each $b \in B$.

**Proof** $(\Rightarrow)$ Let $rs \in [E + soc(B): b]$, where $r, s \in R$, implies that $rsb \in E + soc(B)$. But $soc(B) \subseteq E$, it follows that $E + soc(B) = E$, hence $rsb \in E$. But $E$ is an app-quasi-prime submodule of $B$, implies that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. Thus either $r \in [E + soc(B): b]$ or $s \in [E + soc(B): b]$.

$(\Leftarrow)$ Suppose that $rsb \in E$, where $r, s \in R, b \in B$ implies that either $rb \in E \subseteq E + soc(B)$, it follows that $rb \subseteq E + soc(B)$, hence $rs \in [E + soc(B): b]$. But $[E + soc(B): b]$ is a prime ideal of $R$, implies that either $r \in [E + soc(B): b]$ or $s \in [E + soc(B): b]$, it follows that either $rb \in E + soc(B)$ or $sb \in E + soc(B)$.

**Proposition 2.9** Let $B$ be an $R$-module, and $E$ be a proper submodule of $B$. Then $E$ is an app-quasi-prime submodule of $B$ if and only if whenever $[E: rsb] \subseteq [E + soc(B): r] \cup [E + soc(B): s]$ for all $r, s \in R$.

**Proof** $(\Rightarrow)$ Let $b \in [E: rsb]$, implies that $rsb \in E$. But $E$ is an app-quasi-prime submodule of $B$, then either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. It follows that either $b \in [E + soc(B): r]$ or $b \in [E + soc(B): s]$. Thus $[E: rsb] \subseteq [E + soc(B): r] \cup [E + soc(B): s]$

$(\Leftarrow)$ Now, let $rsb \in E$, where $r, s \in R, b \in B$, then $b \in [E: rsb] \subseteq [E + soc(B): r] \cup [E + soc(B): s]$, implies that $b \in [E + soc(B): r]$ or $b \in [E + soc(B): s]$. Hence $rb \in E + soc(B)$ or $sb \in E + soc(B)$. Thus $E$ is an app-quasi-prime submodule of $B$.

**Proposition 2.10** Let $B$ be an $R$-module, and $E$ be a proper submodule of $B$ such that $E$ is an app-quasi-prime submodule of $B$. Then $[E: rsb] \subseteq [E + soc(B): r] \cup [E + soc(B): s]$ for all $r, s \in R, b \in B$.

**Proof** Let $x \in [E: rsb]$, where $r, s \in R, b \in B$, implies that $rs(xb) \in E$. But $E$ is an app-quasi-prime submodule of $B$, then either $x(xb) \in E + soc(B)$ or $s(xb) \in E + soc(B)$, it follows that either $x \in [E + soc(B): r]$ or $x \in [E + soc(B): s]$. Hence $x \in [E + soc(B): r] \cup [E + soc(B): s]$. Thus $[E: rsb] \subseteq [E + soc(B): r] \cup [E + soc(B): s]$.

**Remark 2.11** Let $B$ be an $R$-module, and $E$ is an app-quasi-prime submodule of $B$, it is not necessary that $[E: B]$ is an app-quasi-prime ideal of $R$. For example in a $Z$-module $Z_{12}$, $(0)$ is an app-quasi-prime submodule, but $[0: Z_{12}] = 12Z$ is not app-quasi-prime ideal of $Z$-module $Z$. Since $2.2.3 \in 12Z$, but $2.3 \in 12Z + soc(Z) = 12Z + 0 = 12Z$. 

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Proposition 2.12 Let $B$ be an $R$-module, and $E$ is an app-quasi-prime submodule of $B$ with $soc(B) \subseteq E$. Then $[E:B]$ is an app-quasi-prime ideal of $R$.

Proof Let $rst \in [E:B]$, where $r,s,t \in R$, implies that $rs(tB) \subseteq E$. Thus since $E$ is an app-quasi-prime submodule of $B$, then by Corollary 2.5 either $r(tB) \subseteq E + soc(B)$ or $s(tB) \subseteq E + soc(B)$. But $soc(B) \subseteq E$, implies that $E + soc(B) = E$. Hence either $r(tB) \subseteq E$ or $s(tB) \subseteq E$. That is either $rt \in [E:B] \subseteq [E:B] + soc(R)$ or $st \in [E:B] \subseteq [E:B] + soc(R)$. Therefore $[E:B]$ is an app-quasi-prime ideal of $R$.

Recall that an $R$-module $B$ is faithful if $ann_B = (0)$ [4].

Before we introduce the converse of Proposition 2.12 we recall the following lemmas:

Lemma 2.13 [4, Coro. 2.14] Let $B$ be a faithful multiplication $R$-module then $soc(R)B = soc(B)$.

Proposition 2.16 Let $B$ be a non-singular multiplication $R$-module and $E$ is a proper submodule of $B$. If $[E:B]$ is an app-quasi-prime ideal of $R$, then $E$ is an app-quasi-prime submodule of $B$.

Proof Follows in similar way of Proposition 2.15 and using Lemma 2.14.

Lemma 2.17 [13, Coro. Of Theo. 9] Let $I$ and $J$ be ideals of a ring $R$, and $B$ is a finitely generated multiplication $R$-module. Then $IB \subseteq JB$ if and only if $I \subseteq J + annB$.

Proposition 2.18 Let $B$ be a faithful finitely generated multiplication $R$-module. If $J$ is an app-quasi-prime ideal of $R$, then $JB$ is an app-quasi-prime submodule of $B$.

Proof Suppose that $rsb \in JB$, where $r,s \in R, b \in B$. Then $rs(b) \subseteq JB$. Since $B$ is a multiplication, implies that $(b) = IB$ for some ideal $I$ of $R$. Thus $rsIB \subseteq JB$. But $B$ is a finitely generated, so by Lemma 2.17 $rsI \subseteq J + annB$. But $B$ is faithful, it follows that $annB = (0)$, hence $rsI \subseteq J$, but $J$ is an app-quasi-prime ideal of $R$, then by Corollary 2.5 either $rl \subseteq J + soc(R)$ or $sl \subseteq J + soc(R)$, it follows that either $rlB \subseteq JB + soc(R)B$ or $slB \subseteq JB + soc(R)B$. But by Lemma 2.13 $soc(R)B = soc(B)$, therefore $rIB \subseteq JB + soc(B)$ or $sIB \subseteq JB + soc(B)$, it follows that either $rb \in JB + soc(B) or sb \in JB + soc(B)$. Therefore $JB$ is an app-quasi-prime submodule of $B$.

Proposition 2.19 Let $B$ be a finitely generated multiplication non-singular $R$-module, and $J$ is an app-quasi-prime ideal of $R$ with $annB \subseteq J$. Then $JB$ is an app-quasi-prime submodule of $B$. 

soc(R)B = soc(B)$, and since $B$ is multiplication $[E:B] = E$, therefore either $rb \in E + soc(B)$ or $sb \in E + soc(B)$. Hence $E$ is an app-quasi-prime submodule of $B$. 

Proposition 2.15 Let $B$ be a faithful multiplication $R$-module and $E$ is a proper submodule of $B$. If $[E:B]$ is an app-quasi-prime ideal of $R$, then $E$ is an app-quasi-prime submodule of $B$.

Proof Suppose that $rsb \in E$, where $r,s \in R, b \in B$ implies that $rs(b) \subseteq E$. Since $B$ is a multiplication, then $(b) = JB$ for some ideal $J$ of $R$. That is $rsJ \subseteq E$, it follows that $rsj \subseteq [E:B]$. But $[E:B]$ is an app-quasi-prime ideal of $R$, then by Corollary 2.5 either $rJ \subseteq [E:B] + soc(R)$ or $sJ \subseteq [E:B] + soc(R)$, it follows that either $rJB \subseteq [E:B]B + soc(R)B$ or $sJB \subseteq [E:B]B + soc(R)B$. Hence, by Lemma 2.13
Proof Similar steps of Proposition 2.18 and using Lemma 2.14 and the condition \( \text{ann}B \subseteq J \) implies that \( J + \text{ann}B = J \).

Remark 2.20 The intersection of any two app-quasi-prime submodules of an \( R \)-module \( B \) not necessary app-quasi-prime submodule of \( B \), the following example shows that.

Example 2.21 Let \( B \) be the \( Z \)-module \( Z \) and \( E = 2Z \) and \( D = 3Z \). It is clear that \( E \) and \( D \) are app-quasi-prime submodules of \( B \), but \( E \cap D = 6Z \) is not app-quasi-prime submodule of \( B \) since \( 2.3.1 \in Z \), but \( 2.1 \notin 6Z + \text{soc}(B) \) and \( 3.1 \notin 6Z + \text{soc}(B) \), where \( \text{soc}(B) = (0) \).

Proposition 2.22 Let \( B \) be an \( R \)-module, and \( E, D \) are app-quasi-prime submodules with \( E \subseteq \text{soc}(B) \) and \( D \subseteq \text{soc}(B) \). Then \( E \cap D \) is an app-quasi-prime submodule of \( B \).

Proof Suppose that \( rsb \in E \cap D \), where \( r, s \in R \), \( b \in B \), then \( rsb \in E \) and \( rsb \in D \). Since both \( E \) and \( D \) are app-quasi-prime submodules of \( B \), so either \( rb \in E + \text{soc}(B) \) or \( sb \in E + \text{soc}(B) \) and either \( rb \in D + \text{soc}(B) \) or \( sb \in D + \text{soc}(B) \). But \( E \subseteq \text{soc}(B) \) and \( D \subseteq \text{soc}(B) \), it follows that \( E + \text{soc}(B) = \text{soc}(B) \) and \( D + \text{soc}(B) = \text{soc}(B) \) and \( E \cap D \subseteq \text{soc}(B) \), implies that \( E \cap D + \text{soc}(B) = \text{soc}(B) \). Thus we have either \( rb \in E \cap D + \text{soc}(B) \) or \( sb \in E \cap D + \text{soc}(B) \). That is \( E \cap D \) is an app-quasi-prime submodule of \( B \).

Lemma 2.23 [11, Lemma 2.3.15] Let \( B \) be an \( R \)-module, and \( E, D, F \) are submodules of \( B \) with \( D \) is contained in \( F \) then \( (E + D) \cap F = (E \cap F) + (D \cap F) = (E \cap F) + D \).

Lemma 2.24 [2, Coro. 9.9] Let \( B \) be an \( R \)-module, and \( E \) submodule of \( B \), then \( \text{soc}(E) = E \cap \text{soc}(B) \).

Proposition 2.25 Let \( B \) be an \( R \)-module, and \( E, D \) are two submodules of \( B \) with \( D \) is not contained in \( E \) and \( \text{soc}(B) \subseteq D \). If \( E \) is an app-quasi-prime submodule of \( B \), then \( E \cap D \) is an app-quasi-prime submodule of \( D \).

Proof Since \( D \) is not contained in \( E \), then \( E \cap D \) is a proper submodule of \( D \). Now, let \( rsb \in E \cap D \) where \( r, s \in R \), \( b \in D \subseteq B \), then \( rsb \in E \) and \( rsb \in D \). But \( E \) is an app-quasi-prime submodule of \( B \), then either \( rb \in E + \text{soc}(B) \) or \( sb \in E + \text{soc}(B) \), since \( b \in D \), it follows that either \( rb \in (E + \text{soc}(B)) \cap D \) or \( sb \in (E + \text{soc}(B)) \cap D \). Since \( \text{soc}(B) \subseteq D \), then by Lemma 2.23 we have \( (E + \text{soc}(B)) \cap D \subseteq (E \cap D) + \text{soc}(B) \cap D \), and by Lemma 2.24 we have \( \text{soc}(B) \cap D = \text{soc}(D) \). Hence either \( rb \in E \cap D + \text{soc}(D) \) or \( sb \in E \cap D + \text{soc}(D) \). Thus \( E \cap D \) is an app-quasi-prime submodule of \( D \).

Proposition 2.26 Let \( f \in \text{Hom}(B, B') \) be an \( R \)-epimorphism, and \( E \) is an app-quasi-prime submodule of \( B' \). Then \( f^{-1}(E) \) is an app-quasi-prime submodule of \( B \).

Proof It is clear that \( f^{-1}(E) \) is a proper submodule of \( B \). Now, let \( rsb \in f^{-1}(E) \), where \( r, s \in R, b \in B \), implies that \( rsb(b) \in E \). Since \( E \) is an app-quasi-prime submodule of \( B' \), so either \( rf(b) \in E + \text{soc}(B') \) or \( sf(b) \in E + \text{soc}(B') \). Hence either \( rb \in f^{-1}(E) + f^{-1}(\text{soc}(B')) \subseteq f^{-1}(E) + \text{soc}(B) \) or \( sb \in f^{-1}(E) + f^{-1}(\text{soc}(B')) \subseteq f^{-1}(E) + \text{soc}(B) \). That is either \( rb \in f^{-1}(E) + \text{soc}(B) \) or \( sb \in f^{-1}(E) + \text{soc}(B) \). Therefore \( f^{-1}(E) \) is an app-quasi-prime submodule of \( B \).

Proposition 2.27 Let \( f \in \text{Hom}(B, B') \) be an \( R \)-epimorphism, and \( E \) be an app-quasi-prime submodule of \( B \) with \( \text{Ker} f \subseteq E \). Then \( f(E) \) is an app-quasi-prime submodule of \( B' \).

Proof \( f(E) \) is a proper submodule of \( B' \). If not, suppose that \( f(E) = B' \), let \( b \in B \), then \( f(b) = f(E) = B \), implies that \( f(b) = f(e) \) for some \( e \in E \), it follows that \( f(b - e) = 0 \), so \( b - e \in \text{Ker} f \subseteq E \), hence \( b \in E \), that is \( E = B \) contradiction. Now let
Proposition 2.28 Let $B, B'$ be $R$-modules, and $E$ be a proper submodule of $B'$, such that $E + \text{soc}(B')$ is a quasi-prime submodule of $B'$, with $\text{Hom}_R(B, E + \text{soc}(B'))$ is a proper submodule of $\text{Hom}_R(B, B')$. Then $\text{Hom}_R(B, E + \text{soc}(B'))$ is app-quasi-prime submodule of $\text{Hom}_R(B, B')$.

Proof Suppose that $rsf \in \text{Hom}_R(B, E + \text{soc}(B'))$ where $r, s \in R$, $f \in \text{Hom}_R(B, B')$. Then for each $b \in B$, we have $rsf(b) \in E + \text{soc}(B')$. But $E + \text{soc}(B')$ is quasi-prime submodule of $B'$, then either $r*(f(b)) \in E + \text{soc}(B')$ or $sf(b) \in E + \text{soc}(B')$. Hence $r*(f(b)) \in E + \text{soc}(B')$ or $sf(b) \in E + \text{soc}(B')$. Thus $r*(f(b)) \in E + \text{soc}(B')$. Hence $r*(f(b)) \in E + \text{soc}(B')$.

Proposition 2.29 Let $B = B_1 \oplus B_2$ be an $R$-module, where $B_1, B_2$ be modules, and $E = E_1 \oplus E_2$ be submodules of $B_1, B_2$ respectively with $E \subseteq \text{soc}(B)$. If $E$ is an app-quasi-prime submodule of $B$, then $E_1$ and $E_2$ are app-quasi-prime submodules of $B_1$ and $B_2$ respectively.

Proof Suppose that $b_i \in E_1$, then for each $b_2 \in B_2$ $rs(b_1, b_2) \in E \oplus B_2$. But $E \oplus B_2$ is an app-quasi-prime submodule of $B$, then for each $b_2 \in B_2$ $rs(b_1, b_2) \in E \oplus B_2$. But $E \oplus B_2$ is an app-quasi-prime submodule of $B$, so, either $r*(b_i, b_2) \in E \oplus B_2 + \text{soc}(B_1 \oplus B_2)$ or $s(b_1, b_2) \in E \oplus B_2 + \text{soc}(B_1 \oplus B_2)$. If $r*(b_i, b_2) \in E \oplus B_2 + \text{soc}(B_1 \oplus B_2)$, since $E \subseteq \text{soc}(B_1)$, then $E + \text{soc}(B_1) = \text{soc}(B_1)$, and $\text{soc}(B_2) = B_2$ so, $r*(b_i, b_2) \in E \oplus B_2 + \text{soc}(B_1 \oplus B_2) \oplus \text{soc}(B_2)$, implies that $r*(b_i, b_2) \in (E + \text{soc}(B_1)) \oplus \text{soc}(B_2)$. But $E \subseteq \text{soc}(B)$, implies that $E + \text{soc}(B) = \text{soc}(B) = \text{soc}(B_1) \oplus \text{soc}(B_2)$. If $s(b_1, b_2) \in \text{soc}(B_1) \oplus \text{soc}(B_2)$, implies that $s(b_1, b_2) \in E + \text{soc}(B_1)$, hence $r*(b_i, b_2) \in E \oplus B_2$. It follows that $r*(b_i, b_2) \in E + \text{soc}(B_1)$. Similarly if $s(b_1, b_2) \in E \oplus B_2$, implies that $s(b_1, b_2) \in E + \text{soc}(B_1)$. Therefore $E$ is an app-quasi-prime submodule of $B_1$.
In similar way we can prove (2).

**Remark 2.31** It is clear that every prime submodule is an app-quasi-prime submodule while the converse is not true in general as the following example shows that.

**Example 2.32** Consider the $Z$-module $Z_4$, the submodule $E = \langle \overline{0} \rangle$ is an app-quasi-prime submodule of $Z_4$, since for each $r, s \in Z$, and $b \in Z_4$, with $rsb \in E$, we have either $rb \in E + \text{soc}(Z_4) = E + \langle \overline{2} \rangle$ or $sb \in E + \text{soc}(Z_4) = E + \langle \overline{2} \rangle$. But $\langle \overline{0} \rangle$ is not prime submodule of $Z_4$, because $2, 2 \in \langle \overline{0} \rangle$, $2 \in Z$, $2 \in Z_4$, but $2 \not\in \langle \overline{0} \rangle$ and $2 \not\in \langle \overline{0} \rangle: Z_4 = 4Z$.

Recall that a proper submodule $E$ of an $R$-module $B$ is called an app-prime submodule of $B$, if whenever $rb \in E$, with $r \in R$, $b \in B$, implies that either $b \in E + \text{soc}(B)$ or $rb \subseteq E + \text{soc}(B)$ [10].

**Remark 2.33** It is clear that every app-prime submodule is an app-quasi-prime submodule, while the converse is not true in general, as the following example shows that.

**Example 2.34** Consider the $Z$-module $Z\oplus Z$, and $E = \langle 0 \rangle \oplus \langle 2 \rangle$, $E$ is not app-prime, since $2(0, 1) \in E$, but $\langle 0, 1 \rangle \not\subseteq E + \text{soc}(Z\oplus Z)$, and $2 \not\in [(0)\oplus 2Z + \text{soc}(Z\oplus Z) : Z\oplus Z] = \langle 0 \rangle$. But $E$ is an app-quasi-prime because $E$ is a quasi-prime submodule of $Z\oplus Z$.

**Proposition 2.35** Let $B$ be an $R$-module, and $E$ be a proper submodule of $B$, with $\text{soc}(B) \subseteq E$. Then $E$ is an app-quasi-prime submodule of $B$ if and only if $[E : R I]$ is an app-quasi-prime submodule of $B$ for every ideal $I$ of $R$.

**Proof** ($\Rightarrow$) Let $rsb \in [E : R I]$, with $r, s \in R$, $b \in B$, implies that $rsbI \subseteq E$, that is $rsba \in E$ for each $a \in I$. Since $E$ is an app-quasi-prime submodule of $B$, it follows that either $rba \in E + \text{soc}(B)$ or $sba \in E + \text{soc}(B)$, but $\text{soc}(B) \subseteq E$, implies that $E + \text{soc}(B) = E$. Thus either $rba \in E$ or $sba \in E$. That is either $rb \in [E : R I] \subseteq [E : R I] + \text{soc}(B)$ or $rb \in [E : R I] \subseteq [E : R I] + \text{soc}(B)$. Hence $[E : R I]$ is an app-quasi-prime submodule of $B$.

($\Leftarrow$) Since $[E : R I]$ is an app-quasi-prime submodule of $B$ for each ideal $I$ of $R$, thus put $I = R$, we get $[E : R] = E$ is an app-quasi-prime submodule of $B$.

**Proposition 2.36** Let $B$ be a multiplication $R$-module, and $E$ be a proper submodule of $B$. Then $E$ is an app-quasi-prime submodule of $B$ if and only if whenever $FDb \subseteq E$, for some submodules $F$ and $D$ of $B$ and $b \in B$, then either $Fb \subseteq E + \text{soc}(B)$ or $Db \subseteq E + \text{soc}(B)$.

**Proof** ($\Rightarrow$) Suppose that $FDb \subseteq E$, for some submodules $F$ and $D$ of $B$ and $b \in B$. But $B$ is a multiplication then $F = IB$ and $D = JB$ for some ideals $I, J$ of $R$, thus $FDb = IJb \subseteq E$. But $E$ is an app-quasi-prime submodule of $B$, then by Corollary 2.7 either $Ib \subseteq E + \text{soc}(B)$ or $Jb \subseteq E + \text{soc}(B)$. It follows that either $Fb \subseteq E + \text{soc}(B)$ or $Db \subseteq E + \text{soc}(B)$.

($\Leftarrow$) Assume that $IJb \subseteq E$, where $I, J$ are ideals in $R$ and $b \in B$. Since $B$ is a multiplication it follows that, $IDb = FDb \subseteq E$, so by hypothesis either $Db \subseteq E + \text{soc}(B)$ or $Fb \subseteq E + \text{soc}(B)$, that is either $Ib \subseteq E + \text{soc}(B)$ or $Jb \subseteq E + \text{soc}(B)$. Hence by Corollary 2.7 Then $E$ is an app-quasi-prime submodule of $B$.

**Proposition 2.37** Let $B$ be a multiplication $R$-module, and $E$ be a proper submodule of $B$. Then $E$ is an app-quasi-prime submodule of $B$ if and only if whenever $FDL \subseteq E$, for some submodules $F, D$ and $L$ of $B$, then either $FL \subseteq E + \text{soc}(B)$ or $DL \subseteq E + \text{soc}(B)$.

**Proof** ($\Rightarrow$) Suppose that $FDL \subseteq E$, for some submodules $F, D$ and $L$ of $B$. But $B$ is a multiplication then $F = IB$ and $D = JB$ for some ideals $I, J$ of $R$, thus $FDL = IJL \subseteq E$. Since $E$ is an
app-quasi-prime submodule of $B$, then by Proposition 2.4 either $IL \subseteq E + soc(B)$ or $JL \subseteq E + soc(B)$. It follows that either $FL \subseteq E + soc(B)$ or $DL \subseteq E + soc(B)$.

($\iff$) Assume that $IL \subseteq E$, where $I, J$ are ideals in $R$ and $L$ is a submodule of $B$. Since $B$ is a multiplication it follows that, $IDL = FDL \subseteq E$, so by hypothesis either $FL \subseteq E + soc(B)$ or $DL \subseteq E + soc(B)$, that is either $IL \subseteq E + soc(B)$ or $JL \subseteq E + soc(B)$. Hence by Proposition 2.4 $E$ is an app-quasi-prime submodule of $B$.

**Proposition 2.38** Let $B$ be a faithful finitely generated multiplication $R$-module, and $E$ be a proper submodule of $B$ with $soc(B) \subseteq E$. then the following statements are equivalent.
1) $E$ is an app-quasi-prime submodule of $B$.
2) $[E:_RB]$ is an app-quasi-prime ideal of $R$.
3) $E = IB$ for some app-quasi-prime ideal $I$ of $R$.

**Proof** (1) $\implies$ (2) Follows by Proposition 2.12
(2) $\implies$ (1) Follows by Proposition 2.15
(2) $\iff$ (3) Since $[E:_RB]$ is an app-quasi-prime ideal of $R$, and $E = [E:_RB]B$, it is follows that $E = IB$ and $I = [E:_RB]$ an app-quasi-prime ideal of $R$.
(3) $\implies$ (2) Suppose that $E = IB$ for some app-quasi-prime ideal $I$ of $R$. But $B$ is a multiplication we have $E = [E:_RB]B = IB$. Thus since $B$ is faithful finitely generated multiplication, then by Lemma 2.17 we have $I = [E:_RB]$, it follows that $[E:_RB]$ is an app-quasi-prime ideal of $R$.

**References**
المقاسات الجزئية الأولية الظاهرة تقريباً ومفاهيم ذات علاقة

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المستخلص:

للحقق ابداله بمحاد ومقاس جزئي أولي ظاهري إذا كان رحذة
مقاس جزئي اولي ظاهري بيلا. يدعو المقياس الجزئي الفعلي E من المقياس B مقاس جزئي اولي ظاهري إذا كان
لحذف ابداً بمحاد ورب E ليؤذي إلى اما
المقاس B، rb ∈ Eจร جزءاً منه جزءاً منه
ب E + soc(B) حيث أن
و مقاس جزئي اولي ظاهري تقريباً إذا كان
كل المقاسات الجزئية الأولية الظاهرة تقريباً تحت تأثير التشابكات. واخيراً قدمنا العديد
من المكافات للمقاسات الجزئية الظاهرة تقريباً في صنف المقاسات الضربية.