Some Properties of Topology Fuzzy Modular Space

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Abstract:

In the present paper, the authors have introduced and studied fuzzy modular space. They have investigated some properties of this space in the open and closed balls. Also the authors discussed the convex set and the locally convex in fuzzy modular space. The result obtained are correct and the methods used are interesting.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [5] in 1965 and thereafter several authors applied it to different branches of pure and applied mathematics. The concept of modular space was introduced by S.S. Abed, K.A. Abdul Sada in 2017. The concept of fuzzy modular space was introduced by Young Shen and Wei Chen [7] in 2013.

Definition (1.1) [4]
A fuzzy set \( A \) in \( X \) (or a fuzzy subset in \( X \)) is a function from \( X \) into \( I = [0,1] \) that is \( A \in I^X \).

Definition (1.2) [6]
Let \( X \) be a linear space over \( F \). A function \( M: X \to [0, \infty] \) is called modular if:
1. \( M(x) = 0 \iff x = 0 \),
2. \( M(ax) = M(x) \) for \( a \in F \) with \( |a| = 1 \)
   for all \( a \in F \).
3. \( M(ax + \beta y) \leq M(x) + M(y) \)
   iff \( \alpha, \beta \geq 0 \) for all \( x, y \in X \).

Definition (1.3) [2]
A binary operation \( *: [0,1] \times [0,1] \to [0,1] \)
is called a continuous t-norm if it satisfies the following:
1. \( * \) is commutative and associative.
2. \( * \) is continuous.
3. \( a \ast b \leq c \ast d \) whenever \( a \leq c, b \leq d \)
   and \( a, b, c, d \in [0,1] \).

Three common examples of the continuous \( t \)-norm are:
1. \( a \ast_M b = \min \{a, b\} \).
2. \( a \ast_p b = a \cdot b \).
3. \( a \ast_1 b = \max \{a + b - 1, 0\} \).

Lemma (1.4) [2]
If the \( t \)-norm \( * \) is continuous, then
1. for every \( y_1, y_2 \in (0,1) \) with \( y_1 \geq y_2 \), there exist \( y_3 \in (0,1) \) such that \( y_3 \geq y_2 \).
2. for every \( y_3 \in (0,1) \), there exist \( y_4 \in (0,1) \)
such that \( y_3 \ast y_4 \geq y_4 \).

Definition (1.5) [7]
A fuzzy modular space is an ordered triple \((X, \mu, *)\) such that \( X \) is a vector space, \( * \) is continuous \( t \)-norm and \( \mu \) is a fuzzy set on \( X \times (0, \infty) \)
satisfying the following condition for all \( x, y \in X \) and \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \):
1. \( \mu(x, t) > 0 \).
2. \( \mu(x, t) = 1 \) for all \( t > 0 \) if and only if \( x = 0 \).
3. \( \mu(x, t) = \mu(-x, t) \).
4. \( \mu(ax + \beta y, s + t) \geq \mu(x, s) \ast \mu(y, t) \).
5. \( \mu(x, \cdot) : (0, \infty) \to (0,1] \) is continuous.

Generally, if \((X, \mu, *)\) is fuzzy modular space, we say that \((\mu, *)\) is a fuzzy modular on \( X \).

Moreover, the triple \((X, \mu, *)\) is called \( \beta \)-homogeneous if every \( x \in X, t > 0 \) and \( \lambda \in \mathbb{R} \setminus \{0\} \)
\( \mu(\lambda x, t) = \mu \left(x, \frac{\lambda t}{|\lambda|}\right) \), where \( \beta \in (0,1] \).

Example (1.6) [7]
Let \( X \) be a vector space and let \( \rho \) be a modular on \( X \). Take \( t \)-norm \( a \ast b = a \ast_M b \). For every \( t \in (0, \infty) \) define \( \mu(x, t) = t/(t + p(x)) \)}
for all \( x \in X \). Then \((X, \mu, \ast)\) is a fuzzy modular space.

**Remark (1.7):** [7]

Note the above conclusion still holds even if the t-norm is replaced by \( a \ast b = a \ast_p b \) and \( a \ast b = a \ast_L b \), respectively.

**Example (1.8):** [7]

Let \( X = \mathbb{R} \), take t-norm \( a \ast b = a \ast_m b \). For every \( x, y \in X \) and \( t \in (0, \infty) \), we define

\[
\mu(x, t) = \sqrt[\beta]{x / 2} = \frac{1}{e^{\rho(x) / t}}
\]

for all \( x \in X \). Then \((X, \mu, \ast)\) is \( \beta \)-homogenous fuzzy modular space.

**Definition (1.10):** [1]

A fuzzy metric space is an ordered triple \((X, M, \ast)\) such that \( X \) is a nonempty set, \( \ast \) is a continuous t-norm, and \( M \) is a fuzzy set on \( X \times X \times (0, \infty) \) satisfying the following conditions:

1. \( M(x, y, t) \geq 0 \).
2. \( M(x, y, t) = 0 \) if and only if \( x = y \).
3. \( M(y, x, t) = M(x, y, t) \).
4. \( M(x, z, t) \leq M(x, y, t) + M(y, z, t) \).
5. \( M(x, y):(0, \infty) \rightarrow [0,1] \) is continuous.

**Theorem (1.11):**

Every fuzzy modular space is fuzzy metric space.

**Proof:**

Let \((X, \mu, \ast)\) be a fuzzy modular space defined as follows:

\[
M(x, y, t) = \mu(x - y, t) \quad \forall x, y \in X
\]

1. Let \( x, y \in X \) \( \Rightarrow x - y \in X \Rightarrow \mu(x - y, t) > 0 \Rightarrow M(x, y, t) > 0 \)

2. \( \mu(x - y, t) = 1 \iff 1 = M(x, y, t) \)

3. Let \( x, y \in X \) and \( t > 0 \)

\[
M(x, y, t) = \mu(x - y, t) = \mu(y - x, t)
\]

4. \( \mu(x - y, t) \leq \mu((x - y) + (y - z), t + s) = \mu(x - z, t + s) = M(x, z, t + s) \)

5. \( M(x, y) = \mu(x - y, t):(0, \infty) \rightarrow [0,1] \) is continuous.

**Theorem (1.12):** [7]

If \((X, \mu, \ast)\) is a fuzzy modular space,
then $\mu(x,.)$ is nondecreasing for all $x \in X$.

**Definition (1.13):** [7]

Let $(X,\mu,\ast)$ be a fuzzy modular space. We define the open ball $B(x,r,t)$ and the closed ball $B[x,r,t]$ with center $x \in X$ and radius $0 < r < 1$ as follows: For $t > 0$

\[ B(x,r,t) = \{ y \in X : \mu(x-y,t) > 1 - r \} \]

open balls

\[ B[x,r,t] = \{ y \in X : \mu(x-y,t) \geq 1 - r \} \]

closed balls

**Theorem (1.14)**

If $(X,\mu,\ast)$ is a $\beta$-homogenous fuzzy modular space then $B(x,r,t) \subset B(x,r_{1/2})$.

**Proof:**

Let $B(x,r_1,t)$ and $B(x,r_2,t)$ be open balls with the same center $x \in X$ and $t > 0$ with the radius $0 < r_1 < 1$ and $0 < r_2 < 1$, respectively.

Then we either have:

$B(x,r_1,t) \subset B(x,r_2,t)$ or $B(x,r_2,t) \subset B(x,r_1,t)$.

Let $x \in X$ and $t > 0$. Consider the open ball

$B(x,r_1,t)$ and $B(x,r_2,t)$ with $0 < r_1 < 1$,

$0 < r_2 < 1$. If $r_1 \neq r_2$, then the Theorem holds.

Next, we assume that $r_1 \neq r_2$. We may assume without loss of generality that

$0 < r_1 < r_2 < 1$.

Now let $y \in B(x,r_1,t) \Rightarrow \mu(x-y,t) < r_1 < r_2$.

Hence $y \in B(x,r_2,t)$. This shows that

$B(x,r_1,t) \subset B(x,r_2,t)$. By assuming that

$0 < r_2 < r_1 < 1$, we can similarly show

$B(x,r_2,t) \subset B(x,r_1,t)$.

**Definition (1.15):**

Let $(X,\mu,\ast)$ be a fuzzy modular space. A subset $A$ of $X$ is said to be open set, if for all $x \in A \exists \ r \in (0,1), t \in (0,\infty)$ such that $B(x,r,t) \subset A$.

**Theorem (1.16):** [7]

Let $(X,\mu,\ast)$ be a $\beta$-homogenous fuzzy modular space. Every $\mu$-ball $B(x,r,t)$ in $(X,\mu,\ast)$ is a $\mu$-open set.

**Theorem (1.17):**

The intersection number of open sets in fuzzy modular space is open sets.

**Proof:**

Let $(X,\mu,\ast)$ be a fuzzy modular space and let 

$\{ G_i : i=1,2,\ldots,m \}$ be a finite collection of open set in the fuzzy modular space. Let

$H \cap \{ G_i, i = 1,2, \ldots, m \}$

To prove that $H$ is an open set

Let $x \in H \Rightarrow x \in G_i \ \forall i=1,2,\ldots,m$

[Since $G_i$ open set : $\forall i \Rightarrow \exists r_i \in (0,1)$ and $t_i > 0 \Rightarrow \exists B(x,r_i,t_i) \subset G_i$

Let $t_k = \max\{t_1, t_2, \ldots, t_m\}$ and

$r_k = \min\{r_1, r_2, \ldots, r_n\}$

$\Rightarrow B(x,r_k,t_k) \subset G_i$

$\Rightarrow B(x,r_k,t_k) \cap G_i \Rightarrow B(x,r_k,t_k) \subset H$

Then $H$ is open set.

**Theorem (1.18):** The union of an arbitrary collection of open set in fuzzy modular space is open sets.

**Proof:**

Let $(X,\mu,\ast)$ be a fuzzy modular space and
let \( \{ \gamma_\lambda : \lambda \in \Lambda \} \) be an arbitrary collection of open sets in \( X \). Let \( G = \bigcup \{ \gamma_\lambda : \lambda \in \Lambda \} \) We must to prove \( G \) is open

Let \( X \in G \Rightarrow X \in \gamma_\lambda \) for some \( \lambda \in \Lambda \)

Since \( \gamma_\lambda \) is open set

\( \Rightarrow \) there exist \( r \in (0,1) \) such that \( B(x, r, t) \subset \gamma_\lambda \)

Since \( \gamma_\lambda \subset G \Rightarrow \) Then \( B(x, r, t) \subset G \)

\( \Rightarrow G \) is open set.

**Theorem (1.19):**

Let \((X, \mu, *)\) be afuzzy modular space

if \( C \) and \( D \) are open sets in a vector space \( X \) then \( C + D \) is open set in \( X \).

**Proof:**

Let \( x \in X \) and \( a \in C \), since \( A \) is open set then

there exist \( r \in (0,1) \) such that

\( B(a, r, t) \subset C \).

then \( B(a, r, t) + x \subset C + x \)

\( \Rightarrow B(a + x, r, t) \subset C + x \Rightarrow C + x \)

is open set in \( X \) for all \( x \in X \)

Since \( C + D = \bigcup \{ C + d : d \in D \} \)

Then \( C + D \) is open set in \( X \).

**Theorem (1.20):**

Every single set in fuzzy modular space is closed set.

**Proof:**

Let \( X \) be a fuzzy modular space

Let \( B = \{ x \} \) be a set in \( X \), to prove \( B \) is closed set

Let \( y \in A^c \Rightarrow y \neq X \)

\( \mu(y - x, t) > 0 \) (since \( X \) is fuzzy modular space)

\( X \notin B(y, r, t) = \{ a \in X : \mu(a - x, t) > 1 - r \} \)

\( B \cap B(y, r, t) = \emptyset \Rightarrow B(y, r, t) \subseteq A^c \)

Then \( y \in B(y, r, t) \subseteq B^c \Rightarrow y \) is interior point

Then \( B^c \) is open set

Then \( B \) is closed set.

**Corollary (1.21)** Every finite set in fuzzy modular space is closed set.

**Definition (1.22):**

A subset \( U \) of \( X \) is said to be a neighborhood of \( x \in X \) in \((X, \mu, *)\) if there exist \( r \in (0,1) \) and \( t \in (0, \infty) \) such that \( B(x, r, t) \subset U \).

**Definition (1.23):**

A subset \( A \) of a vector space \( X \) over \( F \) is called convex set if

\( \alpha A + (1 - \alpha)A \subseteq A \) for all \( 0 \leq \alpha \leq 1 \).

**Theorem (1.24):**

Every open and closed balls in fuzzy modular space are convex sets.

**Proof:**

Let \( y_1, y_2 \in B(x, r, t) \) such that \( r \in (0,1), t > 0, \mu(x - y_1, t) > 1 - r \) and \( \mu(x - y_2, t) > 1 - r \).

Now, we have to prove \( \alpha y_1 + (1 - \alpha)y_2 \in B(x, r, t) \)

\( \mu(x - \alpha y_1 + (1 - \alpha)y_2, t) \)

\( = \mu(x - \alpha y_1 + (1 - \alpha)x - (1 - \alpha)y_2, t) \)

\( = \mu(x - (1 - \alpha)(x - y_2), t) \)

\( = \mu(x - y_1, t) \ast \mu(x - y_2, t) \)

\( > 1 - r \ast 1 - r - r \)

\( = 1 - r \)
Then $B(x, r, t)$ is convex set.

 similary, we can prove $B[x, r, t]$ is convex set.

$x_n \to x$ since $\lim_{n \to \infty} \mu(x_n - x, t) = 1$

**Theorem (1.25):**

Conversely: Suppose the condition is true, to prove $x \in A$.

Let $X$ be a vector space. If $A$ is convex set in $X$.

fuzzy modular space then $\overline{A}$ is convex set.

Proof:

Let $x, y \in \overline{A}, 0 \leq \lambda \leq 1 \implies \exists a, b \in A$ such that $\mu(x - a) < r, \mu(x - b) < r$.

Since $A$ is convex $\Rightarrow \alpha a + (1 - \alpha)b \in A$

$ax + (1 - \alpha)y - (aa + (1 - \alpha)b)$

$= a(x - a) + (1 - \alpha)(y - b)$

$\mu(ax + (1 - \alpha)y - (aa + (1 - \alpha)b))$

$\leq \alpha \mu(x - a) + (1 - \alpha)\mu(y - b)$

$< ar + (1 - a)r = r$

$\Rightarrow ax + (1 - \alpha)y \in \overline{A} \Rightarrow \overline{A}$ is convex set.

**Lemma (1.26):**

Let $(X, \mu, \ast)$ be a fuzzy modular space and $A \subset X$.

if for any $x \in \overline{A}$, then there exist a sequence $\{x_n\}$ in $A$ such that $\lim_{n \to \infty} \mu(x, t) = 1$

for all $t > 0$.

Proof:

Suppose $x \in \overline{A}$

$\Rightarrow A \cup A' \Rightarrow x \in A$ and $x \in A'$

If $x \in A$, the $\{x, x, \ldots\} \to x$

If $x \in A$' and $x \notin A \Rightarrow \forall n \in Z^+, r = \frac{1}{n}$.

$B(x, r, t) - \{x\} \neq \emptyset$

$x_n \in B(x, r, t) \cap A \Rightarrow x_n \in A \in A$

$\Rightarrow \mu(x_n - x, t) > 1 - \frac{1}{n}$

Then $x \in \overline{A}$.

**Definition (1.27):**

A fuzzy modular space $(X, \mu, \ast)$ is called a locally convex if there is a local base $\beta$ at 0 in $X$ such that every member of $\beta$ are convex sets.

**Example (1.28):**

Every fuzzy modular space is locally convex.

**Solution:** Let $(X, \mu, \ast)$ be a fuzzy modular space $\beta = \{B(r, 0, t): r > 0\}$, where $B(r, 0, t) = \{x \in X: \mu(x, t) > 1 - r\}$.

Let $G$ be an open set in $X$, then $G$ is the union of open balls, so $0 \in B(r, 0, t) \subset G$ for some $r > 0$, then $\beta$ is a local base at 0 in $X$.

Since every open ball is convex set, then $B(x, 0, t)$ is convex set for all $r > 0$, then $\beta$ is a convex local base at 0 in $X$.

Therefore $(X, \mu, \ast)$ is locally convex space.
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بعض خصائص فضاء الوحدات الضبابي التبولوجي

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المستخلص:

في هذا البحث، عرفنا فضاء الوحدات الضبابي التبولوجي والفهمايم المتعلقة به مثل الكرات المفتوحة والمغلقة، المجموعة المحدبة، التحدي المحلي، المجموعة المفتوحة والمغلقة، وبرهنا بعض النتائج المتعلقة بهذا الفضاء.