Coefficient estimates for some subclasses of bi-univalent functions related to m-fold symmetry

WaggasGalibAtshan¹ SalwaKalfKazim²

Department of Mathematics, College of Computer Science and Information Technology, University of AL-Qadisiyah, Diwaniyah-Iraq

E-mail: waggas.galib@qu.edu.iqwaggashnd@gmail.com

Abstract:
The purpose of present paper is to introduce and investigate two new subclasses \( \mathcal{N}_m^{\sum}(\tau, \gamma, \alpha) \) and \( \mathcal{N}_m^{\sum}(\tau, \gamma, \beta) \) of analytic and m-fold symmetric bi-univalent functions in the open unit disk. Among other results belonging to these subclasses upper coefficients bounds \( |a_{m+1}| \) and \( |a_{2m+1}| \) are obtained in this study. Certain special cases are also indicated.

Keywords: m-fold symmetry, bi-univalent functions, coefficient estimates.

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1. Introduction

Let $S$ denote the family of functions analytic in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1)$$

Also let $\mathcal{A}$ denote the subclass of functions in $S$ which are univalent in $U$.

The Koebe One Quarter Theorem (e.g., see [6]) ensures that the image of $U$ under every univalent function $f(z) \in S$ contains the disk of radius $1/4$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r(f), r(f) \geq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (2)$$

A function $f \in S$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$.

Let $\Sigma$ denotes the class of analytic and bi-univalent functions in $U$. Some examples of functions in class $\Sigma$ are

$$h_1(z) = \frac{z}{1-z}, \quad h_2(z) = -\log(1-z), \quad h_3(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \quad z \in U.$$ 

For each function $f \in \mathcal{A}$, the function $h(z) = \left(f(z^m)\right)^{\frac{1}{m}}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk $U$ into a region with $m$-fold symmetry. A function is said to be $m$-fold symmetric (see [9,10]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{m} a_{mk+1} z^{mk+1}, \quad (z \in U, m \in \mathbb{N}). \quad (3)$$

We denote $S_m$ the class of $m$-fold symmetric univalent functions in $U$, which are normalized by the series expansion (3). In fact , the functions in the class $\mathcal{A}$ are one-fold symmetric . Analogous to the concept of $m$-fold symmetric univalent functions , we here introduced the concept of $m$-fold symmetric bi-univalent functions . Each function $f \in \Sigma$ generates an $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. Furthermore, for the normalized form of $f$ is given by (3) , they obtained the series expansion for $f^{-1}$ as follows:

$$g(w) = w - a_{m+1} w^{m+1} + \left(\frac{1}{2} (m+1)(m+2) a_{m+1}^2 - a_{2m+1}\right) w^{2m+1} - \left(\frac{1}{2} (m+1)(3m+2) a_{m+1}^2 a_{m+1} - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right) w^{3m+1} + \cdots. \quad (4)$$

where $f^{-1} = g$. We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $U$. It is easily seen that for $m=1$, the formula (4) coincides with the formula (2) of the class $\Sigma$. Some examples of $m$-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \left[\frac{1}{2} \log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}}$$

and

$$\left(-\log(1-z^m)\right)^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \left(\frac{e^{w^m}-1}{e^{w^m}+1}\right)^{\frac{1}{m}}$$

respectively.

Recently , many authors investigated bounds for various subclass of $m$-fold bi-univalent functions (see [1,2,3,4,5,7,9,12,13,15]). The aim of the present paper is to introduce the new subclass $\mathcal{N}_{\Sigma_m}(r, y; \alpha)$ and $\mathcal{N}_{\Sigma_m}(r, y; \beta)$ of $\Sigma_m$ and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclass.

In order to prove our main results , we require the following lemma.

**Lemma 1.** ([6]) If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions $h$ analytic in $U$ for which

$$Re(h(z)) > 0, \quad (z \in U)$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots. \quad (z \in U)$$

**Definition 1.** A function $f(z) \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{N}_{\Sigma_m}(r, y; \alpha)$ if the following condition are satisfied:

$$\left|\arg\left(1 + \frac{1}{r} \frac{(1+y)z f''(z) + zf'(z)}{(1+y)z f''(z) - zf'(z)} - 1\right)\right| < \frac{\alpha \pi}{2}, \quad (z \in U)$$

and

$$\left|\sum_{k=1}^{m} a_{mk+1} z^{mk+1}\right| < r(z), \quad (z \in U).$$

(5)
\[ \arg\left(1 + \frac{1}{\tau}\left[(1 + \gamma)w^2g''(w) + wg'(w) - \gamma g(w) - 1\right]\right) \leq \frac{\alpha\pi}{2} \quad \text{(w \in U)} \quad (6) \]

\[ (0 < \alpha \leq 1; \tau \in \mathcal{C}\setminus\{0\}; 0 \leq \gamma < 1), \]
where the function \( g = f^{-1} \) is given by (4).

**Definition 2.** A function \( f(z) \in \Sigma_m \) given by (3) is said to be in the class \( \mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta) \) if the following conditions are satisfied:

\[ \Re\left(1 + \frac{1}{\tau}\left[(1 + \gamma)z^2f''(z) + zf'(z) - \gamma f(z) - 1\right]\right) > \beta, \]
\[ (z \in U) \quad (7) \]

and

\[ \Re\left(1 + \frac{1}{\tau}\left[(1 + \gamma)w^2g''(w) + wg'(w) - \gamma g(w) - 1\right]\right) > \beta, \]
\[ (w \in U) \quad (8) \]

\[ (0 \leq \beta < 1; \tau \in \mathcal{C}\setminus\{0\}; 0 \leq \gamma < 1), \]
where the function \( g = f^{-1} \) is given by (4).

2. **Coefficient Estimates for the Functions Class** \( \mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta) \)

We begin this section by finding the estimates on the coefficients \( |a_{m+1}| \) and \( |a_{2m+1}| \) for functions in the class \( \mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta) \).

**Theorem 2.1** Let \( f(z) \in \mathcal{N}_{\Sigma_m}(\tau, \gamma; \beta) \) \((0 < \alpha \leq 1; \tau \in \mathcal{C}\setminus\{0\}; 0 \leq \gamma < 1)\) be of the form (3). Then

\[ |a_{m+1}| \leq \frac{2\alpha t}{\sqrt{\|2m[(2m+2my+1)/(m+1) - (m+my+1)^2) - (\alpha - 1)^2]/(m+my+1)]}}, \]  
\[ \text{(9)} \]

and

\[ |a_{2m+1}| \leq \frac{2\alpha t^2}{\sqrt{\|2m[(2m+2my+1)/(m+1) - (m+my+1)^2) - (\alpha - 1)^2]/(m+my+1)]}}, \]  
\[ \text{(10)} \]

**Proof.** It follows from (5) and (6) that

\[ 1 + \frac{1}{\tau}\left[(1 + \gamma)z^2f''(z) + zf'(z) - \gamma f(z) - 1\right] = |p(z)|^2, \]  
\[ \text{(11)} \]

and

\[ 1 + \frac{1}{\tau}\left[(1 + \gamma)w^2g''(w) + wg'(w) - \gamma g(w) - 1\right] = |q(w)|^2, \]  
\[ \text{(12)} \]

where the functions \( p(z) \) and \( q(w) \) are in \( \mathcal{P} \) and have the following series representations:

\[ p(z) = 1 + p_mz^m + p_{2m}z^{2m} + p_{3m}z^{3m} + \ldots \quad (13) \]

and

\[ q(w) = 1 + q_mw^m + q_{2m}w^{2m} + q_{3m}w^{3m} + \ldots \quad (14) \]

Now, equating the coefficients in (11) and (12), we obtain

\[ \frac{m(m+my+1)a_{m+1}}{\tau} = ap_m, \]  
\[ \text{(15)} \]

\[ \frac{(2m(2m+2my+1)a_{2m+1} - m(m+my+1)^2a_{2m+1})}{\tau} = \alpha p_{2m} + \frac{\alpha(a - 1)}{2}p_m^2, \]  
\[ \text{(16)} \]

and

\[ \frac{-m(m+my+1)a_{m+1}}{\tau} = aq_m, \]  
\[ \text{(17)} \]

\[ \frac{(2m(2m+2my+1)(a_{m+1} - a_{2m+1}) - m(m+my+1)^2a_{2m+1})}{\tau} = \alpha q_{2m} + \frac{\alpha(a - 1)}{2}q_m^2. \]  
\[ \text{(18)} \]

From (15) and (17), we find

\[ p_m = -q_m(19) \]

and

\[ 2\frac{m^2(m+my+1)^2q_{2m+1}a_{2m+1}}{\tau} = a^2(p_m^2 + q_m^2). \]  
\[ \text{(20)} \]

From (16), (18), and (20), we get

\[ \left((2m + 2my + 1)(m + 1) - (m + my + 1)^2\right)2ma^2_{m+1} \]  
\[ \text{(21)} \]

Therefore, we have

\[ a_{m+1}^2 = \frac{a^2\tau^2}{2m[(2m+2my+1)/(m+1) - (m+my+1)^2) - (\alpha - 1)^2]/(m+my+1)]}. \]  
\[ \text{(22)} \]

Applying Lemma 1 for the coefficients \( p_{2m} \) and \( q_{2m} \), we have

\[ |a_{m+1}| \leq \frac{2\alpha t}{\sqrt{\|2m[(2m+2my+1)/(m+1) - (m+my+1)^2) - (\alpha - 1)^2]/(m+my+1)]}}, \]  
\[ \text{(23)} \]

This gives the desired bound for \( |a_{m+1}| \) as asserted in (9). In order to find the bound on \( |a_{2m+1}| \), by subtracting (18) from (16), we get

\[ 2m[(2m+2my+1)a_{2m+1} - (2m+2my+1)(m+1)a_{2m+1}] = \alpha[p_{2m}^2 - q_{2m}^2] + \frac{\alpha(\alpha - 1)}{2}(p_m^2 - q_m^2). \]  
\[ \text{(24)} \]

It follows from (19) and (24) that

\[ a_{2m+1} = \frac{a^2\tau^2(p_m^2 + q_m^2)}{4m[(m+my+1)^2 - (\alpha - 1)^2]/(m+my+1)]}, \]  
\[ \text{(25)} \]
Applying Lemma 1 once again for the coefficients $p_m, p_{2m}, q_m$ and $q_{2m}$, we readily obtain
\[
|a_{2m+1}| \leq \frac{2a^2}{m^4(m+my+1)^2} + \frac{a}{m(2m+2my+1)}.
\] (26)

3. Coefficient Bounds for the Functions Class $\mathcal{N}_S(\tau, \gamma; \beta)$

This section is devoted to find the estimates on the coefficients $|a_{2m+1}|$ and $|a_{m+1}|$ for functions in the class $\mathcal{N}_S(\tau, \gamma; \beta)$.

**Theorem 3.1** Let $f(z) \in \mathcal{N}_S(\tau, \gamma; \beta) (0 \leq \beta < 1; \tau \in \mathbb{C} \setminus \{0\}, 0 \leq \gamma < 1)$ be of the form (3). Then
\[
|a_{m+1}| \leq \frac{2\tau(1-\beta)}{\sqrt{m((2m+2my+1)(m+1)-(m+my+1)^2)}}.
\] (27)

and
\[
|a_{2m+1}| \leq \frac{4\tau^2(1-\beta)^2(m+1)}{m^4(m+my+1)^2} + \frac{2\tau(1-\beta)}{m(2m+2my+1)}.
\] (28)

**Proof.** It follows from (7) and (8) that there exist $p, q \in \mathbb{P}$ such that
\[
1 + \frac{1}{\tau} \left[\frac{(1+y)x'f'(z)+x\gamma f(z)}{(1+y)x'f'(z)-y\gamma f(z)} - 1\right] = \beta + (1-\beta)p(z).
\] (29)

and
\[
1 + \frac{1}{\tau} \left[\frac{(1+y)w^2g''(z)+w\gamma g'(z)}{(1+y)w^2g''(z)-w\gamma g'(z)} - 1\right] = \beta + (1-\beta)q(w).
\] (30)

where $p(z)$ and $q(z)$ have the forms (13) and (14), respectively. By suitably comparing coefficients in (29) and (30), we get
\[
m(m+my+1)a_{m+1} = (1-\beta)p_m, \quad (31)
\]
\[
\frac{(2m(2m+2my+1)a_{2m+1}-m(m+my+1)^2a_m^2)}{\tau} = (1-\beta)p_{2m}, \quad (32)
\]
\[
\frac{-m(m+my+1)a_{m+1}}{\tau} = (1-\beta)q_m. \quad (33)
\]
\[
\frac{(2m(2m+2my+1)(m+1)a_m^2a_{m+1}-m(m+my+1)^2a_m^2a_{m+1})}{\tau} = (1-\beta)q_{2m}. \quad (34)
\]

From (31) and (33), we find
\[
p_m = -q_m. \quad (35)
\]

and
\[
\frac{2m^2(m+my+1)^2a_m^2}{\tau^2} = (1-\beta)^2(p_m^2 + q_m^2). \quad (36)
\]

Applying Lemma 1 once again for the coefficients $\frac{(2m+2my+1)(m+1)-(m+my+1)^2)2ma_{m+1}}{\tau}$, we have
\[
|a_{m+1}| \leq \frac{2\tau(1-\beta)}{\sqrt{m((2m+2my+1)(m+1)-(m+my+1)^2)}}.
\] (37)

This is the bound on $|a_{m+1}|$ asserted in (27).

In order to find the bound on $|a_{2m+1}|$ by subtracting (34) form (32), we get
\[
2m(2m+2my+1)a_{2m+1}-(2m+2my+1)(m+1)a_m^2a_{m+1} = (1-\beta)(p_{2m} - q_{2m}). \quad (38)
\]

Or, equivalently,
\[
a_{2m+1} = \frac{2m(2m+2my+1)a_{2m+1}a_m^2}{2m(2m+2my+1) + \frac{(1-\beta)(p_{2m} - q_{2m})}{2m(2m+2my+1)}}. \quad (39)
\]

It follows from (35) and (36) that
\[
a_{2m+1} = \frac{2m^2(m+my+1)^2a_m^2}{\tau^2} + \frac{2\tau(1-\beta)}{m(2m+2my+1)}. \quad (40)
\]

Applying lemma 1 once again for the coefficients $p_m, p_{2m}, q_m$ and $q_{2m}$, we easily obtain
\[
|a_{2m+1}| \leq \frac{4\tau^2(1-\beta)^2(m+1)}{m^4(m+my+1)^2} + \frac{2\tau(1-\beta)}{m(2m+2my+1)}. \quad (41)
\]

4. Corollaries and Consequences

For one-fold symmetric bi-univalent functions and $\tau = 1$, Theorem 2.1 and Theorem 3.1 reduce to Corollary 1 and Corollary 2, respectively, which were proven very recently by Frasin[8] (see also [11]).

**Corollary 4.** Let $f(z) \in \mathcal{N}_S(\alpha, \gamma)(0 < \alpha \leq 1; 0 \leq \gamma < 1)$ be of the form (1).

Then
\[
|a_2| \leq \frac{2\alpha}{\sqrt{2(3-\alpha-\gamma^2)}}. \quad (42)
\]

and
\[
|a_3| \leq \frac{4\alpha^2}{(2+\gamma)^2} + \frac{\alpha}{(3+2\gamma)}. \quad (43)
\]

**Corollary 5.** Let $f(z) \in \mathcal{N}_S(\beta, \gamma)(0 < \alpha \leq 1; 0 \leq \gamma < 1)$ be of the form (1).

Then
\[ |a_2| \leq \sqrt{\frac{2(1-\beta)}{(2+2\gamma+\gamma^2)}} \] (44)

and

\[ |a_3| \leq \frac{\beta(1-\beta)^2}{(2+2\gamma)^2} + \frac{2(1-\beta)}{(3+2\gamma)} \] (45)

The classes \( \mathcal{N}_2'(\alpha, \gamma) \) and \( \mathcal{N}_2'(\beta, \gamma) \) are defined in the following way:

**Definition 3.** A function \( f(z) \in \Sigma \) given by (1) is said to be in the class \( \mathcal{N}_2'(\beta, \gamma) \) if the following conditions are satisfied:

\[ |arg\left(\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}\right) - \frac{\alpha\pi}{2} | (z \in U) \] (46)

And

\[ |arg\left(\frac{(1+\gamma)w^2g''(w)+wg'(w)}{(1+\gamma)wg'(w)-\gamma g(w)}\right) - \frac{\alpha\pi}{2} | (w \in U) \] (47)

\((0 < \alpha \leq 1; 0 \leq \gamma < 1),\)

where the function \( g = f^{-1} \) is given by (2).

**Definition 4.** A function \( f(z) \in \Sigma \) given by (1) is said to be in the class \( \mathcal{N}_2'(\beta, \gamma) \) if the following conditions are satisfied:

\[ Re\left(\frac{(1+\gamma)z^2f''(z)+zf'(z)}{(1+\gamma)zf'(z)-\gamma f(z)}\right) > \beta \] (z \in U) \] (48)

And

\[ Re\left(\frac{(1+\gamma)w^2g''(w)+wg'(w)}{(1+\gamma)wg'(w)-\gamma g(w)}\right) > \beta \] (w \in U) \] (49)

\((0 \leq \beta < 1; 0 \leq \gamma < 1),\)

where the function \( g = f^{-1} \) is given by (2).

If we set \( \gamma = 0 \) and \( \tau = 1 \) in Theorem 2.1 and Theorem 3.1, then the classes \( \mathcal{N}_2'(\tau, \gamma; \alpha) \) and \( \mathcal{N}_2'(\tau, \gamma; \beta) \) reduce to the classes \( \mathcal{N}_2^{\mathcal{a}}_{\Sigma m} \) and \( \mathcal{N}_2^{\mathcal{b}}_{\Sigma m} \) investigated recently by Srivastava et al. [11] and thus, we obtain the following corollaries:

**Corollary 6.** Let \( f(z) \in \mathcal{N}_2^{\mathcal{a}}_{\Sigma m} \) \((0 < \alpha \leq 1)\) be of the form (3) Then

\[ |a_{m+1}| \leq \frac{2\alpha}{\sqrt{m(2m+1)(m+1)!-m(m+1)!m^2(\alpha-1)!}} \] (50)

and

\[ |a_{2m+1}| \leq \frac{\alpha}{m(2m+1)!} + \frac{2\alpha(m+1)!}{m^2(m+1)!^2} \] (51)

**Corollary 7.** Let \( f(z) \in \mathcal{N}_2^{\mathcal{b}}_{\Sigma m} \) \((0 \leq \beta \leq 1)\) be of the form (4). Then

\[ |a_{m+1}| \leq \sqrt{\frac{2(1-\beta)}{m(2m+1)(m+1)!-m^2(m+1)!^2(\alpha-1)!}} \] (52)

and

\[ |a_{2m+1}| \leq \frac{(1-\beta)}{m(2m+1)!} + \frac{2(1-\beta)^2(m+1)!}{m^2(m+1)!^2} \] (53)

The classes \( \mathcal{N}_2^{\mathcal{a}}_{\Sigma m} \) and \( \mathcal{N}_2^{\mathcal{b}}_{\Sigma m} \) are respectively defined as follows:

**Definition 5.** A function \( f(z) \in \Sigma_m \) given by (3) is said to be in the class \( \mathcal{N}_2^{\mathcal{a}}_{\Sigma m} \) if the following conditions are satisfied:

\[ \left| arg \left\{ \frac{z^2f''(z)}{zf'(z)} + 1 \right\} \right| < \frac{\alpha\pi}{2} (z \in U) \] (54)

and

\[ \left| arg \left\{ \frac{w^2g''(w)}{wg'(w)} + 1 \right\} \right| < \frac{\alpha\pi}{2}, (w \in U) \] (55)

and where the function \( g \) is given by (4).

**Definition 6.** A function \( f(z) \in \Sigma_m \) given by (3) is said to be in the class \( \mathcal{N}_2^{\mathcal{b}}_{\Sigma m} \) if the following conditions are satisfied:

\[ Re\left(\frac{z^2f''(z)}{zf'(z)} + 1 \right) > \beta \] (z \in U) \] (56)

and

\[ Re\left(\frac{w^2g''(w)}{wg'(w)} + 1 \right) > \beta \] (w \in U). \] (57)

\((0 \leq \beta < 1),\)

And where the function \( g \) is given by (4).

**References**


الفصل الأول: علاقات وظيفية بين مجموعات الأعداد الموجبة والسلبية

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