On Quadratic Unbiased Estimator for Variance Components of One-Way Repeated Measurements Model

Jasim Mohammed Ali Al-Isawi¹, Abdulhussein Saber Al-Mouel²

¹ Mathematics Department, College of Education for Pure Science Basrah University-Iraq. Email: jasim.ali@uobasrah.edu.iq
² Mathematics Department, College of Education for Pure Science Basrah University-Iraq. Email: abdulhusseinsaber@yahoo.com

ABSTRACT

In this paper, we investigate the estimator of variance components of one-way repeated measurements model (RMM) using MINQUE-principle (Rao 1971a and Rao 1971b) and method of MINQUE (1) which using priori values for components of variance.

ARTICLE INFO

Article history:
Received: 17/04/2019
Revised form: 00/00/0000
Accepted: 12/05/2019
Available online: 30/05/2019

Keywords:
Quadratic Estimator, Repeated Measurements Model, Minimum Norm Quadratic Unbiased Estimator (MINQUE), MINQUE (1), Variance Components, Quadratic Form, Estimation, Moore-Penrose generalized inverse.
1. Introduction

A models of repeated measurements (RMM) are widespread in statistical studies (life, health, social, agricultural and others). And since the study of estimating the components of variance is of great importance in statistical studies, there are many statistical methods for estimating these components. The interclass correlation model is a special case of repeated measurements model introduced by Wilks (1946). Vonesh and Chinchilli (1997) introduce univariate repeated measurements Model (called One-Way Repeated Measurement Model). AL-Mouel (2004) studied the multivariate repeated measurements models and comparison of estimators. AL-Mouel A. H. S. and others(2017) studied Bayesian One-Way Repeated Measurements Model Based on Bayes Quadratic Unbiased Estimator. Al-Isawi J. A. M. A. and Al-Mouel A. H. S. (2018) studied Best Quadratic Unbiased Estimator for Variance Component of One-Way Repeated Measurements Model, in this article we study the quadratic unbiased estimator for variance components of one-way repeated measurements model. Now we introduce some definitions and remarks which used in this article.

Definition 1 [7]: For given matrix \( A \) of size \( n \times m \) we called a matrix \( A^+ \) of size \( m \times n \) is Moore-Penrose generalized inverse (MP-inverse) of \( A \) if satisfy the following conditions

(a) \( AA^+A = A \),

(b) \( A^+AA^+ = A^+ \),

(c) matrix \( AA^+ \) is symmetric

(d) matrix \( A^+A \) is symmetric.

Definition 2 [7]: The Kronecker product (\( \otimes \)) of an \( n \times m \) and \( p \times q \) matrix \( A \) and \( B \), is denoted by \( A \otimes B \). This is an \( np \times mq \) matrix with the \((i,j)\) block \( A_{ij}B \), where \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

Kronecker product have the following properties:

1. \((A \otimes B) (D \otimes C) = AD \otimes BC\)
2. \((A + B) \otimes C = (A \otimes C) + (B \otimes C) \) and \( A \otimes (B + C) = (A \otimes B) + (A \otimes C)\)
3. \((A \otimes B)^+ = A^+ \otimes B^+ \) and \((A \otimes B)' = A' \otimes B'\)
4. \((c_1A) \otimes (c_2B) = (c_1c_2)(A \otimes B)\)
5. \(A \otimes [B_1 : B_2] = [A \otimes B_1 : A \otimes B_2]\)
6. \(tr(A \otimes B) = tr(A) tr(B)\).

Remark 1: If \( j_p \) denotes to \( p \times 1 \) vector of \( 1 \)'s, \( I_p \) denotes to the \( p \times p \) matrix of \( 1 \)'s and \( I_p \) denotes to \( p \times p \) identity matrix then

1. \( j_p = \frac{1}{p} j_p \).
2. \( j_p^+ = \frac{1}{p^2} j_p = (\frac{1}{p} j_p)(\frac{1}{p} j_p^t) \).
3. \((I_p + j_p)^+ = I_p - \frac{1}{p+1} j_p \).
4. \((I_p - \frac{1}{p} j_p)^+ = (I_p - \frac{1}{p} j_p) \).

Remark 2: If \( A \) is any square matrix of \( q \times q \), then

\((I_q \otimes j_p)A(I_q \otimes j_p) = A \otimes j_p\).

Remark 3: If \( H_p \) denotes to \( p \times p \) Idempotent matrix and \( A \) any matrix, then

1. \((A \otimes H_p)^+ = A^+ \otimes H_p\).
2. \((H_p \otimes A)^+ = H_p \otimes A^+\)
3. \(H_p^+ = H_p\)
4. \(Y^+ = \frac{y^t}{y^t y} \) for any \( n \times 1 \) vector \( Y \).
2. **The one-way repeated measurements model**

Consider the following linear model and parameterization for the one-way repeated measurement model with one between-units factor incorporating univariate random effects.

\[ Y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + \varepsilon_{ijk} \quad (1) \]

where

- \( i = 1, \ldots, n \) is an index for an experimental unit within group \( j \),
- \( j = 1, \ldots, q \) is an index for levels of the between-units factor (group),
- \( k = 1, \ldots, p \) is an index for levels of the within-units factor (time),
- \( Y_{ijk} \) is the response measurements at time \( k \) for unit \( i \) within group \( j \),
- \( \mu \) is the overall mean,
- \( \tau_j \) is the added effect for treatment group \( j \),
- \( \delta_{i(j)} \) is the random effect due to experimental unit \( i \) within treatment group \( j \),
- \( \gamma_k \) is the added effect for time \( k \),
- \( (\tau\gamma)_{jk} \) is the added effect for the group \( j \times \) time \( k \) interaction, and
- \( \varepsilon_{ijk} \) is the random error on time \( k \) for unit \( i \) within group \( j \).

For the parameterization to be of full rank, we impose the following set of conditions:

\[
\sum_{j=1}^{q} \tau_j = 0 \quad \sum_{k=1}^{p} \gamma_k = 0, \\
\sum_{j=1}^{q} (\tau\gamma)_{jk} = 0 \quad \forall \; k = 1, \ldots, p, \quad \sum_{k=1}^{p} (\tau\gamma)_{jk} = 0 \quad \forall \; j = 1, \ldots, q. 
\]

We assume that the \( \varepsilon_{ijk} \)'s and the \( \delta_{i(j)} \)'s are independent with

\[
\varepsilon_{ijk} \sim \text{Li.d} \, N(0, \sigma^2) \quad \text{and} \quad \delta_{i(j)} \sim \text{Li.d} \, N(0, \sigma^2_\delta) 
\]

We can write model (1) as follows

\[ Y = X\beta + Z\delta + \varepsilon, \quad (4) \]

where \( Y \) is \( nqp \)-dimensional response vector,
- \( Z \) is a \( nqp \times nq \) design matrix,
- \( \beta \) is a \((q + 1)(p + 1)\)-dimensional vector of fixed effects parameters,
- \( \delta \) is a \( nq \)-dimensional vector of random effects,
- \( \varepsilon \) is error term has length \( nqp \) with \( \varepsilon \sim N_{nqp}(0_{nqp}, \sigma^2 I_{nqp}) \).

And design matrix \( X \) of size \( nqp \times (q + 1)(p + 1) \) is
\[ X = [x_1 : x_2 : x_3 : x_4]_{nq 	imes (qp+q+p+1)} \]

where
\[ x_1 = j_{nq p} , \quad x_2 = j_n \otimes l_q \otimes l_p , \quad x_3 = j_{nq p} \otimes l_p , \quad x_4 = j_n \otimes l_{qp} \]  

(5)

Then from (3)
\[
\begin{align*}
&\mathbf{\varepsilon} \sim N_{nq p} \left( 0_{nq p}, \sigma^2 \mathbf{I}_{nq p} \right) , \quad \delta \sim N_{nq p} \left( 0_{nq p}, \sigma^2 \mathbf{I}_{nq p} \right) \text{ and } \text{cov} \left( \mathbf{\varepsilon} , \delta \right) = 0 \\
&
\mathbf{Y} \sim N_{nq p} \left( \mathbf{X} \beta , \Sigma \right) \text{ where, } \Sigma = \sigma^2 \mathbf{Z} \mathbf{Z}' + \sigma^2 \mathbf{I}_{nq p}
\end{align*}
\]

(6)

*(\Sigma = \text{var}(\mathbf{Y}) \text{ is variance-covariance matrix})*

**Lemma 1 [13]:** Let \( A^+ \) be MP-inverse of \( A \) and put \( X_{i,j} = (I_n - E_j)X_i \), where

\( X_i \) is a given \( n \times m \) matrix and \( E_j = X_j X_j^+ \). Then with \( X = [X_1 : X_2] \)

and \( E_{i,j} = X_{i,j} X_{j,i} \) we have \( XX^+ = E_1 + E_{2,1} = E_2 + E_{1,2} \).

**Proposition 1:** For model (4) and using Lemma 1; if \( X = [X_1 : X_2] \), where \( X_1 \) and \( X_2 \)

are a matrix of size \( nq \times (q+1)(p+1) - m \) and \( nq \times m \), \( 1 \leq m < (q+1)(p+1) \) then

\[
E_2 = \begin{cases} 
\left( \frac{l_n}{n} \right) \otimes l_{qp} & ; m \geq qp \\
\left( \frac{l_n}{n} \right) \otimes H_{qp} & ; m < qp 
\end{cases} , \text{ H is Idempotent matrix}
\]

and

\[
E_{1,2} = \begin{cases} 
0 & ; m \geq qp \\
\left( \frac{l_n}{n} \right) \otimes (l_{qp} - H_{qp}) & ; m < qp, H = H^2
\end{cases}
\]

\[
XX^+ = \frac{I_n \otimes l_{qp}}{n}
\]

(7)

**Proof:**

\[
E_2 = X_2 X_2^+ = \frac{l_n}{n} \otimes \chi \left( j_n^+ \otimes X^+ \right) \text{ where } \chi \text{ is a matrix of size qp} \times m
\]

\[
= j_n j_n^+ \otimes \chi X^+ = \begin{cases} 
\left( \frac{l_n}{n} \right) \otimes l_{qp} & ; m \geq qp \\
\left( \frac{l_n}{n} \right) \otimes H_{qp} & ; m < qp
\end{cases} , \text{ H is Idempotent matrix}
\]

Similarly \( E_1 = X_1 X_1^+ = \begin{cases} 
\left( \frac{l_n}{n} \right) \otimes l_{qp} & ; m < qp \\
\left( \frac{l_n}{n} \right) \otimes H_{qp} & ; m \geq qp
\end{cases} , \text{ H is Idempotent matrix}

And

\[
E_{1,2} = X_{1,2} X_{2,1}^+ = \left[ (l_{nq} - E_2) X_1 \right] \left[ (l_{nq} - E_2) X_1 \right]^+
\]

\[
= \left( I_{nq} - E_2 \right) X_1 \left( I_{nq} - E_2 \right)^+
\]

\[
= \begin{cases} 
\left( I_n - l_n \right) \otimes l_{qp} \left( \left( \frac{l_n}{n} \right) \otimes H_{qp} \right) \left( I_n - l_n \right) \otimes l_{qp} & ; m \geq qp \\
\left( I_{nq} - l_n \right) \otimes H_{qp} \left( \left( \frac{l_n}{n} \right) \otimes l_{qp} \right) \left( I_{nq} - l_n \right) \otimes H_{qp} & ; m < qp
\end{cases}
\]

\( E_{1,2} = \begin{cases} 
0 & ; m \geq qp \\
\left( \frac{l_n}{n} \right) \otimes (l_{qp} - H_{qp}) & ; m < qp
\end{cases} , \text{ H = H}^2
\]

\[
XX^+ = E_2 + E_{1,2} = \frac{I_n \otimes l_{qp}}{n}
\]
It is clear that
\[
X = [j_{nq} \otimes l_q \otimes j_p : j_{nq} \otimes l_p : j_n \otimes l_{qp}]
\]
\[
= j_n \otimes [l_{qp} : l_q \otimes j_p : j_n \otimes l_{qp}]
\]
\[
= (j_n \otimes X) \text{ where } X = [j_{aq} \otimes l_q : j_q \otimes l_p : l_{aq \times (q+1)(p+1)}]
\]
\[
\rightarrow X X^+ = (j_n \otimes X)(j_n^* \otimes X^+)
\]
\[
= j_n j_n^* \otimes X X^+ \quad (j_n^+ = \frac{j_n}{n} \text{ and } X X^+ = l_{qp})
\]
\[
= \frac{j_n \otimes l_{qp}}{n}
\]

Complete proof.

**Proposition 2**: For model (4) and using Lemma 1, with \( U = [U_1 : U_2] \)

where \( U_1 = X : U_2 = Z \), then we have \( E_1 = U_1 U_1^+ = X X^+ = \frac{j_n \otimes l_{qp}}{n} \)

and \( E_{2.1} = U_2 U_{2.1}^+ = \frac{I_{nq} \otimes l_p}{p} - \frac{j_n \otimes I_q \otimes l_p}{np} = (I_n - (J/n)) \otimes I_q \otimes l_p \)

Implies to
\[
U(U'U)^+U' = UU^+ = \frac{j_n \otimes l_{qp}}{n} + \frac{(I_n - (J/n)) \otimes I_q \otimes l_p}{p} \tag{8}
\]

**Proof:**

Since \( U_{2.1} = (I_{nq} - E_1)Z \), \( Z = I_{nq} \otimes l_p \) and \( E_1 = \frac{j_n \otimes l_{qp}}{n} \) (proved in Proposition 1).

\[
U_{2.1} = ((I_n - (J/n)) \otimes I_{qp})Z
\]

\[
\rightarrow E_{2.1} = U_{2.1} U_{2.1}^+ = \left[ ((I_n - (J/n)) \otimes I_{qp})Z \right] \left[ ((I_n - (J/n)) \otimes I_{qp})Z \right]^+
\]
\[
= ((I_n - (J/n)) \otimes I_{qp})(ZZ^+)(I_n - (J/n)) \otimes I_{qp}
\]
\[
= ((I_n - (J/n)) \otimes I_{qp})(I_{nq} \otimes l_p)(I_{nq} \otimes l_p)^+(I_n - (J/n)) \otimes I_{qp}
\]
\[
= ((I_n - (J/n)) \otimes I_{qp})(I_{nq} \otimes l_p)(I_n - (J/n)) \otimes I_{qp}
\]
\[
= ((I_n - (J/n)) \otimes I_{qp})(I_{nq} \otimes l_p)(I_n - (J/n)) \otimes l_{qp}
\]
\[
\rightarrow UU^+ = E_1 + E_{2.1} = \frac{j_n \otimes l_{qp}}{n} + \frac{(I_n - (J/n)) \otimes I_q \otimes l_p}{p} \tag{10}
\]

Complete proof.
3.1- MINQUE for $\sigma^2$

Let $A$ be an $n \times q \times p$ matrix; then a quadratic estimator for $\sigma^2$ is defined as

$$\hat{\sigma}^2 = Y'AY$$  \hspace{1cm} (11)$$

**Note:** For matrix $A$ without loss of generality, we can assume that $A$ is a symmetric matrix and nonnegative definite, to make $\hat{\sigma}^2$ nonnegative for all $Y$.

When a ratio $\frac{\sigma^2}{\sigma^2}$ is known which equal to $\theta$, or $\sigma^2$ and $\sigma^2$ are equal ($\theta = 1$).

We can write model (4) as follows

$$Y = X\beta + \varepsilon,$$  \hspace{1cm} (12)$$

Where $\varepsilon = Z\delta + \varepsilon \rightarrow E(\varepsilon) = 0, \text{var}(\varepsilon) = \sigma^2(I + \theta ZZ')$

Since $E(\varepsilon A\varepsilon') = \text{tr}(A\text{var}(\varepsilon))$ and $E(\varepsilon') = 0$, then

$$E(\hat{\sigma}^2) = E(Y'AY) = E[(X\beta + \varepsilon)'A(X\beta + \varepsilon)]$$

$$= \beta'X'AX\beta + E(\varepsilon' A\varepsilon) + 2\varepsilon'AX\beta$$

$$= \beta'X'AX\beta + \text{tr}(A\text{var}(\varepsilon))$$

$$= \beta'X'AX\beta + \sigma^2\text{tr}(A(I + \theta ZZ'))$$

$$\rightarrow E(\hat{\sigma}^2) = \beta'X'AX\beta + \sigma^2\text{tr}(Z'AZ) + \sigma^2\text{tr}(A)$$

To make $\hat{\sigma}^2$ unbiased that minimizes the norm of matrix $A$ must be have

$$\|A\|^2 = tr(\text{AA}') = \text{min}$$

and $X'AX = 0, Z'AZ = 0, \text{and tr}(A) = 1$. \hspace{1cm} (13)

To solve this problem let we assume that

$$U = [X: Z] \rightarrow U'AU = \begin{bmatrix} X'AX & X'AZ \\ Z'AX & Z'AZ \end{bmatrix}$$  \hspace{1cm} (14)$$

Since $X'AX = 0, Z'AZ = 0$ and $A$ is a symmetric and nonnegative matrix then

$$AX = 0 \text{ and } AZ = 0 \rightarrow X'AZ = 0 \text{ and } Z'AX = 0$$  \hspace{1cm} (15)

From (14) and (15) we have $U'AU = 0$ which implies that problem (13) becomes

$$\text{tr}(\text{AA}') = \text{min},$$  \hspace{1cm} (16)$$

under restrictions

$$U'AU = 0 \text{ and } \text{tr}(A) = 1$$  \hspace{1cm} (17)$$

To solve problems (16-17) using a Lagrange function for multiplier matrix (Lagrange multipliers technique), the Lagrange function can be defined as

$$f(A, L, \lambda) = \frac{1}{2} \text{tr}(\text{AA}') + \text{tr}(U'AU'L') + (1 - \text{tr}(A))\lambda$$

where $L$ is $m \times m$ Lagrange multiplier matrix and $\lambda$ is scalar.

Differentiate function $f$ with respect to $A$,

$$\frac{\partial \text{tr}(A)}{\partial A} = I, \hspace{1cm} \frac{\partial \text{tr}(\text{AA}')}{\partial A} = 2A, \hspace{1cm} \frac{\partial \text{tr}(BAC)}{\partial A} = B'C'$$

From equation $f(L)$ we have

$$A = \lambda I - ULU'$$  \hspace{1cm} (19)$$

To find $L$ and $\lambda$ in (19), we using the conditions in (17), we have
\[ U' \lambda IU - U'ULU'U = 0 \rightarrow L = \lambda(U'U)^+ \] (20)

Substituting the value of \( L \) (20) in (19), we have

\[ A = \lambda(I - U(U'U)^+U') = \lambda(I - UU^+) \] (21)

And
\[ \lambda = \frac{1}{\text{rank}(I - UU^+)} \]

The matrix \([I - UU^+]\) is idempotent (proved by [6] Graybill(1983)) then

\[ \text{rank}[I - UU^+] = \text{tr}(I - UU^+) = np - \text{rank}(U) \]

\[ \therefore \lambda = \frac{1}{np - \text{rank}(U)}, \quad \text{rank}(U) = q(n + p - 1) \]

\[ \rightarrow A = \frac{(I - U(U'U)^+U')}{q(np - n - p + 1)} = \frac{(I - UU^+)}{q(n - 1)(p - 1)} \] (22)

From (11) and (22), we have

\[ \hat{\sigma}_\varepsilon^2 = Y' \frac{(I - UU^+)}{q(n - 1)(p - 1)} Y. \square \] (23)

And

\[ \hat{\sigma}_\delta^2 = Y' \frac{\theta(I - UU^+)}{q(n - 1)(p - 1)} Y. \square \] (24)

From Proposition 2,

Since \( U(U'U)^+U' = E_1 + E_{21} \)

we have \( U(U'U)^+U' = UU^+ = \frac{I_n \otimes I_{ap}}{n} + \frac{(I_n - (J_n/n)) \otimes I_q \otimes I_p}{p} \) (25)

Substituting (25) in (22) \( A = \frac{(I_{nap} - (J_n/n) \otimes I_{q} \otimes (I_p - (J_p/p)))}{q(n - 1)(p - 1)} \)

\[ \rightarrow A = \frac{(I_n - (J_n/n)) \otimes I_q \otimes (I_p - (J_p/p))}{q(n - 1)(p - 1)} \] (26)

From (23), (24) and (26) we have, the quadratic estimator for \( \sigma_\varepsilon^2 \) and \( \sigma_\delta^2 \) are

\[ \hat{\sigma}_\varepsilon^2 = Y' \frac{(I_n - (J_n/n)) \otimes I_q \otimes (I_p - (J_p/p))}{q(n - 1)(p - 1)} Y \] (27)

And

\[ \hat{\sigma}_\delta^2 = Y' \frac{\theta(I_n - (J_n/n)) \otimes I_q \otimes (I_p - (J_p/p))}{q(n - 1)(p - 1)} Y \] (28)
3.2- MINQUE for $\sigma = [\sigma_0^2, \sigma_1^2]'$

The derivation of the law is based on minimizing the Euclidean norm.
If the mixed linear model is expressed as a matrix

For model (4) we have that $Y \sim N(X\beta, \Sigma = \sigma_0^2 I_{nqp} + \sigma_1^2 Z_1')$.

Such that $\sigma_0^2 = \sigma_1^2$, $\sigma_1^2 = \sigma_2^2$ and $Z_0 = I_{nqp}$, $Z_1 = Z$ in model (4)

We can express the model (4) as

$$Y = X\beta + Z\delta$$

(29)

where $Z = [Z_0 : Z_1]$ and $\delta' = [\varepsilon' : \delta']$. The model (29) is called a mixed linear model. Thus generally we have $E(Y) = X\beta$ and $\Sigma = \sigma_0^2 V_0 + \sigma_1^2 V_1$, where $V_r = Z_r Z_r'$, $r = 0, 1$. $\Sigma$ is called the covariance matrix and the parameters $\sigma_0^2, \sigma_1^2$ are the unknown components of variance whose values should be estimated.

We can write a linear combination for the components of variance $\sigma^2_i$, by a quadratic form $Y'AY$, where $A$ is a symmetric matrix chosen subject to the conditions which, guarantee the estimator's unbiasedness and invariance we have

$$E(Y'AY) = c_0 \sigma_0^2 + c_1 \sigma_1^2 = C\sigma$$

(30)

and

$$Y'AY = (X\beta + Z\delta)'A(X\beta + Z\delta)$$

$$= \beta'X'AX\beta + 2\beta'X'AZ\delta + \delta'Z'AZ\delta$$

Under unbiasedness and invariance, the estimator reduces to

$$Y'AY = \delta'Z'AZ\delta$$

(31)

Where $C = [c_0, c_1]'$ and $A$ is chosen to satisfy the restrictions

$$AX = 0 \text{ and } c_r = \text{tr}(Z_r'AZ_r), r = 0, 1.$$

(32)

Clear that: $\delta$ has a normal distribution (since $\varepsilon \sim iid. N(0, \sigma_0^2 I)$ and $\delta \sim iid. N(0, \sigma_1^2 I)$). The components of variance are a linear function of the natural estimated, so it should be $\delta'D\delta$ where $D$ is known diagonal matrix.

The difference between the proposed estimator (30) and the natural unbiased estimator ($\delta'D\delta$) is

$$\delta'(Z'AZ - D)\delta$$

(33)

Remark 4: $\|Z'AZ - D\|^2 = \text{tr}([Z'AZ - D]^2) = \text{min}$. (Rao1971a deduced )[9].

The MINQUE method tries to find minimize the difference in (33) with the restrictions in (32).

19

imize the square of Euclidean norm ($\|\|$) using (Remark 4) inasmuch

$$\|Z'AZ - D\|^2 = \text{tr}([Z'AZ - D]^2) = \text{tr}([AV]^2) - \text{tr}[D^2]$$

Where $V = V_0 + V_1$

Inasmuch as $\text{tr}[D^2]$ does not involve $A$, the problem of MINQUE reduces to minimizing $\text{tr}([AV]^2)$ with the conditions in (32) attained at, according to Rao [11]

$$A = a_0 QV_0 Q + a_1 QV_1 Q$$

(34)

Where

$$Q = V^{-1}(I - X(X'V^{-1}X)^+X'V^{-1})$$

And $a = [a_0, a_1]'$ is determined from the equations $a = S^+ C$, 

with

\[ S = (S_{rs}) = \text{tr}(Q V_r Q V_s), \quad r = 0, 1 \text{ and } s = 0, 1. \]

where \( X \) is the matrix in the model in (29) and \( V \) is a positive definite matrix.

For the problem of MINQUE, choosing \( V = V_0 + V_1 \) or \( \Sigma(t) = t_0 V_0 + t_1 V_1 \),

where \( t = [t_0, t_1]' \) are a priori ratios of unknown components of variance.

On using (34), we have the MINQUE of \( C^t \sigma \) is given by

\[ C^t \hat{\sigma} = Y' A Y = Y' (a_0 Q V_0 Q + a_1 Q V_1 Q) Y \]
\[ \text{(35)} \]

the estimator (35) can be written as

\[ C^t \hat{\sigma} = a^t \gamma, \quad \text{where } \gamma = \begin{bmatrix} Y' Q V_0 Q Y \\ Y' Q V_1 Q Y \end{bmatrix} \]
\[ \text{(36)} \]

On substituting \( a = S^t C \) in (36), we have

\[ C^t \hat{\sigma} = C^t S^t \gamma \rightarrow \hat{\sigma} = S^t \gamma. \]
\[ \text{(37)} \]

The solution vector (37) is unique if and only if the individual components are unbiased.

Now since \( \sigma_0^2 \) and \( \sigma_1^2 \) not equal

Let \( \alpha_0 = \frac{\varepsilon}{\sigma_0} \) and \( \alpha_1 = \frac{\delta}{\sigma_1} \)
\[ \text{(38)} \]

Then the difference in (33) is given by

\[ \alpha^t \Psi^{1/2} (Z'AZ - D) \Psi^{1/2} \alpha \]
\[ \text{(39)} \]

Where \( \alpha' = (\alpha_0', \alpha_1') \) and \( \Psi = \begin{bmatrix} \sigma_0^2 I & 0 \\ 0 & \sigma_1^2 I \end{bmatrix} \)

Now, the minimization of (39) using (Remark 4) is equivalent to minimizing \( tr[(A \Sigma)^2] \) under the restrictions in (32),

Where \( \Sigma \) defined in (29) as.

\[ \Sigma = \sigma_0^2 V_0 + \sigma_1^2 V_1 = \sigma_0^2 (V_0 + \frac{\sigma_1^2}{\sigma_0^2} V_1) \]
\[ \text{(40)} \]

The matrix \( \Sigma \) in (40) have two unknown variance \( (\sigma_r^2, r = 0, 1) \).

Then according to Rao [10], we have two amendments to this problem:

1. If we have a priori knowledge of the approximate ratio \( \frac{\sigma_1^2}{\sigma_0^2} \), we can substitute them in (40) and use the \( \Sigma \) thus computed like as estimator in (section 3.1).

2. We can use a priori estimates in (40) and obtain MINQUEs of \( \sigma_r^2, r = 0, 1. \)

These estimates then may be substituted in (40) many times. The procedure is called iterative MINQUE or I-MINQUE (Rao and Kleffe, 1988) [8]. In this procedure, the MINQUE estimator of the variance components can be obtained by solving the system of equations (37)
\[
\begin{bmatrix}
\text{tr}(Q_{(r)}V_0Q_{(r)}V_0) & \text{tr}(Q_{(r)}V_0Q_{(r)}V_1)
\end{bmatrix} \begin{bmatrix}
\sigma_0^2 \\
\sigma_1^2
\end{bmatrix} = \begin{bmatrix}
\text{tr}(Q_{(r)}V_0Q_{(r)}Y) \\
\text{tr}(Q_{(r)}V_1Q_{(r)}Y)
\end{bmatrix}
\] (41)

Where
\[
Q_{(r)} = \Sigma_{(r)}^{-1} \left( I - X'(X'\Sigma_{(r)}^{-1}X)^+X'\Sigma_{(r)}^{-1} \right)
\]
\[
\Sigma_{(r)} = t_0V_0 + t_1V_1 ; t = [t_0, t_1]', \quad \quad (42)
\]

Although the estimate of the variance component depends on a priori value of the human choice \( t_r \), as long as these a priori values do not depend on the experimental data, the MINQUE estimator is still unbiased. Choose any a priori \( t_r \), can be obtained the variance component estimate \( [\sigma_r^2] \). New estimates can be obtained if the estimates are replaced with priori estimates for reevaluation value. This process is repeated until the new estimate is very close to the old estimate. This iterative estimation method is like to relative maximum likelihood estimator (REML) method, which is a result of the maximum likelihood estimate (EML). In other word, REML estimates and MINQUE estimates are relatively close. For more see [12].

### 3.3 - MINQUE (1) for \( \sigma_r^2 \) and \( \sigma_s^2 \)

The choice of a priori \( t_r \) in (42) can based on experience or even on past analysis. The easier way is to take it all a priori values are 1 (\( t_r = 1 \)). This method is called the MINQUE (1) method, and the variance component obtained is estimated metering is a MINQUE (1) estimate.

The unbiased estimator of the MINQUE (1) (MINQUE on \( t = j_2 \)) is,

Assume \( t = j_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), substituting \( t_r \) in (42) we have

\[
\begin{align*}
\Sigma_{(1)} &= V_0 + V_1 = V = Z_0Z_0' + Z_1Z_1' = I_{nq} \otimes (I_p + J_p), \\
Q_{(1)} &= \Sigma_{(1)}^{-1}(I - X(X'\Sigma_{(1)}^{-1}X)^+X'\Sigma_{(1)}^{-1})
\end{align*} \quad \quad (43)
\]

Let \( U = (\Sigma_{(1)}^{-1})^{1/2} X \) and \( P_U = U(U'U)^{-1}U' \) is projection matrix. Then

\[
\begin{align*}
Q_{(1)} &= (\Sigma_{(1)}^{-1})^{1/2}(I - P_U)(\Sigma_{(1)}^{-1})^{1/2} \\
Q_{(1)}X &= (\Sigma_{(1)}^{-1})^{1/2}(I - P_U)(\Sigma_{(1)}^{-1})^{1/2}X \\
&= (\Sigma_{(1)}^{-1})^{1/2}(I - P_U)U \\
&= (\Sigma_{(1)}^{-1})^{1/2}(U - U) = 0
\end{align*} \quad \quad (44)
\]

Therefor \( r = 0,1 \) and \( s = 0.1 \) we have

\[
E[ Y'Q_{(1)}V_0Q_{(1)}Y] = \text{tr}(Q_{(1)}V_0Q_{(1)}\Sigma) + (X\beta)'Q_{(1)}V_0Q_{(1)}X\beta
\]
\[
= \text{tr}(Q_{(1)}V_0Q_{(1)}(\sigma_0^2V_0 + \sigma_1^2V_1)) \\
= \text{tr}(\sigma_0^2Q_{(1)}V_0Q_{(1)}V_0 + \sigma_1^2Q_{(1)}V_0Q_{(1)}V_1)
\]

Implies to

\[
E \begin{bmatrix}
Y'Q_{(1)}V_0Q_{(1)}Y \\
Y'Q_{(1)}V_1Q_{(1)}Y
\end{bmatrix} = \begin{bmatrix}
\text{tr}(Q_{(1)}V_0Q_{(1)}V_0) & \text{tr}(Q_{(1)}V_0Q_{(1)}V_1) \\
\text{tr}(Q_{(1)}V_1Q_{(1)}V_0) & \text{tr}(Q_{(1)}V_1Q_{(1)}V_1)
\end{bmatrix} \begin{bmatrix}
\sigma_0^2 \\
\sigma_1^2
\end{bmatrix}
\]

Where
\[
S = \begin{bmatrix}
\text{tr}(Q_1 V_0 Q_1 V_0) & \text{tr}(Q_1 V_0 Q_1 V_1) \\
\text{tr}(Q_1 V_1 Q_1 V_0) & \text{tr}(Q_1 V_1 Q_1 V_1)
\end{bmatrix}_{2 \times 2} = \begin{bmatrix}
\lambda_{00} & \lambda_{01} \\
\lambda_{10} & \lambda_{11}
\end{bmatrix} \tag{45}
\]

and \(\sigma = \begin{bmatrix}
\sigma_0^2 \\
\sigma_1^2
\end{bmatrix}, \lambda_{r,s} = \text{sumation all eigenvalues of } Q_1 V_r Q_1 V_s.\)

we have
\[
S \bar{\sigma} = \begin{bmatrix}
Y' Q_1 V_0 Q_1 Y \\
Y' Q_1 V_1 Q_1 Y
\end{bmatrix}_{2 \times 1}
\]
\[
\sigma^2 = S^+ \begin{bmatrix}
Y' Q_1 V_0 Q_1 Y \\
Y' Q_1 V_1 Q_1 Y
\end{bmatrix}, S^+ = \frac{1}{|S|} \begin{bmatrix}
\lambda_{1,1} & -\lambda_{0,1} \\
-\lambda_{1,0} & \lambda_{0,0}
\end{bmatrix}, |S| \text{ is determinant of } S.
\]

We can write as
\[
\begin{bmatrix}
\text{tr}(Q_1 Z_r Q_0 Z_0 Q_1) \\
\text{tr}(Q_1 Z_r Q_0 Z_1 Q_1)
\end{bmatrix} \left(\sigma^2 \right) = \begin{bmatrix}
Y' Q_1 Z_r Q_0 Q_1 Y \\
Y' Q_1 Z_r Q_1 Q_1 Y
\end{bmatrix} \tag{46}
\]

Relationships (43-46) are proof of (41). \(\square\)

Then the MINQUE for \(\sigma_0^2\) and \(\sigma_1^2\) are
\[
\hat{\sigma}_0^2 = \hat{\sigma}_1^2 = Y' \left( s_{0,0} Q_1 Z_0 Z_0 Q_1 + s_{0,1} Q_1 Z_1 Z_0 Q_1 \right) Y. \tag{47}
\]
\[
= Y' \left( \frac{1}{|S|} \left( \lambda_{1,1} Q_1 Q_1 - \lambda_{0,1} Q_1 (I_{nq} \otimes I_p) Q_1 \right) \right) Y. \tag{48}
\]

and
\[
\hat{\sigma}_1^2 = \hat{\sigma}_1^2 = Y' \left( s_{1,0} Q_1 Z_0 Z_0 Q_1 + s_{1,1} Q_1 Z_1 Z_0 Q_1 \right) Y. \tag{49}
\]
\[
= Y' \left( \frac{1}{|S|} \left( -\lambda_{1,0} Q_1 Q_1 + \lambda_{0,0} Q_1 (I_{nq} \otimes I_p) Q_1 \right) \right) Y. \tag{50}
\]

4. Conclusions
The conclusions obtained throughout this work are as follows:

1. The MINQUE for \(\sigma_0^2\) and \(\sigma_1^2\) are
\[
\hat{\sigma}_0^2 = Y' \left( \frac{(l_n - (l_n/n)) \otimes I_q \otimes (I_p - (I_p/p))}{q(n-1)(p-1)} \right) Y.
\]

And
\[
\hat{\sigma}_1^2 = Y' \left( \frac{\Theta(l_n - (l_n/n)) \otimes I_q \otimes (I_p - (I_p/p))}{q(n-1)(p-1)} \right) Y.
\]

2. The MINQUE(1) for \(\sigma_0^2\) and \(\sigma_1^2\) are
\[
\hat{\sigma}_0^2 = Y' \left( \frac{1}{|S|} \left( \lambda_{1,1} Q_1 Q_1 - \lambda_{0,1} Q_1 (I_{nq} \otimes I_p) Q_1 \right) \right) Y.
\]

And
\[
\hat{\sigma}_1^2 = Y' \left( \frac{1}{|S|} \left( -\lambda_{1,0} Q_1 Q_1 + \lambda_{0,0} Q_1 (I_{nq} \otimes I_p) Q_1 \right) \right) Y.
\]
5. References


