Strongly b star (Sb*) – cleavability (splitability)

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ABSTRACT

A. Poongothai, R. Parimelazhagan[5] introduced some new type of separation axioms and study some of their basic properties. Some implications between T_0, T_1 and T_2 axioms are also obtained. In this paper we studied the concept of cleavability over these spaces: (Sb^*T_0, Sb^*T_1, Sb^*T_2) as following:

1. If \( P \) is a class of topological spaces with certain properties and if \( X \) is cleavable over \( P \) then \( X \in P \)

2. If \( P \) is a class of topological spaces with certain properties and if \( Y \) is cleavable over \( P \) then \( Y \in P \)

MSC:

1. Introduction

In 1985 Arhangl' Skii [1] introduced different types of cleavability (originally named splitability) as following: A topological space \( X \) is said to be cleavable over a class of spaces \( P \) if for \( A \subset X \) there exists a continuous mapping \( f: X \to Y \in P \) such that \( f^{-1}(A) = A \), \( f(X) = Y \). Throughout this paper, \( X \) and \( Y \) denote the topological spaces \( (X, \tau) \) and \( (Y, \sigma) \) respectively. Let \( A \) be a subset of the space \( X \). The interior and closure of a set \( A \) in \( X \) are denoted by \( \text{int}(A) \) and \( \text{cl}(A) \) respectively. The complement of \( A \) is denoted by \( (X - A) \) or \( A^c \).
3-Preliminaries

In this section, we recall some definitions and results which are needed in this paper.

**Definition 3.1.** [11]

A topological space $X$ is called a $T_0$-space if and only if it satisfies the following axiom of Kolmogorov. ($T_0$) If $x$ and $y$ are distinct points of $X$, then there exists an open set which contains one of them but not the other.

**Definition 3.2.** [11]

A topological space $X$ is a $T_1$-space if and only if it satisfies the following separation axiom of Frechet. ($T_1$) If $x$ and $y$ are two distinct points of $X$, then there exists two open sets, one containing $x$ but not $y$ and the other containing $y$ but not $x$.

**Definition 3.3.** [11]

A topological space $X$ is said to be a $T_2$-space or Hausdorff space if and only if for every pair of distinct points $x, y$ of $X$, there exists two disjoint open sets one containing $x$ and the other containing $y$.

**Definition 3.4.** [8]

A subset $(X, \tau)$ is said to be Sb*-closed set if $\text{cl}(\text{int}(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is b-open in $X$. The family of all Sb*-open sets of a space $X$ is denoted by $\text{Sb}^*O(X)$.

**Theorem 3.1.** [5]

Let $X$ be a topological space and $A$ be a subset of $X$. Then $A$ is Sb*-open iff $A$ contains a Sb* open neighbourhood of each of its points.

**Definition 3.5.** [6]

A subset $A$ of a topological space $(X, \tau)$ is called b-open set if $A \subseteq (\text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)))$. The complement of a b-open set is said to be b-closed. The family of all b-open subsets of a space $X$ is denoted by $\text{BO}(X)$.

**Definition 3.6.** [11]

A map $f: X \rightarrow Y$ is said to be Continuous function if $f^{-1}(V)$ is closed in $X$ for every closed set $V$ in $Y$.

**Definition 3.7.**

A map $f: X \rightarrow Y$ is said to be Sb*-open map if the image of every open set in $X$ is Sb*-open in $Y$.

**Definition 3.8.** [9]

Let $X$ and $Y$ be topological spaces. A map $f: X \rightarrow Y$ is called strongly b* - continuous (sb*- continuous) if the inverse image of every open set in $Y$ is sb*-open in $X$.

**Definition 3.9.** [3]

Let $X$ and $Y$ be topological spaces. A map $f: X \rightarrow Y$ is called strongly b* - closed (briefly sb*-closed) map if the image of every closed set in $X$ is sb*-closed in $Y$.

**Definition 3.10.**

Let $X$ and $Y$ be topological spaces. A map $f:(X, \tau) \rightarrow (Y, \sigma)$ is said to be sb*-Irresolute if the inverse image of every sb*-closed (respectively sb*-open) set in $Y$ is sb*-closed (respectively sb*-open) set in $X$.

4- sb* – cleavability

**Definition 4.1.**

A topological spaces $X$ is said to be sb*-pointwise cleavable over a class of spaces $\mathcal{P}$ if for every point $x \in X$ there exists a sb*-continuous mapping $f: X \rightarrow Y \in \mathcal{P}$, such that $f^{-1}(\{x\}) = \{x\}$.

**Definition 4.2.**

A topological spaces $X$ is said to be sb* Irresolute - pointwise cleavable over a class of spaces $\mathcal{P}$ if for every point $x \in X$ there exists a sb* - Irresolute - continuous mapping $f: X \rightarrow Y \in \mathcal{P}$, such that $f^{-1}(\{x\}) = \{x\}$. 
Definition 4.3

By a sb*-open(closed) pointwise cleavable, we mean that the sb*- (Irresolute) continuous function \( f: X \to Y \in P \) is an injective and open(closed) respectively

Definition 4.4.[5]

A topological space X is said to be sb*-T_0 if for every pair of distinct points x and y of X, there exists a sb*-open set G such that x \( \in \) G and y \( \notin \) G or y \( \in \) G and x \( \notin \) G.

Proposition 4.1

Let X be a sb* - irresolute pointwise cleavable over a class of sb*-T_0 spaces \( P \), then \( X \in P \).

Proof:

Let \( x \in X \), then there exists sb*-T_0 space Y and sb* irresolute a continuous mapping \( f: X \to Y \in P \), such that \( f^{-1}(x) = \{x\} \). This implies that for every \( y \in X \) with \( x \neq y \), we have \( f(x) \neq f(y) \) since \( Y \) is a sb*-T_0-space, so there exists a sb*-open set G in Y contains one of the two points but not the other. Let \( f(x) \in G, (y) \notin G \), then \( f^{-1}(f(x)) \in f^{-1}(G) \) such that \( x \in f^{-1}(G) \) and \( y \notin f^{-1}(G) \), since \( f \) is a sb*irresolute a continuous, so \( f^{-1}(G) \) is a sb*-open set in X. Therefore \( X \) is a sb*-T_0 - space.

Theorem 4.1.[5]

Every subspace of a sb*-T_0 space is sb*-T_0.

Proof:

Let \( (Y, t^*) \) be a subspace of a space X where \( t^* \) is the relative topology of \( t \) on Y. Let \( y_1, y_2 \) be two distinct points of \( Y \), as \( Y \subseteq X \), \( y_1 \) and \( y_2 \) are distinct points of X and there exists a sb*-open set G such that \( y_1 \in G \) but \( y_2 \notin G \) since \( X \) is sb*-T_0.

Then \( G \cap Y \) is a sb*-open set in \( (Y, t^*) \) which contains \( y_1 \) but does not contain \( y_2 \). Hence \( (Y, t^*) \) is a sb*-T_0 space.

Proposition 4.2

Let \( X \) be a sb* T_0-space is sb* - irresolute pointwise cleavable over a class spaces \( P \), then \( Y \in P \).

Proof:

Let \( y \in Y \), then there exists an sb*-irresolute continuous mapping \( f: X \to Y \in P \) such that \( f^{-1}(f^{-1}(y)) = f^{-1}(y) \), This implies that for every \( x \in Y \) with \( y \neq x \), we have \( f^{-1}(x) \neq f^{-1}(y) \) since \( X \) is a sb*-T_0 space, so there exists a sb*-open sets U contains one of the two points but not the other. Let \( f^{-1}(y) \in U \) and \( f^{-1}(x) \in U \), then \( f^{-1}(y) \in f^{-1}(U) \) and \( f^{-1}(x) \notin f^{-1}(U) \) . This implies that \( y \notin f(U) \) but \( x \in f(U) \) .Therefore \( Y \) is sb*-T_0 - space, then \( Y \in P \).

Definition 4.5.[5] A space \( X \) is said to be sb*-T_1, if for every pair of distinct points x and y in \( X \), there exist sb* - open sets U and V such that \( x \in U \) but \( y \notin U \) and \( y \in V \) but \( x \notin V \).

Proposition 4.3

Let \( X \) be a sb*- irresolute pointwise cleavable over a class of sb*-T_1 spaces \( P \), then \( X \in P \).

Proof:

Let \( x \in X \), then there exists a sb* T_1-space Y and a sb*- irresolute- continuous mapping \( f: X \to Y \in P \) such that \( f^{-1}(x) = \{x\} \). This implies mapping \( f: X \to Y \in P \) such that \( f^{-1}(x) = \{x\} \). This implies that for every \( y \in X \) with \( x \neq y \), we have \( f(x) \neq f(y) \). Since \( Y \) is sb*-T_1 space, so there exist two sb*- open sets U and V such that \( f(x) \in U, (y) \notin U \) and \( f(y) \in V, f(x) \notin V \), then \( f^{-1}(x) \in f^{-1}(U) \), \( f^{-1}(y) \notin f^{-1}(U) \) and \( f^{-1}(y) \in f^{-1}(V) \), \( f^{-1}(y) \notin f^{-1}(V) \). This implies that \( x \in f^{-1}(U) \), \( y \notin f^{-1}(U) \) and \( y \in f^{-1}(V) \), \( x \notin f^{-1}(V) \). By a sb*- irresolute - continuity of \( f \), \( f^{-1}(U), f^{-1}(V) \) are sb*- open sub sets in \( X \). Then \( X \in P \).

Proposition 4.4

Let \( X \) be a sb*-pointwise cleavable over a class of T_1 spaces \( P \), then \( X \in sb*-T_1 \).

Proof:

Let \( x \in X \), then there exists a T_1 space Y and a sb*- continuous mapping \( f: X \to Y \in P \) such that \( f^{-1}(x) = \{x\} \). This implies mapping \( f: X \to Y \in P \) such that \( f^{-1}(x) = \{x\} \). This implies that for every \( x' \in X \) with \( x \neq x' \), we have \( f(x) \neq f(x') \). Since \( Y \) is T_1 - space, so there exist two open sets G and H such that \( f(x) \in G, f(x') \notin G \) and \( f(x') \in H, f(x) \notin H \), then \( f^{-1}(x) \in f^{-1}(G) \), \( f^{-1}(x') \notin f^{-1}(G) \) and \( f^{-1}(x') \in f^{-1}(H) \),\( f^{-1}(x) \notin f^{-1}(H) \). This implies that \( x \in f^{-1}(H) \), \( x' \notin f^{-1}(G) \) and \( x' \notin f^{-1}(H) \), \( x \notin f^{-1}(H) \) .By a sb*- continuity of \( f \) then \( f^{-1}(G), f^{-1}(H) \) are sb*-open sub sets in \( X \). Thus \( X \) is sb*- T_1 - space, then \( X \in P \).
Proposition 4.5

Let \( X \) be \( sb^* T_1 \)-space is an \( sb^* \) - open pointwise cleavable over a class of spaces \( \mathcal{P} \), then \( Y \in \mathcal{P} \).

Proof:

Let \( y \in Y \), then there exists a \( sb^* T_1 \)-space \( X \) and \( sb^* \) - open continuous mapping \( f : X \to Y \in \mathcal{P} \), such that \( ff^{-1}(f^{-1}(y)) = f^{-1}(y) \). This implies that for every \( x \in X \) with \( y \neq x \), we have \( f^{-1}(y) \neq f^{-1}(x) \). Since \( X \) is \( sb^* T_1 \)-space, so there exist two \( sb^* \)-open sets \( V \) and \( W \) such that \( f^{-1}(y) \in V, f^{-1}(x) \in V \) and \( f^{-1}(x) \in W, f^{-1}(y) \in W \). Then \( ff^{-1}(y) \in f(V) \), \( ff^{-1}(x) \in f(V) \) and \( ff^{-1}(x) \in f(W), ff^{-1}(y) \in f(W) \). This implies that \( y \in f(V), x \in f(V) \) and \( x \in f(W), y \in f(W) \), since \( f \) is a \( sb^* \) open, so \( f(V), f(W) \) are open \( sb^* \) sets of \( Y \). Therefore \( Y \in \mathcal{P} \).

Definition 4.6[5].

A space \( X \) is said to be \( sb^* T_2 \) if for every pair of distinct points \( x \) and \( y \) in \( X \), there are disjoint \( sb^* \)-open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \) respectively.

Theorem 4.2[5] Every \( sb^* T_2 \) space is \( sb^* T_1 \).

Proof:

Let \( X \) be a \( sb^* T_2 \) space. Let \( x \) and \( y \) be two distinct points in \( X \). Since \( X \) is \( sb^* T_2 \), there exist disjoint \( sb^* \)-open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \). Since \( U \) and \( V \) are disjoint, \( x \in U \) but \( y \in V \) but \( x \notin V \). Hence \( X \) is \( sb^* T_1 \).

Proposition 4.6

Let \( X \) be \( sb^* T_2 \)-space is a \( sb^* \) - open pointwise cleavable over a class of spaces \( \mathcal{P} \), then \( Y \in \mathcal{P} \).

Proof:

Let \( y_1 \in Y \), then there exists a \( sb^* T_2 \)-space \( X \) and a \( sb^* \) open continuous mapping \( f : X \to Y \in \mathcal{P} \) such that \( f^{-1}(f(y_1)) = f^{-1}(y_1) \). This implies that for every \( y_2 \in Y \), with \( y_1 \neq y_2 \), we have \( f^{-1}(y_1) \neq f^{-1}(y_2) \), so there exist \( x_1, x_2 \) in \( X \), such that \( x_1 = f^{-1}(y_1) \), \( x_2 = f^{-1}(y_2) \) with \( x_1 \neq x_2 \). Since \( X \) is \( sb^* T_2 \), so there exist two \( sb^* \) open sets \( G, H \) such that \( f^{-1}(y_1) \in G, f^{-1}(y_2) \in H \) and \( G \cap H = \emptyset \), then \( ff^{-1}(x_1) \in f(G), ff^{-1}(x_2) \in f(H) \). Since \( f \) is \( sb^* \) open, then \( f(G), f(H) \) are \( sb^* \) open sets of \( Y \) and \( y_1 \in f(G), y_2 \in f(H) \) and \( f(G) \cap f(H) = f(G \cap H) = f(\emptyset) = \emptyset \). Then \( Y \in \mathcal{P} \).

Proposition 4.7

Let \( X \) be \( sb^* \) - open pointwise cleavable over a class of \( sb^*- T_2 \)-spaces \( \mathcal{P} \), then \( X \in \mathcal{P} \).

Proof:

Let \( x \in X \), then there exists a \( sb^* T_2 \)-space \( Y \) and a \( sb^* \)-continuous mapping \( f : X \to Y \in \mathcal{P} \) such that \( f^{-1}(x) = \{x\} \). This implies that for every \( y \in Y \) with \( x \neq y \), we have \( f(x) \neq f(y) \). Since \( Y \) is \( sb^* T_2 \), so there exist two \( sb^* \) open sets \( U \) and \( V \) such that \( f(x) \in U, f(y) \in V \) and \( U \cap V = \emptyset \), then \( f^{-1}(U), f^{-1}(V) \) are \( sb^* \) open sets of \( X \) and \( U \cap V = \emptyset \). Then \( X \in \mathcal{P} \).

5-conclusion:

In this paper we have studied and proved these cases:

1) If \( \mathcal{P} \) is a class of \( (sb^*- T_0, \ sb^*- T_1) \) spaces with certain properties and if \( X \) is a \( sb^* \) - irresolute pointwise cleavable over \( \mathcal{P} \), then \( X \in \mathcal{P} \). also if \( \mathcal{P} \) is a class of \( (sb^*- T_0, \ sb^*- T_1) \) spaces with certain properties and if \( Y \) is a \( sb^* \) - irresolute pointwise cleavable over \( \mathcal{P} \), then \( Y \in \mathcal{P} \).

2) If \( \mathcal{P} \) is a class of \( (sb^*- T_1, \ sb^*- T_2) \) spaces with certain properties and if \( X \) is point wise \( sb^* \) - cleavable over \( \mathcal{P} \), then \( X \in \mathcal{P} \). also If \( \mathcal{P} \) is a class of \( sb^* T_2 \) spaces with certain properties and if \( X \) is a \( sb^* \) - irresolute pointwise cleavable over \( \mathcal{P} \), then \( X \in \mathcal{P} \).
3) If $\mathcal{P}$ is a class of $(s_b^*-T_1 \cdot s_b^*-T_2)$ spaces with certain properties and if $Y$ is point wise $s_b^*$ cleavable over $\mathcal{P}$, then $Y \in \mathcal{P}$.

References

$s_b^*$ - Separation axioms 163.