Strongly b star(Sb*) – cleavability(splitability)

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**Abstract**

A. Poongothai and R. Parimelazhagan[5] introduced some new type of separation axioms and study some of their basic properties. Some implications between $T_0$, $T_1$ and $T_2$ axioms are also obtained.

In this paper we studied the concept of cleavability over these spaces: (sb*T₀, sb*T₁, sb*T₂) as following:

1. If $\mathcal{P}$ is a class of topological spaces with certain properties and if $X$ is cleavable over $\mathcal{P}$, then $X \in \mathcal{P}$.

2. If $\mathcal{P}$ is a class of topological spaces with certain properties and if $Y$ is cleavable over $\mathcal{P}$, then $Y \in \mathcal{P}$.

**1. Introduction**

In 1985 Arhangl’ Skii [1] introduced different types of cleavability(originally named splitability) as following:

A topological space $X$ is said to be cleavable over a class of spaces $\mathcal{P}$, if for $A \subset X$ there exists a continuous mapping $f:X \rightarrow Y \in \mathcal{P}$ such that $f^{-1}(f(A)) = A$, $f(X) = Y$.

Throughout this paper, $X$ and $Y$ denote the topological spaces $(X, \tau)$ and $(Y, \sigma)$ respectively. Let $A$ be a subset of the space $X$, the interior and closure of a set $A$ in $X$ are denoted by $int(A)$ and $cl(A)$ respectively. The complement of $A$ is denoted by $A^c$.

**2. Preliminaries**

In this section, we recall some definitions and results which are needed in this paper.

**Definition 2.1.** [3]

A topological space $X$ is called a $T_0$ - space if and only if it satisfies the following axiom of Kolmogorov. ($T_0$) If $x$ and $y$ are distinct points of $X$, then there exists an open set which contains one of them but not the other.
Definition 2.2. [3]
A topological space $X$ is a $T_1$-space, if and only if it satisfies the following separation axiom of Frechet. $(T_1)$, if $x$ and $y$ are two distinct points of $X$, then there exist two open sets, one containing $x$ but not $y$ and the other containing $y$ but not $x$.

Definition 2.3 [6]
A topological space $X$ is said to be a $T_2$-space or Hausdorff space, if and only if for every pair of distinct points $x$ and $y$ of $X$, there exist two disjoint open sets, one containing $x$ and the other containing $y$.

Definition 2.4 [3]
A subset $A \subseteq X$ is said to be $Sb^*$-closed if $\text{cl}(\text{int}(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is $b$-open in $X$. The complements of closed sets $Sb^*$-closed is $Sb^*$-open sets. The family of all $sb^*$-open sets of a space $X$ is denoted by $sb^*O(X)$.

Theorem 2.5 [5]
Let $X$ be a topological space and $A$ be a subset of $X$. Then $A$ is $Sb^*$-open iff $A$ contains a $Sb^*$ open neighbourhood of each of its points.

Definition 2.5 [3]
A map $f : X \rightarrow Y$ is said to be $Sb^*$-open map if the image of every open set in $X$ is $Sb^*$-open in $Y$.

Definition 2.6 [3]
Let $X$ and $Y$ be topological spaces. A map $f : X \rightarrow Y$ is called strongly $b^*$-continuous ($sb^*$-continuous) if the inverse image of every open set in $Y$ is $sb^*$-open in $X$.

Definition 2.7 [3]
Let $X$ and $Y$ be topological spaces. A map $f : X \rightarrow Y$ is called strongly $b^*$-closed (briefly $sb^*$-closed) map if the image of every closed set in $X$ is $sb^*$-closed in $Y$.

Definition 2.8 [3]
A topological space $X$ is said to be $sb^*$-$T_3$ if for every pair of distinct points $x$ and $y$ of $X$, there exists a $sb^*$-open set $G$ such that $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$.

Definition 2.9 [5]
A topological space $X$ is said to be $sb^*$-$T_4$ if for every pair of distinct points $x$ and $y$ of $X$, there exist $sb^*$-open sets $U$ and $V$ containing $x$ and $y$ respectively.

3- $sb^*$ - cleavability

Definition 3.1
A topological spaces $X$ is said to be $sb^*$-pointwise cleavable over a class of spaces $\mathcal{P}$ if for every point $x \in X$ there exists a $sb^*$-continuous mapping $f : X \rightarrow Y \in \mathcal{P}$, such that $f^{-1}(x) = \{x\}$.

Definition 3.2
A topological spaces $X$ is said to be $sb^*$-Irresolute - pointwise cleavable over a class of spaces $\mathcal{P}$ if for every point $x \in X$ there exists a $sb^*$-Irresolute - continuous mapping $f : X \rightarrow Y \in \mathcal{P}$, such that $f^{-1}(x) = \{x\}$.
Remark 3.1  
By a sb*-open(closed) pointwise cleavable, we mean that the sb*-Irresolute continuous function \(f: X \to Y \in \rho\) is an injective and open(opened) respectively.

Theorem 3.1 [3]  
If a map \(f: X \to Y \in \rho\) is continuous, then it is sb* - continuous but not conversely.

Example 3.1  
Let \(X = \{a, b\}\), \(\tau = \{X, \phi, \{a\}, \{b\}\}\). \(Y = \{p, q\}\), \(\sigma = \{Y, \phi, \{p\}, \{q\}\}\), let \(f: (X, \tau) \to (Y, \sigma)\) be a sb*-continuous map defined by \(f(a) = p, f(b) = q\).

To show that sb* pointwise cleavable over \(Y\) as follows:

Clearly \(f\) is sb* - continuous function.

Now, \(a \in X, f(a) = \{p\}\) and \(f^{-1}(p) = \{a\}\).

Proposition 3.1  
Let \(X\) be a sb* - irresolute pointwise cleavable over a class of sb*-T0 spaces \(\mathcal{P}\), then \(X \in \mathcal{P}\).

Proof:  
Let \(x \in X\), then there exist \(sb^* T_0\) space \(Y\) and \(sb^*\) irresolute continuous mapping \(f: X \to Y \in \rho\), such that \(f^{-1}(x) = \{x\}\). This implies that for every \(y \in X\) with \(x \neq y\), we have \(f(x) \neq f(y)\), since \(Y\) is a sb*-T0-space, there exist two \(sb^*\)-open sets \(U \in Y\) containing one of the two points but not the other. Let \(f^{-1}(y) \in G, f(y) \notin G\), such that \(f^{-1}(G) \subset f^{-1}(G)\) and \(f^{-1}(G) \subset f^{-1}(G)\). This implies that \(x \in f^{-1}(G)\) and \(y \notin f^{-1}(G)\). Since \(f\) is a sb* irresolute a continuous, \(f^{-1}(G)\) is a \(sb^*\)-open sets in \(X\). Therefore \(X\) is sb* T0-space.

Theorem 3.1 [5]  
Every subspace of a sb*-T0 space is sb*-T0.

Proposition 3.2  
Let \(X\) be a sb* T0-space is a sb* - irresolute pointwise cleavable over a class \(\mathcal{P}\) of space, then \(Y \in \mathcal{P}\).

Proof:  
Let \(y \in Y\), then there exists a \(sb^*\)- irresolute continuous mapping \(f: X \to Y \in \rho\) such that \(f^{-1}(y) \subset f^{-1}(y)\). This implies that for every \(x \in X\) with \(x \neq y\), we have \(f(x) \neq f(y)\). Since \(Y\) is a sb*-T0-space, there exist two \(sb^*\)-open sets \(U\) and \(V\) such that \(f(x) \in U, f(y) \notin U\) and \(f(y) \in V, f(x) \notin V\), such that \(f^{-1}(U) \subset f^{-1}(V)\) and \(f^{-1}(V) \subset f^{-1}(U)\). This implies that \(x \in f^{-1}(U)\) and \(y \notin f^{-1}(U)\) and \(y \in f^{-1}(V)\), \(x \in f^{-1}(V)\).

Proposition 3.3  
Let \(X\) be a sb* - irresolute pointwisecleavable over a class of sb*-T1 spaces \(\mathcal{P}\), then \(X \in \mathcal{P}\).

Proof:  
Let \(x \in X\), then there exist \(sb^*\) and \(a\) sb*- T1-space \(Y\) and sb*- irresolute continuous mapping \(f: X \to Y \in \mathcal{P}\) such that \(f^{-1}(x) = \{x\}\). This implies that for every \(y \in X\) with \(x \neq y\), we have \(f(x) \neq f(y)\). Since \(Y\) is sb*-T1-space, there exist two \(sb^*\)-open sets \(U \in Y\) and \(V\) such that \(f(x) \in U, f(y) \notin U\) and \(f(y) \in V, f(x) \notin V\), such that \(f^{-1}(U) \subset f^{-1}(V)\) and \(f^{-1}(V) \subset f^{-1}(U)\) and \(f^{-1}(V) \subset f^{-1}(U)\). This implies that \(x \in f^{-1}(U)\) and \(y \notin f^{-1}(U)\) and \(y \in f^{-1}(V)\), \(x \in f^{-1}(V)\).

Proposition 3.4  
Let \(X\) be a sb* - pointwisecleavable over a class of T1 space \(\mathcal{P}\), then \(X \in \mathcal{P}\).

Proof:  
Let \(x \in X\), then there exists a \(T_1\) space \(Y\) and sb*- continuous mapping \(f: X \to Y \in \mathcal{P}\) such that \(f^{-1}(x) = \{x\}\). This implies mapping \(f: X \to Y \in \mathcal{P}\) such that \(f^{-1}(x) = \{x\}\). This implies that for every \(x \in X\) with \(x \neq x^*\), we have \(f(x) \neq f(x^*)\). Since \(Y\) is \(T_1\)-space, there exist two \(sb^*\)-open sets \(G \in Y\) and \(H\) such that \(f(x) \in G, f(x^*) \notin G\) and \(f(x^*) \in H, f(x) \notin H\). Then \(f^{-1}(x) \subset f^{-1}(G)\) and \(f^{-1}(x^*) \subset f^{-1}(H)\). Since \(f\) is a sb* - irresolute continuous mapping, \(f^{-1}(G)\) and \(f^{-1}(H)\) are sb*- open sets in \(X\). Thus \(X\) is sb*-T1-space, hence \(X \in \mathcal{P}\)
Proposition 3.5
Let X be sb*-T₁-space is a sb* - open pointwise cleavable over a class of spaces \( \mathcal{P} \), then Y is sb*-T₁-space, thus \( Y \in \mathcal{P} \).

Proof:
Let \( y \in Y \), then there exist a sb* T₁-space X and sb* - open continuous mapping \( f : X \to Y \in \mathcal{P} \), such that \( f^{-1}(f^{-1}(y)) = f^{-1}(y) \). This implies that for every \( x \in X \), we have \( f^{-1}(y) \neq f^{-1}(x) \). Since X is sb* T₁-space, there exist two sb*-open sets V and W, such that \( f^{-1}(y) \notin V, f^{-1}(x) \notin V \) and \( f^{-1}(y) \notin W, f^{-1}(x) \notin W \). Then \( f^{-1}(y) \in f(V), f^{-1}(x) \notin f(V) \) and \( f^{-1}(y) \in f(W), f^{-1}(x) \notin f(W) \). This implies that \( y \in f(V), x \notin f(V) \) and \( x \in f(W), y \notin f(W) \), since f is a sb*-open, so f(V), f(W) are open sb* sets of Y, then \( Y \in \mathcal{P} \). Therefore \( Y \in \mathcal{P} \).

Proposition 3.6
Let X be sb*-T₂-space is a sb* - open pointwise cleavable over a class \( \mathcal{P} \) of spaces Y, then Y is sb*-T₂-space, thus \( Y \in \mathcal{P} \).

Proof:
Let \( y_1 \in Y \), then there exist a sb*-T₂-space X and a sb* continuous mapping \( f : X \to Y \in \mathcal{P} \) such that \( f^{-1}(f^{-1}(y_1)) = f^{-1}(y_1) \). This implies that for every \( y_2 \in Y \), with \( y_1 \neq y_2 \), we have \( f^{-1}(y_1) \neq f^{-1}(y_2) \), so there exist \( x_1, x_2 \) in X, such that \( x_1 = f^{-1}(y_1), x_2 = f^{-1}(y_2) \), \( x_1 \neq x_2 \). Since X is sb*-T₂-space, there exist two sb* open sets \( G, H \), such that \( f^{-1}(y_1) \in G, f^{-1}(y_2) \in H \) and \( G \cap H = \emptyset \), then \( f^{-1}(y_1) \in f(G), f^{-1}(y_2) \in f(H) \). Since f is sb* open, then \( f(G), f(H) \) are sb* open sets of Y and \( y_1 \in f(G), y_2 \in f(H) \) and \( f(G) \cap f(H) = f(G \cap H) = \emptyset \). Thus Y is sb*-T₂-space, then \( Y \in \mathcal{P} \).

Proposition 3.7
Let X be sb* - open pointwise cleavable over a class of sb*-T₂-spaces \( \mathcal{P} \), then \( X \in \mathcal{P} \).

Proof:
Let \( x \in X \), then there exist a sb*-T₂-space Y and a sb*-continuous mapping \( f : X \to Y \in \mathcal{P} \) such that \( f^{-1}(f(x)) = \{ x \} \). This implies that for every \( y \in Y \) with \( x \neq y \), we have \( f(x) \neq f(y) \). Since Y is sb*-T₂, so there exist two sb*open sets \( U \) and \( V \) such that \( f(x) \in U, f(y) \in V \) and \( U \cap V = \emptyset \), then \( f^{-1}(f(x)) \in f^{-1}(U), f^{-1}(f(y)) \in f^{-1}(V) \), this implies that \( x \in f^{-1}(U), y \in f^{-1}(V) \), since f is sb*-continuous, so \( f^{-1}(U), f^{-1}(V) \) are sb* open sets of X and \( f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset \). Thus X is sb*-T₂-space, then \( X \in \mathcal{P} \).

4-conclusion:
In this paper we have studied and proved these cases:
1) If \( \mathcal{P} \) is a class of (sb*-T₀, sb*-T₁) spaces with certain properties and if X is a sb*-irresolute pointwise cleavable over \( \mathcal{P} \), then \( X \in \mathcal{P} \), also if \( \mathcal{P} \) is a class of (sb*-T₀, sb*-T₁) spaces with certain properties and if X is a sb*-irresolute pointwise cleavable over \( \mathcal{P} \), then \( Y \in \mathcal{P} \).
2) If \( \mathcal{P} \) is a class of (sb*-T₁, sb*-T₂) spaces with certain properties and if X is pointwise sb*-cleavable over \( \mathcal{P} \), then \( X \in \mathcal{P} \), also if \( \mathcal{P} \) is a class of sb*-T₁ spaces with certain properties and if X is a sb*-irresolute pointwise cleavable over \( \mathcal{P} \), then \( X \in \mathcal{P} \).
3) If \( \mathcal{P} \) is a class of (sb*-T₁, sb*-T₂) spaces with certain properties and if X is pointwise sb*-cleavable over \( \mathcal{P} \), then Y is (sb*-T₂) respectively, then \( Y \in \mathcal{P} \).

References