On Differential Subordination Theorems of Analytic Multivalent Functions Defined by Generalized Integral Operator

Authors Names

a. Waggas Galib Atshan
b. Ali Hussein Battor,
Abeer Farhan Abaas

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ABSTRACT

In this paper, we obtain some applications of second-order differential Subordination results involving a generalized integral operator for certain normalized analytic functions.

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1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0, p \in N = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic and p -valent in the open unit disk $U = \{z: z \in \mathbb{C}, |z| < 1\}$. If f and g are analytic functions in U , we say that f is subordinate to g in U , written $f < g$ or $f(z) < g(z)$, if there exists a Schwarz function $w(z)$ analytic in U , with $w(0) = 0$ and $|w(z)| < 1$ such that

^a Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq, E-Mail: waggas.galib@qu.edu.iq

^b Department of Mathematics, College of Education for Girls, University of Al-Kufa, Najaf, Iraq, E-Mail: alih.battoor@uokufa.edu.iq,
abeerfarhan688@gmail.com

$$f(z) = g(w(z)), (z \in U).$$

In particular, if the function g is univalent in U , then $f < g$ if

$$f(0) = g(0), \text{ and } f(U) \subset g(U) \quad ([9,18]).$$

For the function f given by (1.1) and $g \in A(p)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k.$$

The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

The set of all functions f that are analytic and injective on $\bar{U} \setminus E(f)$, Denote by Q where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty \right\},$$

and are such that $\dot{f}(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$ (see [19]).

Let $\psi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$, and h is univalent in U with $q \in Q$. Miller and Mocanu [12] consider the problem of determining conditions on admissible functions ψ such that

$$\psi(p(z), zp'(z), z^2 \dot{p}(z); z) < h(z) \quad (1.2)$$

implies $p(z) < q(z)$, for all functions $p(z) \in H[a, n] = \{f \in H: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$, where H be the linear space of all analytic functions in U , $a \in \mathbb{C}$ and $n \in \mathbb{Z}^+$ that satisfy the differential subordination (1.2), moreover, they found conditions so that q is the smallest function with this property, called the best dominant of the subordination (1.2).

Let $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$, and $h \in H$ with $q \in H[a, n]$. Recently Miller and Mocanu [13,14] studied the dual problem and determined conditions on ϕ such that

$$h(z) < \phi(p(z), zp'(z), z^2 \dot{p}(z); z) \quad (1.3)$$

implies $q(z) < p(z)$, for all functions $p \in Q$ that satisfy the above superordination. They also found conditions so that the function q is the largest function with this property, called the best subordinant of the superordination (1.3). See [1,2,3,4,5,6], the authors studied differential subordination results for multivalent functions for other classes.

We define the integral operator $\mathcal{A}_p^q(\Psi, \Phi, T)f(z)$, $f(z) \in A(p)$ as follows :

$$\mathcal{A}_p^0(\Psi, \Phi, T)f(z) = f(z)$$

$$\mathcal{A}_p^1(\Psi, \Phi, T)f(z) = \mathcal{A}_p(\Psi, \Phi, T)f(z) = \left(\frac{p+\Phi}{\Psi+T}\right) z^{p-\left(\frac{p+\Phi}{\Psi+T}\right)} \int_0^z t^{\left(\frac{p+\Phi}{\Psi+T}\right)-(p+1)} f(t) dt$$

$$\mathcal{A}_p^2(\Psi, \Phi, T)f(z) = \left(\frac{p+\Phi}{\Psi+T}\right) z^{p-(\frac{p+\Phi}{\Psi+T})} \int_0^z t^{\left(\frac{p+\Phi}{\Psi+T}\right)-(p+1)} \mathcal{A}_p^1(\Psi, \Phi, T)f(t) dt$$

and, in general

$$\mathcal{A}_p^q(\Psi, \Phi, T)f(z) = \left(\frac{p+\Phi}{\Psi+T}\right) z^{p-(\frac{p+\Phi}{\Psi+T})} \int_0^z t^{\left(\frac{p+\Phi}{\Psi+T}\right)-(p+1)} \mathcal{A}_p^{q-1}(\Psi, \Phi, T)f(t) dt$$

$$(f(z) \in A(p); q \in N_0; z \in U). \quad (1.4)$$

We see that for $f(z) \in A(p)$, we have that

$$\mathcal{A}_p^q(\Psi, \Phi, T)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\Phi}{p+\Phi+(\Psi+T)(k-p)} \right)^q a_k z^k,$$

$$0 < \Psi < 1, \Phi, T \geq 0; p \in N, q \in N_0. \quad (1.5)$$

From (1.5), it easy to verify that

$$(\Psi+T)z \left(\mathcal{A}_p^{q+2}f(z) \right)' = (\Phi+p) \left(\mathcal{A}_p^{q+1}f(z) \right) - \left(\Phi+p(1-(\Psi+T)) \right) \left(\mathcal{A}_p^{q+2}f(z) \right). \quad (1.6)$$

We note that :

$$1- \mathcal{A}^q(\Psi, 0, 0)f(z) = I_{\lambda}^{-q}f(z) \quad (\text{see [17]})$$

$$2- \mathcal{A}_1^{\alpha}(1, 1, 0)f(z) = I^{\alpha}f(z) \quad (\text{see [11]}).$$

$$3- \mathcal{A}_p^q(1, 1, 0)f(z) = I_p^qf(z) \quad (\text{see [19]}).$$

$$4- \mathcal{A}_1^q(1, 1, 0)f(z) = D^qf(z) \quad (\text{see [16]}).$$

$$5- \mathcal{A}_1^q(1, 1, 0)f(z) = I^qf(z) \quad (\text{see [10]}).$$

$$6- \mathcal{A}_1^q(1, 0, 0)f(z) = I^qf(z) \quad (\text{see [18]}).$$

Also we note that :

$$1- \mathcal{A}_p^q(1, 0, 0)f(z) = J_p^qf(z)$$

$$= \left\{ f(z): J_p^qf(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p}{k} \right)^q a_k z^k, q \in N_0, z \in U \right\}.$$

$$2- \mathcal{A}_p^q(1, l, 0)f(z) = J_p^q(l)f(z) =$$

$$\left\{ f(z): J_p^q(l)f(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p+l}{k+l} \right)^q a_k z^k, q \in N_0, l > 0, z \in U \right\}$$

$$3- \mathcal{A}_p^q(\lambda, 0, 0)f(z) = J_{p,\lambda}^q f(z)$$

$$= \left\{ f(z): J_{p,\lambda}^q f(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p}{k + \lambda(k-p)} \right)^q a_k z^k, q \in N_0, \lambda \geq 0, z \in U \right\}.$$

$$4- \mathcal{A}_p^q(\lambda, \alpha\delta, 0)f(z) = I_p^q(\lambda, \alpha, \delta)f(z) [1]$$

$$\left\{ f(z): J_{p,\lambda}^q f(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p + \alpha\delta}{p + \alpha\delta + \lambda(k-p)} \right)^q a_k z^k, q \in N_0, \lambda \geq 0, \alpha, \delta > 0, z \in U \right\}.$$

In this paper, we shall determine some properties on the admissible functions defined with operator $\mathcal{A}_p^q(\Psi, \Phi, T)$.

2- Preliminaries:

In order to prove our results, we shall make use of the following known results.

Lemma (2.1)[9] : Let q be univalent in U , $\zeta \in \mathbb{C}^*$ and suppose that

$$Re \left\{ 1 + \frac{z\dot{q}(z)}{\dot{q}(z)} \right\} > \max \left\{ 0, -Re \left(\frac{1}{\zeta} \right) \right\}. \quad (2.1)$$

If $p(z)$ is analytic in U , with $p(0) = q(0)$ and

$$p(z) + \zeta z\dot{p}(z) < q(z) + \zeta z\dot{q}(z), \quad (2.2)$$

then $p(z) < q(z)$, and $q(z)$ is the best dominant.

Lemma (2.2)[17] : Let the function $q(z)$ be univalent in the unit disk, and let θ, φ be analytic in domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set

$Q(z) = z\dot{q}(z)\varphi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

1- Q is starlike univalent in U .

2- $Re \left\{ \frac{zh(z)}{Q(z)} \right\} > 0$ for $z \in U$.

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + z\dot{p}(z)\varphi(p(z)) < \theta(q(z)) + z\dot{q}(z)\varphi(q(z)), \quad (2.3)$$

then $p < q$, and $q(z)$ is the best dominant.

Lemma (2.3)[7] : Let $q(z)$ be convex in U , $q(0) = a$ and $\zeta \in \mathbb{C}$, $Re(\zeta) > 0$.

If $p \in H[a, 1]$ and $p(z) + \gamma z\dot{q}(z)$ is univalent in U , then

$$q(z) + \zeta z\dot{q}(z) < p(z) + \zeta z\dot{p}(z), \quad (2.4)$$

implies $q(z) < p(z)$, and $q(z)$ is the best subordinant.

Lemma (2.4)[8] : Let $q(z)$ be convex univalent in the unit disk U and let θ, φ be analytic in a domain D containing $q(U)$. Suppose that

$$1 - \operatorname{Re} \left\{ \frac{\dot{\theta}(q(z))}{\varphi(q(z))} \right\} > 0, \quad \text{for } z \in U.$$

$$2 - z\dot{q}(z)\varphi(q(z)) \text{ is starlike univalent in } U.$$

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U , and

$$\theta(q(z)) + z\dot{q}(z)\varphi(q(z)) < \theta(p(z)) + zp'(z)\varphi(p(z)), \quad (2.5)$$

then $q(z) < p(z)$, and $q(z)$ is the best subordinant.

3- Main results:

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\Psi > 0, \Phi, T \geq 0; p \in N, q \in N_0 = N \cup \{0\}; z \in U$ and the powers are understood as principle values.

Theorem (3.1): Let $q(z)$ be univalent in U with $q(0) = 0, \gamma > 0$ and suppose that

$$\operatorname{Re} \left\{ 1 + \frac{z\dot{q}(z)}{\dot{q}(z)} \right\} > \max \left\{ 0, -\operatorname{Re} \left(\frac{\sigma(\Phi + p)}{T + \Psi} \right) \right\}. \quad (3.1)$$

If $f \in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma + \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma \left(\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1 \right) < q(z) + \frac{T + \Psi}{\sigma(\Phi + p)} z\dot{q}(z), \quad (3.2)$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma < q(z)$$

and $q(z)$ is the best dominant.

Proof : If we consider the analytic function

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma, \quad \sigma > 0, \quad z \in U. \quad (3.3)$$

Differentiating (3.3) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\frac{z\dot{p}(z)}{p(z)} = \frac{\sigma(\Phi + p)}{\Psi + T} \left(\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1 \right), \quad (3.4)$$

that is

$$\frac{\Psi + T}{\sigma(\Phi + p)} z\dot{p}(z) = \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma \left(\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1 \right).$$

Thus, the subordination (3.2) is equivalent to

$$p(z) + \frac{\Psi + T}{\sigma(\Phi + p)} z\dot{p}(z) < q(z) + \frac{\Psi + T}{\sigma(\Phi + p)} z\dot{q}(z). \quad (3.5)$$

Applying Lemma (2.1), with $\zeta = \frac{\Psi + T}{\sigma(\Phi + p)}$, the proof of Theorem (3.1) is complete.

Taking the convex function $q(z) = \frac{1+Az}{1+Bz}$, in the Theorem (3.1), we have the following corollary.

Corollary (3.1): Let $A, B \in \mathbb{C}, A \neq B, |B| < 1, \operatorname{Re}(\zeta) > 0$ and $\sigma > 0$. If $f(z) \in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma + \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma \left(\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1 \right) < \frac{1 + Az}{1 + Bz} + \frac{\Psi + T}{\sigma(\Phi + p)} \frac{(A - B)z}{(1 + Bz)^2},$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma < \frac{1 + Az}{1 + Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $q = 0$ in Theorem (3.1), we obtain the following result:

Corollary (3.2): Let $q(z)$ be univalent in U , with $q(0) = 1, \sigma > 0$, and suppose that (3.1) holds. If $f(z) \in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^2 f(z)}{z^p} \right)^\sigma + \left(\frac{\mathcal{A}_p^2 f(z)}{z^p} \right)^\sigma \left(\frac{\mathcal{A}_p^1 f(z)}{\mathcal{A}_p^2 f(z)} - 1 \right) < q(z) + \frac{\Psi + T}{\sigma(\Psi + p)} z\dot{q}(z),$$

then

$$\left(\frac{\mathcal{A}_p^2 f(z)}{z^p} \right)^\sigma < q(z).$$

and $q(z)$ is the best dominant.

Taking $\Phi = \Psi = 1$ in the Theorem (3.1), we have the following result.

Corollary (3.3): Let $q(z)$ be univalent in U , with $q(0) = 1, \beta \in \mathbb{C}^*, \sigma > 0$, and suppose that (3.1) holds. If $f(z) \in A(p)$ satisfies the subordination

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma + \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma \left(\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} - 1 \right) < q(z) + \frac{1+T}{\sigma(1+p)} z \dot{q}(z),$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma < q(z).$$

and $q(z)$ is the best dominant.

Theorem (3.2): Let $q(z)$ be univalent in U , with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$, let $\lambda, \sigma \in \mathbb{C}^*$, $f \in A(p)$ and suppose that f and q satisfy the next conditions:

$$\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \neq 0, \quad (3.6)$$

and

$$\operatorname{Re} \left\{ 1 + \frac{z \dot{q}(z)}{\dot{q}(z)} - \frac{z \dot{q}(z)}{q(z)} \right\} > 0, \quad (z \in U). \quad (3.7)$$

If

$$\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} < 1 + \frac{(\Psi + T)z \dot{q}(z)}{\sigma(\Phi + p)q(z)}, \quad (3.8)$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma < q(z)$$

and $q(z)$ is the best dominant of (3.6).

proof: Let

$$p(z) = \left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma, \quad z \in U. \quad (3.9)$$

According to (3.4) the function $p(z)$ is analytic in U , and differentiating (3.9) logarithmically with respect to z , we obtain

$$\frac{z\dot{p}(z)}{p(z)} = \frac{\sigma(\Phi + p)}{\Psi + T} \left(\frac{\mathcal{A}_p^{q+1}f(z)}{\mathcal{A}_p^{q+2}f(z)} - 1 \right). \quad (3.10)$$

In order to prove our result we will use Lemma (2.2). In this lemma consider

$$\theta(w) = 1 \text{ and } \varphi(w) = \frac{\Psi + T}{\sigma(\Phi + p)w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = z\dot{q}(z)\varphi(q(z)) = \frac{(\Psi + T)z\dot{q}(z)}{\sigma(\Phi + p)q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \frac{(\Psi + T)z\dot{q}(z)}{\sigma(\Phi + p)q(z)},$$

from (3.7), we see that $Q(z)$ is a starlike function in U . We also have

$$Re \left\{ \frac{zh(z)}{Q(z)} \right\} = Re \left\{ 1 + \frac{z\dot{q}(z)}{\dot{q}(z)} - \frac{z\dot{q}(z)}{q(z)} \right\} > 0, \quad (z \in U)$$

and then, by using Lemma (2.2), we deduce that the subordination (3.6) implies

$$p(z) \prec q(z)$$

and the function $q(z)$ is the best dominant of (3.8).

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

Corollary (3.4): Let $\sigma \in \mathbb{C}^*$. Let $f(z) \in A(p)$ and suppose that

$$\frac{\mathcal{A}_p^{q+2}f(z)}{z^p} \neq 0, \quad (z \in U).$$

If

$$\frac{\mathcal{A}_p^{q+1}f(z)}{\mathcal{A}_p^{q+2}f(z)} \prec 1 + \frac{(\Psi + T)z(A - B)}{\sigma(\Phi + p)(1 + Az)(1 + Bz)},$$

then

$$\left(\frac{\mathcal{A}_p^{q+2}f(z)}{z^p} \right)^\sigma \prec \frac{1 + Az}{1 + Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem (3.2), it is easy to check that the assumption (3.5) holds, hence we obtain the next result.

Corollary (3.5): Let $\sigma \in \mathbb{C}^*$, $f(z) \in A(p)$ and suppose that

$$\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \neq 0, \quad (z \in U).$$

If

$$\frac{\mathcal{A}_p^{q+1} f(z)}{\mathcal{A}_p^{q+2} f(z)} < 1 + \frac{2(\Psi + T)z}{\sigma(\Phi + p)(1-z)(1+z)},$$

then

$$\left(\frac{\mathcal{A}_p^{q+2} f(z)}{z^p} \right)^\sigma < \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant.

Theorem (3.3): Let $q(z)$ be univalent in U , with $q(0) = 1$, let $\sigma \in \mathbb{C}^*$, and let $\psi, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$. Let $f \in A(p)$ and suppose that f and q satisfy the next conditions:

$$\frac{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)}{(\nu + \eta)z^p} \neq 0, \quad (z \in U) \quad (3.11)$$

and

$$\operatorname{Re} \left\{ 1 + \frac{z\dot{q}(z)}{\dot{q}(z)} \right\} > \max\{0, -\operatorname{Re}(\psi)\}, \quad (z \in U). \quad (3.12)$$

If

$$K(z) = \Upsilon \left[\frac{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)}{(\nu + \eta)z^p} \right]^\sigma + \sigma \left[\left(\frac{\nu z(\mathcal{A}_p^{q+1} f(z))' + \eta z(\mathcal{A}_p^{q+2} f(z))'}{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)} - p \right) \right] \quad (3.13)$$

and

$$K(z) < \Upsilon q(z) + \frac{z\dot{q}(z)}{q(z)}, \quad (3.14)$$

then

$$\left[\frac{\nu \mathcal{A}_p^{q+1} f(z) + \eta \mathcal{A}_p^{q+2} f(z)}{(\nu + \eta)z^p} \right]^\sigma < q(z)$$

and $q(z)$ is the best dominant of (3.11).

Proof : Let

$$p(z) = \left[\frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu + \eta) z^p} \right]^\sigma, \quad z \in U. \quad (3.15)$$

According to (3.8) the function $p(z)$ is analytic in U , and differentiating (3.15) logarithmically with respect to z , we obtain

$$\frac{zp'(z)}{p(z)} = \sigma \left[\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right], \quad (3.16)$$

and hence

$$zp'(z) = \sigma \left[\frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu + \eta) z^p} \right]^\sigma \left[\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right].$$

In order to prove our result, we will use Lemma (2.2). In this lemma consider

$$\theta(w) = \gamma w \quad \text{and} \quad \varphi(w) = \frac{1}{w},$$

then θ is analytic in C and $\varphi(w) \neq 0$ is analytic in C^* . Also if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma \left[\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right]$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \gamma \left[\frac{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)}{(\nu + \eta) z^p} \right]^\sigma + \sigma \left[\left(\frac{\nu z(A_p^{q+1} f(z))' + \eta z(A_p^{q+2} f(z))'}{\nu A_p^{q+1} f(z) + \eta A_p^{q+2} f(z)} - p \right) \right] \end{aligned}$$

from (3.11), we see that $Q(z)$ is a starlike function in U . We also have

$$\operatorname{Re} \left\{ \frac{zh(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \gamma + 1 + \frac{zq'(z)}{q(z)} \right\} > 0, \quad (z \in U)$$

and then, by using Lemma (2.2), we deduce that the subordination (3.14) implies

$$p(z) < q(z).$$

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Theorem (3.3) and according to (3.4), the condition (3.12) becomes

$$\max\{0, -\operatorname{Re}(Y)\} \leq \frac{1 - |B|}{1 + |B|}.$$

Hence, for the special case $\nu = 1$ and $\eta = 0$, we obtain the following result.

Corollary (3.6) : Let $Y \in \mathcal{C}$ with

$$\max\{0, -\operatorname{Re}(Y)\} \leq \frac{1 - |B|}{1 + |B|}.$$

Let $f(z) \in A(p)$ and suppose that

$$\frac{A_p^{q+1} f(z)}{z^p} \neq 0, \quad (z \in U).$$

If

$$Y \left[\frac{A_p^{q+1} f(z)}{z^p} \right]^\sigma + \sigma \left[\left(\frac{z(A_p^{q+1} f(z))}{A_p^{q+1} f(z)} - p \right) \right] < Y \frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Az)(1 + Bz)},$$

then

$$\left(\frac{A_p^{q+1} f(z)}{z^p} \right)^\gamma < \frac{1 + Az}{1 + Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $p = \nu = q = 1, \eta = 0$ and $q(z) = \frac{1+z}{1-z}$ in Theorem (3.3), we obtain the next result.

Corollary (3.7) : Let $f(z) \in A(p)$ and suppose that

$$\frac{A_p^2 f(z)}{z^p} \neq 0, \quad (z \in U).$$

and $\sigma \in \mathcal{C}^*$. If

$$Y \left[\frac{A^2 f(z)}{z} \right]^\sigma + \sigma \left[\left(\frac{z(A^2 f(z))}{A^2 f(z)} - 1 \right) \right] < Y \frac{1 + z}{1 - z} + \frac{2z}{(1 + z)(1 - z)},$$

then

$$\left(\frac{A^2 f(z)}{z} \right)^\gamma < \frac{1 + z}{1 - z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant.

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