AL-Qadisiyah Journal Of Pure Science (QJPS)

Vol .25, No 1 ,math pp. 50 –54, Year2020

Received 22 /1/2020

Accepted 3/3/2020

On Differential Subordination and Superordination Results of Multivalent Functions Defined by a Linear Operator

Waggas Galib Atshan ^a , Haneen Zaghir Hassan ^b

Department of Mathematics,

College of Science,

University of Al-Qadisiyah , Diwaniyah-Iraq,

E-mail: waggashnd@gmail.com , waggas.galib@qu.edu.iq ^a , haneenzaghir@gmail.com ^b

Abstract :

The main idea of this paper is to derive some subordination and superordination results defined by a linear operator for multivalent functions in the open unit disk . Several sandwich-type results are also obtained.

Keywords : Analytic function, multivalent function, subordination, superordination , linear operator .

2019 Mathematics Subject Classification: 30C45, 30C50.

1- Introduction :

Let $A(p)$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, 3, \dots\}, \quad z \in U \quad (1.1)$$

which are analytic multivalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions f and g are analytic in U , we say that the functions f is subordinate to g in U , written $f < g$, if there exists Schwarz function w , analytic in U with $w(0) = 0$ and

$|w(z)| < 1$ in U such that $f(z) = g(w(z))$, $z \in U$. If g is univalent and $g(0) = f(0)$, then $f(U) \subset g(U)$.

If $f \in A(p)$ is given by (1.1), then the linear operator

$I_p(n, \lambda) : A(p) \rightarrow A(p)$ ([2]) is defined by

$$I_p(n, \lambda)f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right) a_n z^n, \quad \lambda > -p, \quad p \in \mathbb{N} = \{1, 2, 3, \dots\}. \quad (1.2)$$

It is easily verified from (1.2) that

$$z[I_p(n, \lambda)f(z)]' = (p+\lambda) I_p(n+1, \lambda)f(z) - \lambda I_p(n, \lambda)f(z). \quad (1.3)$$

The main object of this idea is to find sufficient conditions for certain normalized analytic functions f to satisfy :

$$q_1(z) < \left(\frac{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} < q_2(z)$$

$$q_1(z) < \left(\frac{\operatorname{Ip}(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} < q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

Several authors studied differential subordination and superordination for different conditions (see [3,4,5,6,7,8,9]).

2- Preliminaries : In order to prove our subordinations and superordinations results, we need the following definitions and lemmas.

Definition (2.1)[11] : Let Q the set of all functions q that are analytic and injective on $\bar{U} / E(q)$ where $\bar{U} = U \cup \{z \in \partial U\}$, and $E(q) = \{\varepsilon \in \partial U : \lim_{z \rightarrow \varepsilon} q(z) = \infty\}$ (2.1)

such that $q'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U / E(q)$. Further, let the subclass

of Q for which $q(0) = a$ be denoted by $Q(a)$, and $Q(0) = Q_0$, $Q(1) = Q_1$.

Lemma (2.1) [1]: Let $q(z)$ be convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} / \{0\}$ and suppose that

$$\operatorname{Re} \left\{ 1 + \frac{z q''(z)}{q'(z)} \right\} > \max \{0, -\operatorname{Re} \left(\frac{\alpha}{\beta} \right)\}.$$

If $p(z)$ is analytic in U , and

$$\alpha p(z) + \beta z p'(z) < \alpha q(z) + \beta z q'(z),$$

then $p(z) < q(z)$ and q is the best dominant.

Lemma (2.2)[10]: Let q be univalent function in U and let ϕ and θ be analytic in the domain D containing $q(U)$ with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = z q'(z) \phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that,

(i) Q is starlike univalent in U .

(ii) $\operatorname{Re} \left\{ \frac{z h'(z)}{Q(z)} \right\} > 0$ for $z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + z p'(z) \phi(p(z)) < \theta(q(z)) + z p'(z) \phi(q(z)),$$

then $p < q$ and q is the best dominant.

Lemma (2.3) [12]: Let $q(z)$ be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(i) $\operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} > 0$ for $z \in U$,

(ii) $z q'(z) \phi(q(z))$ is starlike univalent in $z \in U$.

If $p \in H[q(0), 1] \cap Q$ with $p(U) \subseteq D$, and $\theta(p(z)) + z p'(z) \phi(p(z))$ is univalent in U , and $\theta(q(z)) + z q'(z) \phi(q(z)) < \theta(p(z)) + z p'(z) \phi(p(z))$, (2.2)

then $q < p$ and q is the best subdominant.

Lemma (2.4) [12]: Let $q(z)$ be convex univalent in U and $q(0) = 1$. Let $\beta \in \mathbb{C}$, that $\operatorname{Re}(\beta) > 0$. If $p(z) \in H[q(0), 1] \cap Q$ and $P(z) + \beta z p'(z)$ is univalent in U , then $q(z) + \beta z q'(z) < p(z) + \beta z p'(z)$, which implies that $q(z) < p(z)$ and $q(z)$ is the best subdominant.

3- Subordination Results :

Theorem (3.1) : Let $q(z)$ be convex univalent function in U with $q(0) = 1, \eta, \delta \in \mathbb{C} \setminus \{0\}$. Suppose that

$$\operatorname{Re} \left\{ 1 + \frac{z q''(z)}{q'(z)} \right\} > \max \{0, -\operatorname{Re} \left(\frac{1}{\delta \eta} \right)\}. \quad (3.1)$$

If $f \in A(p)$ is satisfies the subordination $G(z) < q(z) + \delta \eta z q'(z)$,

$$(3.2)$$

where

$$G(z) = \left(\frac{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} \times \left(1 + \eta \left(\frac{t_1 (p + \lambda) \operatorname{Ip}(n + 1, \lambda) f(z) - \lambda \operatorname{Ip}(n, \lambda) f(z) + t_2 (p + \lambda) \operatorname{Ip}(n + 1, \lambda) f(z) - \lambda \operatorname{Ip}(n, \lambda) f(z)}{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)} \right) \right), \quad (3.3)$$

then

$$\left(\frac{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} < q(z), \quad (3.4)$$

and $q(z)$ is the best dominant.

Proof : Define a function $k(z)$ by

$$k(z) = \left(\frac{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}}, \quad (3.5)$$

then the function $k(z)$ is analytic in U and $q(0) = 1$, therefore, differentiating (3.5) logarithmically with respect to z and using the identity (1.3) in the resulting equation,

$$G(z) = \left(\frac{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} \times \left(1 + \eta \left(\frac{t_1 (p + \lambda) \operatorname{Ip}(n + 1, \lambda) f(z) - \lambda \operatorname{Ip}(n, \lambda) f(z) + t_2 (p + \lambda) \operatorname{Ip}(n + 1, \lambda) f(z) - \lambda \operatorname{Ip}(n, \lambda) f(z)}{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)} \right) \right).$$

Thus The subordination (3.2) is equivalent to

$$k(z) + \delta \eta z k'(z) < q(z) + \delta \eta z q'(z).$$

An application of Lemma (2.1) with $\beta = \delta\eta$ and $\alpha = 1$, we obtain (3.4).

Taking $q(z) = \left(\frac{1+Bz}{1+Bz}\right)$ ($-1 \leq B < A \leq 1$) in Theorem (3.1), we obtain the following corollary.

Corollary (3.1): Let $\eta, \delta \in \mathbb{C} \setminus \{0\}$ and $(-1 \leq B < A \leq 1)$. Suppose that

$$\operatorname{Re} \left(\frac{1+Bz}{1+Bz} \right) > \max \left\{ 0, -\operatorname{Re} \left(\frac{1}{\delta\eta} \right) \right\}.$$

If $f \in A(p)$ is satisfy the following subordination condition :

$$G(z) < \frac{1+Bz}{1+Bz} + \delta\eta \frac{(A-B)z}{(1+Bz)^2},$$

where $G(z)$ is given by (3.3), then

$$\left(\frac{t_1 I_p(n, \lambda) f(z) + t_2 I_p(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} < \frac{1+Bz}{1+Bz}$$

and $\frac{1+Bz}{1+Bz}$ is the best dominant.

Taking $A = 1$ and $B = -1$ in Corollary(3.1), we get following result.

Corollary (3.2) : Let $\eta, \delta \in \mathbb{C} \setminus \{0\}$ and suppose that

$$\operatorname{Re} \left\{ \frac{1+z}{1-z} \right\} > \max \left\{ 0, -\operatorname{Re} \left(\frac{1}{\delta\eta} \right) \right\}.$$

If $f \in A(p)$ satisfy, the following subordination ,

$$G(z) < \frac{1+z}{1-z} + \delta\eta \frac{2z}{\delta(1-z)^2},$$

where $G(z)$ is given by (3.3) then

$$\left(\frac{t_1 I_p(n, \lambda) f(z) + t_2 I_p(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} < \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant .

Theorem (3.2): Let $q(z)$ be convex univalent in unit disk U with $q(0)=1, \delta > 0$, let $\eta, \delta, s \in \mathbb{C} \setminus \{0\}$, $\gamma, t, \psi, \tau \in \mathbb{C}, f \in A(p)$, and suppose that f and q satisfy the following conditions :

$$\operatorname{Re} \left\{ \frac{\psi}{s} q(z) + \frac{2\tau\gamma}{s} q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)} \right\} > 0, \quad (3.6)$$

and

$$\frac{I_p(n, \lambda) f(z)}{z^p} \neq 0. \quad (3.7)$$

$$\text{If } r(z) < t + \psi q(z) + \tau\gamma q^2(z) + s \frac{zq'(z)}{q(z)}, \quad (3.8)$$

where

$$\begin{aligned} r(z) &= \left(\frac{I_p(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} \left(\psi + t\gamma \left(\frac{I_p(n, \lambda) f(z)}{z^p} \right) + t \right. \\ &\quad \left. + s \frac{1}{\delta} (p+\lambda) \left(\frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)} - 1 \right) \right), \end{aligned} \quad (3.9)$$

then

$$\left(\frac{I_p(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} < q(z), \text{ and } q(z) \text{ is the best dominant.}$$

Proof : Define analytic function $k(z)$ by

$$k(z) = \left(\frac{I_p(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}}, \quad (3.10)$$

then the function $k(z)$ is analytic in U and $g(0) = 1$,

differentiating (3.10) logarithmically with respect to z , we get

$$\frac{zk'(z)}{k(z)} = \frac{1}{\delta} (p + \lambda) \left(\frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)} - 1 \right). \quad (3.11)$$

By setting $\theta(w) = t + \psi w + \tau\gamma w^2$ and $\phi(w) = \frac{s}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$.

Also, if we let .

$$\Phi(z) = zq'(z) \phi(q(z)) = s \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = t + \psi q(z) + \tau\gamma q^2(z) + s \frac{zq'(z)}{q(z)},$$

we find $Q(z)$ is starlike univalent in U , we have

$$h'(z) = \psi q'(z) + 2\tau\gamma q(z)q'(z) + s \frac{q'(z)}{q(z)} + sz \frac{q''(z)}{q(z)} - sz \left(\frac{q'(z)}{q(z)} \right)^2,$$

and

$$\frac{zh'(z)}{Q(z)} = \frac{\psi}{s} q(z) + \frac{2\tau\gamma}{s} q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)},$$

hence that

$$\operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) = \operatorname{Re} \left(\frac{\psi}{s} q(z) + \frac{2\tau\gamma}{s} q^2(z) + 1 + z \frac{q''(z)}{q'(z)} - z \frac{q'(z)}{q(z)} \right) > 0.$$

By using (3.11), we obtain

$$\begin{aligned} \Psi k(z) + \tau\gamma k^2(z) + s \frac{zk'(z)}{k(z)} &= \left(\frac{I_p(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} \left(\psi + \right. \\ &\quad \left. \gamma\tau\gamma \left(\frac{I_p(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} + t + \left(s \frac{1}{\delta} (p + \lambda) \left(\frac{I_p(n+1, \lambda) f(z)}{I_p(n, \lambda) f(z)} - 1 \right) \right) \right). \end{aligned}$$

By using (3.8), we have

$$\Psi k(z) + \tau\gamma k^2(z) + s \frac{zk'(z)}{k(z)} < \psi q(z) + \tau\gamma q^2(z) + s \frac{zq'(z)}{q(z)},$$

and by using Lemma (2.2), we deduce that subordination (3.8) implies that $k(z) < q(z)$ and the function $q(z)$ is the best dominant .

and

Taking the function $q(z) = \frac{1+Bz}{1+Bz}$ ($-1 \leq B < A \leq 1$), in Theorem (3.2), the condition (3.6) becomes.

$$\operatorname{Re} \left(\frac{\psi}{s} \frac{1+Bz}{1+Bz} + \frac{2\tau\gamma}{s} \left(\frac{1+Bz}{1+Bz} \right)^2 + 1 + \frac{(A-B)z}{(1+Bz)(1+Bz)} - \frac{2Bz}{1+Bz} \right) > 0, \quad (3.12)$$

hence, we have the following Corollary.

Corollary (3.3) : Let $(-1 \leq B < A \leq 1)$, $s, \delta \in \mathbb{C} \setminus \{0\}$, $\gamma, t, \tau, \psi \in \mathbb{C}$. Assume that (3.12) holds. If $f \in A(p)$ and

$$r(z) < t + \psi \frac{1+Bz}{1+Bz} + \tau\gamma \left(\frac{1+Bz}{1+Bz} \right)^2 + s \frac{(A-B)z}{(1+Bz)(1+Bz)},$$

where $r(z)$ is defined in (3.9), then

$$\left(\frac{I_p(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} < \frac{1+Bz}{1+Bz}, \text{ and } \frac{1+Bz}{1+Bz} \text{ is the best dominant.}$$

Taking the function $q(z) = \left(\frac{1+z}{1-z} \right)^{\rho}$, ($0 < \rho \leq 1$), In Theorem (3.2), the condition (3.6) becomes

$$\operatorname{Re} \left\{ \frac{\psi}{s} \left(\frac{1+z}{1-z} \right)^\rho + \frac{2\tau\gamma}{s} \left(\frac{1+z}{1-z} \right)^{2\rho} + \frac{2z^2}{1-z^2} \right\} > 0, (s \in \mathbb{C} \setminus \{0\}), \quad (3.13)$$

hence, we have the following Corollary.

Corollary (3.4): Let $0 < \rho \leq 1$, $s, \delta \in \mathbb{C} \setminus \{0\}$, $\gamma, t, \tau, \psi \in \mathbb{C}$. Assume that (3.13) holds. If $f \in A(p)$ and

$$r(z) < t + \psi \left(\frac{1+z}{1-z} \right)^\rho + \tau\gamma \left(\frac{1+z}{1-z} \right)^{2\rho} + s \frac{2pz}{1-z^2},$$

where $r(z)$ is defined in (3.9), then

$$\left(\frac{I_p(n, \lambda)f(z)}{z^p} \right)^{\frac{1}{\delta}} < \left(\frac{1+z}{1-z} \right)^\rho, \text{ and } \left(\frac{1+z}{1-z} \right)^\rho \text{ is the best dominant.}$$

4- Superordination Results :

Theorem (4.1): Let $q(z)$ be convex univalent in U with $q(0) = 1$, $\delta \in \mathbb{C} \setminus \{0\}$, $\operatorname{Re} \{\eta\} > 0$, if $f \in A(p)$, such that

$$\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \neq 0,$$

and

$$\left(\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} \in H[q(0), 1] \cap Q. \quad (4.1)$$

If the function $G(z)$ defined by (3.3) is univalent and the following superordination condition:

$$q(z) + \delta \eta z q'(z) < G(z), \quad (4.2)$$

holds, then

$$q(z) < \left(\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} \quad (4.3)$$

and $q(z)$ is the best subdominant.

Proof : Define a function $k(z)$ by

$$k(z) = \left(\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}}. \quad (4.4)$$

Differentiating (4.4) with respect to z logarithmically, we get.

$$\frac{zk(z)}{k(z)} = \frac{1}{\delta} \left(\frac{t_1(z(I_p(n, \lambda)f(z)))' + t_2(z(I_p(n, \lambda)f(z)))'}{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)} - \frac{pt_1 I_p(n, \lambda)f(z) + pt_2 I_p(n, \lambda)f(z)}{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)} \right). \quad (4.5)$$

A simple computation and using (1.3) from (4.5), we get

$$G(z) = \left(\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} \times \left(1 + \eta \left(\frac{t_1(p+\lambda) I_p(n+1, \lambda)f(z) - \lambda I_p(n, \lambda)f(z) + t_2(p+\lambda) I_p(n+1, \lambda)f(z) - \lambda I_p(n, \lambda)f(z)}{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)} \right) \right)$$

$$= k(z) + \delta \eta z k'(z).$$

Now, by using Lemma (2.4), we get the desired result.

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$), in Theorem (4.1), we get the following Corollary.

Corollary (4.2): Let $\operatorname{Re}\{\eta\} > 0$, $\delta \in \mathbb{C} \setminus \{0\}$ and $-1 \leq B < A \leq 1$, such that

$$\left(\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} \in H[q(0), 1] \cap Q.$$

If the function $G(z)$ given by (3.3) is univalent in U and

$f \in A(p)$ satisfies the following superordination condition :

$$\frac{1+Az}{1+Bz} + \delta \eta \frac{(A-B)Z}{(1+BZ)^2} < G(z),$$

then

$$\frac{1+Az}{1+Bz} < \left(\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}},$$

and the function $\frac{1+Az}{1+Bz}$ is the best subdominant.

Theorem (4.2): Let $q(z)$ be convex univalent in unit disk U , let $\delta, s \in \mathbb{C} \setminus \{0\}$, $\gamma, t, \psi, \tau \in \mathbb{C}$, $q(z) \neq 0$ and $f \in A(p)$. Suppose that

$$\operatorname{Re} \left\{ \frac{q(z)}{s} (2\tau\gamma q(z) + \psi) \right\} q'(z) > 0,$$

and satisfies the next conditions

$$\left(\frac{I_p(n, \lambda)f(z)}{z^p} \right)^{\frac{1}{\delta}} \in H[q(0), 1] \cap Q, \quad (4.6)$$

and

$$\frac{I_p(n, \lambda)f(z)}{z^p} \neq 0.$$

If the function $r(z)$ is given by (3.9) is univalent in U ,

$$t + \psi q(z) + \tau\gamma q^2(z) + s \frac{zq'(z)}{q(z)} < r(z) \quad (4.7)$$

implies

$$q(z) < \left(\frac{I_p(n, \lambda)f(z)}{z^p} \right)^{\frac{1}{\delta}}, \text{ and } q(z) \text{ is the best subdominant.}$$

Proof : Let the function $k(z)$ defined on U by (4.1).

then a computation show that

$$\frac{zk'(z)}{k(z)} = \frac{1}{\delta} (p + \lambda) \left(\frac{I_p(n+1, \lambda)f(z)}{I_p(n, \lambda)f(z)} - 1 \right). \quad (4.8)$$

By setting $\theta(w) = t + \psi w + \tau\gamma w^2$ and $\phi(w) = \frac{s}{w}$, it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$ $w \in \mathbb{C} \setminus \{0\}$.

Also, we get $Q(z) = zq'(z) \phi(q(z)) = s \frac{zq'(z)}{q(z)}$, it observed that $Q(z)$ is starlike univalent in U , since $q(z)$ is convex, it follows that

$$\operatorname{Re} \left(\frac{z\theta'(q(z))}{\phi(q(z))} \right) = \operatorname{Re} \left\{ \frac{q(z)}{s} (2\tau\gamma q(z) + \psi) q'(z) \right\} > 0.$$

By making use of (4.8) the hypothesis (4.7) can be equivalently written as

$\theta(q(z) + zq'(z) \phi(q(z))) = \theta(k(z) + zk'(z) \phi(k(z)))$, thus, by applying Lemma (2.3), the proof is complete.

5- Sandwich Results :

Combining Theorem (3.1) with Theorem (4.1), we obtain the following sandwich theorem.

Theorem (5.1): Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$ and q_2 satisfies (3.1). Suppose that $\operatorname{Re}\{\eta\} > 0$, $\eta, \delta \in \mathbb{C} \setminus \{0\}$. If $f \in A(p)$, such that

$$\left(\frac{t_1 I_p(n, \lambda)f(z) + t_2 I_p(n, \lambda)f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} \in H[q(0), 1] \cap Q,$$

and the function $G(z)$ defined by (3.3) is univalent and satisfies

$$q_1(z) + \delta \eta z q_1'(z) < G(z) < q_2(z) + \frac{\eta}{\delta} z q_2'(z), \quad (5.1)$$

then

$$q_1(z) < \left(\frac{t_1 \operatorname{Ip}(n, \lambda) f(z) + t_2 \operatorname{Ip}(n, \lambda) f(z)}{(t_1 + t_2) z^p} \right)^{\frac{1}{\delta}} < q_2(z),$$

where q_1 and q_2 are respectively, the best subdominant and the best dominant of (5.1).

Combining Theorem (3.2) with Theorem (4.2), we obtain the following sandwich theorem.

Theorem (5.2): Let q_1 be two convex univalent function in U , such that $q_i(0) = 1$ and $q_i(0) \neq 0$ ($i=1,2$). Suppose that q_1 and q_2 satisfies (3.8) and (4.8), respectively. If $f \in A(p)$ and suppose that f satisfies the next conditions :

$$\left(\frac{\operatorname{Ip}(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} \in H[Q(0), 1] \cap Q,$$

and

$$\frac{\operatorname{Ip}(n, \lambda) f(z)}{z^p} \neq 0,$$

and $r(z)$ is univalent in U , then

$$t + \psi q_1(z) + \tau \gamma q_1^2(z) + s \frac{z q_1'(z)}{q_1(z)} < t + \psi q_1(z) + \tau \gamma q_1^2(z) + s \frac{z q_1'(z)}{q_1(z)},$$

implies

$$q_1(z) < \left(\frac{\operatorname{Ip}(n, \lambda) f(z)}{z^p} \right)^{\frac{1}{\delta}} < q_2(z),$$

and q_1 and q_2 are the best subdominant and the best dominant respectively and $r(z)$ is given by (3.9).

References

- [1] R. M. Ali, V. Ravichandran and K. G. Subramanian, Differential sandwich theorem for Certain analytic functions, Far East J. Math. Sci., (2004), 15,87-94.
- [2] R. M. Ali, V. Ravichandran and N. Seenivasagan, Differential subordination and superordination of analytic functions defined by the multiplier transformation, Math. Inequal. Appl., 12 (2009), 123-139.
- [3] W. G. Atshan and H. K. Abid Zaid, Subordination results for multivalent functions involving a multiplier Transformation, Analele Universității Oradea Fasc. Matematica, Tom XXII(2015), Issue No. 1,167-172.
- [4] W. G. Atshan and A. A. Husien, Some results of second order differential subordination for fractional integral of Dziok-Srivastava operator, Analele Universității Oradea Fasc. Matematica, Tom XXI (2014), Issue No. 1,145-152.
- [5] W. G. Atshan and I. A. Abbas, Some properties of differential sandwich results of p -valent functions defined by Liu-Srivastava operator, International Journal of Advances in Mathematics, Volume 2017, No. 6,(2017), 101-113.
- [6] W. G. Atshan and N. A. Jiben, Differential subordination and superordination for multivalent functions involving a generalized differential operator, Inter. J. Adv. Res. Sci., Eng. Tech., 4(10)(2017), 4767-4775.
- [7] W. G. Atshan and S. K. Kazim, On differential sandwich theorems of multivalent functions defined by a linear operator,

[8] W. G. Atshan and S. R. Kulkarni, Application of linear operator on meromorphically univalent functions involving differential subordination, Analele Universității Oradea Fasc. Matematica, Tom XVI (2009), 31- 42.

[9] W. G. Atshan, A. K. Wanas and G. Murugusundaramoorthy, Properties and Characteristics of certain subclass of multivalent prestartlike functions with negative coefficients, Analele Universității Oradea Fasc. Matematica, Tom XXVI (2014), Issue No. 2,17-24.

[10] T. Bulbocaco, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book publ., Cluj-Napoca, (2005).

[11] S. S. Miller and P.T. Mocanu, Differential subordinations :Theory and Applications, Series on Monographs and Text Books in pure and Applied Mathematics, 225, Marcel Dekker, New York and Basel, (2000).

[12] S. S. Miller and P. T. Mocanu, Subordinations of differential superordinations, complex variables, 48(10) (2003), 815-826.