New and Extended Results On Fourth-Order Differential Subordination for Univalent Analytic Functions

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1. Introduction

Let $H(U)$ be the class of function which are analytic in the open unit disk

$$U = \{z: z \in \mathbb{C}: |z| < 1\}.$$

For $n \in N = \{1, 2, 3, \dots\}$, and $a \in \mathbb{C}$, let

$$H[a, n] = \{f \in H: f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

and also, let $H_0 = [0, 1]$.

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Let Σ denote the subclass of $H(U)$ consisting of all univalent and analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \quad (1.1)$$

Komatu [8] introduced and investigated a family of integral operator

$$J_{\mu}^{\lambda}: \Sigma \rightarrow \Sigma.$$

That is obtained as follows

$$J_{\mu}^{\lambda} f(z) = z + \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu+n-1} \right)^{\lambda} a_n z^n, \quad (z \in U^*, n > 1, \lambda \geq 0). \quad (1.2)$$

The Hurwitz - Lerch Zeta function

$$\phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(k+a)^s}, \quad a \in \mathbb{C}/z_0^-, s \in \mathbb{C} \text{ when } 0 < |z| < 1.$$

Interims of Hadamard product (or convolution) where $G_{s,a(z)}$ is given by

$$G_{s,a(z)} = (1+a)^s [\phi(z, s, a) - a^{-s}], \quad (z \in U).$$

Then a linear operator $I_{s,a,\mu}^{\lambda} f(z): \Sigma \rightarrow \Sigma$ (see[2]) is defined

$$I_{s,a,\mu}^{\lambda} f(z) = G_{s,a(z)} * J_{\mu}^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+a}{k+a} \right)^s \left(\frac{\mu}{\mu+n-1} \right)^{\lambda} a_n z^n. \quad (1.3)$$

It is easily verified from (1.3) that

$$z \left(I_{s,a,\mu}^{\lambda+1} f(z) \right)' = \mu I_{s,a,\mu}^{\lambda} f(z) - (\mu-1) I_{s,a,\mu}^{\lambda+1} f(z) \quad (1.4)$$

For several past years, there are many authors introduce and dealing with the theory of second-order differential subordination and superordination for example ([3,4,10]). Recently, many authors discussed the theory of third-order differential subordination and superordination for example ([5,6,7,11,12,13,14,15]). In the present paper, we investigated the extended theory of second-order differential subordination in the open unit disk introduced by Miller and Mocanu [9] to third-order case. Now, we extend this to fourth-order differential subordination and determined properties of functions p that satisfy the following fourth-order differential subordination:

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z): z \in U\}.$$

To prove our main results, we need the basic concepts in theory of the fourth-order.

Definition(1.1): [9]. Let $f(z)$ and $F(z)$ be members of the analytic function class $H(U)$. The function $f(z)$ is said to be subordinate to $f(z)$ or $F(z)$ is superordinate to $f(z)$ if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$), and such that $f(z) = F(w(z))$. In such case, we write

$$f < F, \text{ or } f(z) < F(z).$$

If $F(z)$ is univalent in U , then $f(z) < F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Definition (1.2): [1]. Let $\mathbb{Q}$ be the set of all functions q that are analytic and univalent on $\bar{U}/E(q)$, where

$$E(q) = \left\{ \xi: \xi \in \partial U: \lim_{z \rightarrow \xi} \{q(z)\} = \infty \right\},$$

and are such that $\min |q'(\xi)| = p > 0$ for $\xi \in \partial U/E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$ with

$$\mathbb{Q}(0) = \mathbb{Q}_0 \text{ and } \mathbb{Q}(1) = \mathbb{Q}_1.$$

Definition (1.3): Let $\psi: \mathbb{C}^5 \times U \rightarrow \mathbb{C}$ and the function $h(z)$ be univalent in U . If the function $p(z)$ is analytic in U and satisfies the following fourth-order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) < h(z), \quad (1.5)$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) < q(z)$ for all $p(z)$ satisfying (1.5). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) < q(z)$ for all dominants $q(z)$ of (1.5) is said to be the best dominant.

Lemma (1.4): Let $z_0 \in U$ with $r_0 = |z_0|$. For $n \geq 1$. Let

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots$$

be continuous on $\bar{U}_{r_0}$ and analytic in $U_{r_0} \cup \{z_0\}$, with $f(z) \neq 0$. If

$$|f(z_0)| = \max\{|f(z)|: z \in \bar{U}_{r_0}\}, \quad (1.6)$$

then there exists $m, n \geq 1$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = m, \quad (1.7)$$

$$\operatorname{Re} \left\{ \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right\} \geq m, \quad (1.8)$$

and

$$Re \left\{ \frac{z_0 f'(z_0) + 3z_0^2 f''(z_0) + z_0^3 f^{(3)}(z_0)}{z_0 f'(z_0)} \right\} \geq m^2. \quad (1.9)$$

Then

$$Re \left\{ \frac{z_0 f'(z_0) + 7z_0^2 f''(z_0) + 6z_0^3 f^{(3)}(z_0) + z_0^4 f^{(4)}(z_0)}{z_0 f'(z_0)} \right\} \geq m^3. \quad (1.10)$$

Proof: The relations (1.7), (1.8) were proved in Lemma [9, Chap 2, page. 19] and (1.9) were proved in Lemma [9, Chap 6, page. 322]. We only need to prove (1.10).

If we $f(z) = R(r_0, \theta)e^{i\phi(r_0, \theta)}$, for $z = r_0 e^{i\theta}$,

then

$$\frac{zf'(z)}{f(z)} = \frac{\partial \phi}{\partial \theta} - i \frac{1}{R} \frac{\partial R}{\partial \theta}. \quad (1.11)$$

Differentiating (1.11) with respect to θ , we obtain

$$i \frac{zf'(z)}{f(z)} \left[\frac{z(zf'(z))'}{zf'(z)} - \frac{zf'(z)}{f(z)} \right] = \frac{\partial^2 \phi}{\partial \theta^2} - i \left[\frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left(\frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right]. \quad (1.12)$$

Another differentiation with respect θ leads to

$$\begin{aligned} \frac{zf'(z)}{f(z)} \left[\frac{z(z(zf'(z)))'}{zf'(z)} - 3 \frac{z(zf'(z))'}{zf'(z)} \cdot \frac{zf'(z)}{f(z)} + 2 \left(\frac{zf'(z)}{f(z)} \right)^2 \right] \\ = -\frac{\partial^3 \phi}{\partial \theta^3} - i \frac{\partial}{\partial \theta} \left[\frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left(\frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right]. \end{aligned} \quad (1.13)$$

Also differentiation (1.13) with respect to θ , we obtain

$$\begin{aligned} i \frac{zf'(z)}{f(z)} \left[\frac{z(z(z(zf'(z)))')'}{zf'(z)} - 4 \frac{z(z(zf'(z)))'}{zf'} \cdot \frac{zf'(z)}{f} - 3 \frac{z(zf'(z))'}{zf'} \cdot \frac{z(zf')'}{zf'} \cdot \frac{zf'}{f} \right. \\ \left. + 12 \frac{z(zf')'}{zf'} \cdot \left(\frac{zf'}{f} \right)^2 - 6 \left(\frac{zf'}{f} \right)^3 \right] = -\frac{\partial^4 \phi}{\partial \theta^4} + i \frac{\partial^2}{\partial \theta^2} \left[\frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left(\frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right]. \end{aligned}$$

Taking imaginary parts of this expression at z_0 and using (1.7) leads to

$$\begin{aligned} Re \left[\frac{z_0 f'(z_0) + 7z_0^2 f''(z_0) + 6z_0^3 f'''(z_0) + z_0^4 f^{(4)}(z_0)}{z_0 f'(z_0)} \right] m - 4m^2 Re \left[\frac{z_0 f'(z_0) + 3z_0^2 f''(z_0) + z_0^3 f^{(3)}(z_0)}{z_0 f'(z_0)} \right] - \\ 3m^2 Re \left[\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right]^2 + 12m^2 Re \left[\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right] - 6m^4 = \frac{\partial^2}{\partial \theta^2} \left[\frac{1}{R} \frac{\partial^2 R}{\partial \theta^2} - \left(\frac{1}{R} \frac{\partial R}{\partial \theta} \right)^2 \right]. \end{aligned}$$

Finally, by applying (1.8), (1.9), we obtain the desired result (1.10).

Lemma (1.5): Let $p \in H[a, n]$ and $q \in Q$ with $q(0) = a$ for $z \in \overline{U_{r_0}}$. Let

$$S = q^{-1}[p(z)] = f(z). \quad (1.14)$$

If there exists points $z_0 \in U$ and $s_0 \in \partial U/E(q)$ such that $p(z_0) = q(s_0)$ and $P(\overline{U_{r_0}}) \subset q(U)$,

$$Re \left\{ \frac{s_0 q''(s_0)}{q'(s_0)} \right\} \geq 0, \quad \left| \frac{z p'(z)}{q'(s)} \right| \leq k \quad (1.15)$$

$$\text{and} \quad Re \left\{ \frac{s_0^2 q^{(3)}(s_0)}{q'(s_0)} \right\} \geq 0, \quad \left| \frac{z^2 p''(z)}{q'(s)} \right| \leq k^2, \quad (1.16)$$

where $r_0 = |z_0|$. Then there exists an $m \geq n \geq 1$ such that

$$z_0 p'(z_0) = m s_0 q'(s_0) \quad (1.17)$$

$$Re \left\{ \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \right\} \geq m Re \left\{ \frac{s_0 q''(s_0)}{q'(s_0)} + 1 \right\} \quad (1.18)$$

and

$$Re \left\{ \frac{z_0 p'(z_0) + 3z_0^2 p''(z_0) + z_0^3 p^{(3)}(z_0)}{z_0 p'(z_0)} \right\} \geq m^2 Re \left\{ \frac{s_0 q'(s_0) + 3s_0^2 q''(s_0) + s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)} \right\}. \quad (1.19)$$

Then

$$\begin{aligned} & Re \left\{ \frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)} \right\} \\ & \geq m^3 Re \left\{ \frac{s_0 q'(s_0) + 7s_0^2 q''(s_0) + 6s_0^3 q^{(3)}(s_0) + s_0^4 q^{(4)}(s_0)}{s_0 q'(s_0)} \right\}, \end{aligned} \quad (1.20)$$

$$\text{or} \quad Re \left\{ \frac{z_0^3 p^{(4)}(z_0)}{p'(z_0)} \right\} \geq k^3 Re \left\{ \frac{s_0^3 q^{(4)}(s_0)}{q'(s_0)} \right\}. \quad (1.21)$$

Proof: The relations (1.17), (1.18) proved in [9, Chap. 2, page 22] and (1.19) proved in [9, Chap. 6, pages 325 and 327]. We only need to prove (1.20). Note that q is univalent at s_0 and hence $q'(s_0) \neq 0$. Since p is analytic in U , the set $p(\overline{U_{r_0}})$ is a bounded set and $p(\overline{U_{r_0}}) \subset q(\overline{U})/E(q)$. From (1.14) we see that f is analytic in $(\overline{U_{r_0}})$ and satisfies $|f(z_0)| = |s_0| = 1, f(0) = 0$ and $|f(z)| \leq 1$ for $|z| \leq r_0$. A further calculation show that $f^{(k)}(0) = p^{(k)}(0)$ for $k = 1, 2, \dots, n-1$. Thus f satisfies the conditions of Lemma (1.4) and deduce that there exists an m, n such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = m, \quad (1.22)$$

and

$$\operatorname{Re} \left\{ \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right\} \geq m. \quad (1.23)$$

Since from (1.14) we have $p(z) = q(s)$, with $s = f(z)$, we obtain

$$zp'(z) = q'(z)zf'(z). \quad (1.24)$$

Differentiating (1.20) leads to

$$\frac{z(zp'(z))'}{zp'(z)} = \frac{sq''(s)}{q'(s)} \cdot \frac{zf'(z)}{f(z)} + \frac{z(zf'(z))'}{zf'(z)}, \quad (1.25)$$

Another differentiation leads to

$$\frac{z(z(zp'(z)))'}{zp'(z)} = \frac{s^3 q^{(3)}(s)}{sq'(s)} \left(\frac{zf'(z)}{f(z)} \right)^2 + 3 \frac{s^2 q''(s)}{sq'(s)} \cdot \frac{zf'(z)}{f(z)} \cdot \frac{z(zf'(z))'}{zf'(z)} + \frac{z(z(zf'(z)))'}{zf'(z)}. \quad (1.26)$$

While another differentiation leads to

$$\begin{aligned} \frac{z(z(z(zp'(z)))')')}{zp'(z)} &= \frac{z(z(z(zf'(z)))')')}{zf'(z)} \\ &+ 4 \frac{s^2 q''(s)}{q'(s)} \cdot \frac{z(z(zf'(z)))'}{zf'(z)} \cdot \frac{zf'(z)}{f(z)} + 6 \frac{s^3 q^{(3)}(s)}{sq'(s)} \cdot \frac{z(zf'(z))'}{zf'(z)} \cdot \left(\frac{zf'(z)}{f(z)} \right)^2 \\ &+ 3 \frac{s^2 q''(s)}{sq'(s)} \cdot \left(\frac{z(zf'(z))'}{zf'(z)} \right)^2 \cdot \frac{zf'(z)}{f(z)} + \frac{s^4 q^{(4)}(s)}{sq'(s)} \cdot \left(\frac{zf'(z)}{f(z)} \right)^3. \end{aligned}$$

If we evaluate the real part of this expression at z_0 and use (1.23) and (1.25), we obtain

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)} \right\} &= \operatorname{Re} \left\{ \frac{z_0 f'(z_0) + 7z_0^2 f''(z_0) + 6z_0^3 f^{(3)}(z_0) + z_0^4 f^{(4)}(z_0)}{z_0 f'(z_0)} \right\} + \\ 4m \frac{s_0^2 q''(s_0)}{s_0 q'(s_0)} \operatorname{Re} \left\{ \frac{z_0 f'(z_0) + 3z_0^2 f''(z_0) + z_0^3 f^{(3)}(z_0)}{z_0 f'(z_0)} \right\} &+ 3m \frac{s_0^2 q''(s_0)}{s_0 q'(s_0)} \operatorname{Re} \left\{ \left(\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right)^2 \right\} + \\ 6m^2 \frac{s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)} \operatorname{Re} \left\{ \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right\} &+ \frac{m^3 s_0^4 q^{(4)}(s_0)}{s_0 q'(s_0)}. \end{aligned}$$

By use condition (1.9) and (1.10) in Lemma(1.4) and use (1.7), (1.8), we obtain

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)} \right\} \\ \geq m^3 + 4m^3 \frac{s_0^2 q''(s_0)}{s_0 q'(s_0)} + 3m^3 \frac{s_0^2 q''(s_0)}{s_0 q'(s_0)} + 6m^3 \frac{s_0^3 q^{(3)}(s_0)}{s_0 q'(s_0)} + m^3 \frac{s_0^4 q^{(4)}(s_0)}{s_0 q'(s_0)}, \end{aligned}$$

hence

$$Re \left\{ \frac{z_0 p'(z_0) + 7z_0^2 p''(z_0) + 6z_0^3 p^{(3)}(z_0) + z_0^4 p^{(4)}(z_0)}{z_0 p'(z_0)} \right\} \\ \geq m^3 \left[\frac{s_0 q'(s_0) + 7s_0^2 q''(s_0) + 6s_0^3 q^{(3)}(s_0) + s_0^4 q^{(4)}(s_0)}{s_0 q'(s_0)} \right] \\ \text{or } Re \left\{ \frac{z_0^3 p^{(4)}(z_0)}{p'(z_0)} \right\} \geq k^3 Re \left\{ \frac{s_0^3 q^{(4)}(s_0)}{q'(s_0)} \right\}.$$

Definition(1.6): Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in N/\{2\}$. The class of admissible function $A_n[\Omega, q]$ consists of those function $\psi: \mathbb{C}^5 \times U \rightarrow \mathbb{C}$ that satisfies the following admissibility condition $\psi(r, s, t, w, b; z) \notin \Omega$, whenever

$$r = q(\xi), s = k\xi q'(\xi), \Re \left(\frac{t}{s} + 1 \right) \geq k \Re \left(\frac{\xi q''(\xi)}{q'(\xi)} + 1 \right), \Re \left(\frac{w}{s} \right) \geq k^2 \Re \left(\frac{\xi^2 q^{(3)}(\xi)}{q'(\xi)} \right) \\ \text{and } \Re \left(\frac{b}{s} \right) \geq k^3 \Re \left(\frac{\xi^3 q^{(4)}(\xi)}{q'(\xi)} \right),$$

where $z \in U$, $\xi \in \partial U/E(q)$ and $k \geq n$.

The next theorem is a foundation result in the theory of fourth-order differential subordinations. Its proof is very short because of we use Lemma (1.5) and the special conditions given in the definition of the class of admissible functions $A_n[\Omega, q]$.

Theorem (1.7): Let $p \in H[a, n]$ with $n \in N/\{2\}$. Also, let $q \in Q(a)$ and satisfy the following conditions:

$$\Re \left(\frac{\xi^2 q'''(\xi)}{q'(\xi)} \right) \geq 0 \quad \left| \frac{z^2 p''(z)}{q'(\xi)} \right| \leq k^2, \quad (1.27)$$

where $z \in U$, $\xi \in \partial U/E(q)$ and $k \geq n$. If Ω a set in $\mathbb{C}$, $\psi \in A_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z), z^4 p''''(z); z) \in \Omega, \quad (1.28)$$

then

$$p(z) < q(z) \quad (z \in U).$$

Proof: If we assume that $p \not\prec q$, then there exist point $z_0 = r_0 e^{i\theta_0} \in U$ and $s_0 \in \partial U/E(q)$ such that $p(z_0) = q(s_0)$ and $p(\overline{U_{r_0}}) \subset q(U)$. From (1.27), we see that the condition (1.16) of Lemma (1.5) is satisfied when $z \in \overline{U_{r_0}}$ and $s_0 \in \partial U/E(q)$. Since all the conditions of that Lemma are satisfied, conclusions (1.17), (1.18), (1.19) and (1.21) follow. Using these last four results of Definition (1.6) leads to

$$\psi(p(z_0), zp'(z_0), z^2 p''(z_0), z^3 p^{(3)}(z_0), z^4 p^{(4)}(z_0); z) \notin \Omega,$$

since this contradicts (1.28), we must have $p \prec q$.

2. Fourth-order differential subordination with $I_{s,a,\mu}^\lambda$

We first define the following class of admissible function, which are required in proving the differential subordination theorem involving the operator $I_{s,a,\mu}^\lambda f(z)$ defined by (1.3).

Definition (2.1): Let Ω be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap H_0$. The class function $B_I[\Omega, q]$ consists of those function $\phi: \mathbb{C}^5 \times U \rightarrow \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi) + (\mu - 1)q(\xi)}{\mu}$$

$$\mathcal{R}\left(\frac{x\mu^2 - (\mu - 1)^2u}{v\mu - u(\mu - 1)} + (2 - 2\mu)\right) \geq \mathcal{R}\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)$$

$$\mathcal{R}\left(\frac{y\mu^3 - 3x\mu^3 + (2\mu + 1)(\mu - 1)^2}{v\mu - u(\mu - 1)} + (3\mu^2 - 1)\right) \geq k^2 \mathcal{R}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right)$$

and

$$\mathcal{R}\left(\frac{g\mu^4 - (\mu - 1)^4 + (4\mu^4 - 2\mu^3)y + (\mu - 1)^3(4\mu + 2)u + (12\mu^2 + 6\mu)(x\mu^2 - (\mu - 1)^2u}{v\mu - u(\mu - 1)} - (12\mu^3 + 6\mu^2 + 8\mu - 2)\right) \geq k^3 \mathcal{R}\left(\frac{\xi^3 q''''(\xi)}{q'(\xi)}\right).$$

Theorem (2.2): Let $\phi \in B_I[\Omega, q]$. If the function $f(z) \in \Sigma$ and $q \in \mathbb{Q}_0$ and satisfy the following conditions:

$$\mathcal{R}\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right) \geq 0 \quad \left| \frac{I_{s,a,\mu}^{\lambda-1} f(z)}{q'(\xi)} \right| \leq k^2, \quad (2.1)$$

and

$$\{\phi(I_{s,a,\mu}^{\lambda+1} f(z), I_{s,a,\mu}^\lambda f(z), I_{s,a,\mu}^{\lambda-1} f(z), I_{s,a,\mu}^{\lambda-2} f(z), I_{s,a,\mu}^{\lambda-3} f(z)); z \in U\} \subset \Omega, \quad (2.2)$$

then

$$I_{s,a,\mu}^{\lambda+1} f(z) \prec q(z) \quad (z \in U).$$

Proof: Define the analytic function $p(z)$ in U by

$$p(z) = I_{s,a,\mu}^{\lambda+1} f(z), \quad (z \in U). \quad (2.3)$$

Then, differentiating (2.3) with respect to z and using (1.4), we have

$$I_{s,a,\mu}^\lambda f(z) = \frac{zp'(z) + (\mu - 1)p(z)}{\mu}. \quad (2.4)$$

Further computations show that

$$I_{s,a,\mu}^{\lambda-1} f(z) = \frac{z^2 p''(z) + (2\mu - 1)zp'(z) + (\mu - 1)^2 p(z)}{\mu^2} \quad (2.5)$$

$$I_{s,a,\mu}^{\lambda-2} f(z) = \frac{z^3 p'''(z) + 3\mu z^2 p''(z) + (3\mu^2 - 3\mu + 1)zp'(z) + (\mu - 1)^3 p(z)}{\mu^3}. \quad (2.6)$$

and

$$I_{s,a,\mu}^{\lambda-3} f(z) = \frac{z^4 p''''(z) + 2(2\mu + 1)z^3 p'''(z) + (6\mu^2 + 1)z^2 p''(z) + (4\mu^3 - 3\mu^2 + \mu - 1)zp'(z) + (\mu - 1)^4 p(z)}{\mu^4}. \quad (2.7)$$

We now define the transformation from $\mathbb{C}^5$ to $\mathbb{C}$ by

$$\begin{aligned} u(r, s, t, w, b) &= r, \quad v(r, s, t, w, b) = \frac{s + (\mu - 1)r}{\mu}, \\ x(r, s, t, w, b) &= \frac{t + (2\mu - 1)s + (\mu - 1)^2 r}{\mu^2}, \\ y(r, s, t, w, b) &= \frac{w + 3\mu t + (3\mu^2 - 3\mu + 1)s + (\mu - 1)^3 r}{\mu^3}, \\ g(r, s, t, w, b) &= \frac{b + 2(2\mu + 1)w + (6\mu^2 + 1)t + (4\mu^3 - 3\mu^2 + \mu - 1)s + (\mu - 1)^2 r}{\mu^4}. \end{aligned} \quad (2.8)$$

Let

$$\begin{aligned} \psi(r, s, t, w, b; z) &= \phi(u, v, x, y, g; z) = \\ &= \phi \left(\frac{s + (\mu - 1)r}{\mu}, \frac{t + (2\mu - 1)s + (\mu - 1)^2 r}{\mu^2}, \frac{w + 3\mu t + (3\mu^2 - 3\mu + 1)s + (\mu - 1)^3 r}{\mu^3}, \right. \\ &\quad \left. \frac{b + 2(2\mu + 1)w + (6\mu^2 + 1)t + (4\mu^3 - 3\mu^2 + \mu - 1)s + (\mu - 1)^2 r}{\mu^4}; z \right). \end{aligned} \quad (2.9)$$

The proof will make use of Theorem(1.7). Using the equations (2.3) to (2.7), we have from (2.9) that

$$\begin{aligned} \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z), z^4 p''''(z); z) = \\ \phi(I_{s,a,\mu}^{\lambda+1} f(z), I_{s,a,\mu}^\lambda f(z), I_{s,a,\mu}^{\lambda-1} f(z), I_{s,a,\mu}^{\lambda-2} f(z), I_{s,a,\mu}^{\lambda-3} f(z)). \end{aligned} \quad (2.10)$$

Hence, clearly, (2.2) becomes

$$\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z), z^4 p''''(z); z) \in \Omega,$$

we note that

$$\frac{t}{s} + 1 = \frac{x\mu^2 - (\mu - 1)^2 u}{v\mu - u(\mu - 1)} + (2 - 2\mu)$$

$$\frac{w}{s} = \frac{y\mu^3 - 3x\mu^3 + (2\mu + 1)(\mu - 1)^2}{v\mu - u(\mu - 1)} + (3\mu^2 - 1),$$

and

$$\frac{b}{s} = \frac{g\mu - (\mu - 1)^4 + (4\mu^4 - 2\mu^3)y + (\mu - 1)^3(4\mu + 2)u + (12\mu^2 + 6\mu)(x\mu^2 - (\mu - 1)^2 u)}{v\mu - u(\mu - 1)} - (12\mu^3 + 6\mu^2 + 8\mu - 2).$$

Therefore, the admissibility condition for $\phi \in B_I[\Omega, q]$ in Definition (2.1) is equivalent to admissibility condition for $\psi \in A_3[\Omega, q]$ as given in Definition (1.6) with $n = 3$.

Therefore, by using (2.1) and Theorem (1.7), we obtain

$$p(z) = I_{s,a,\mu}^{\lambda+1} f(z) < q(z).$$

This completes the proof of Theorem (2.2).

Our next corollary is an extension of Theorem (2.2) to the case when the behavior of $q(z)$ on ∂U is not known.

Corollary (2.3): Let $\Omega \subset \mathbb{C}$ and let function $q(z)$ be univalent in U with $q(0) = 0$. Let $\phi \in B_I[\Omega, q]$ for some $p \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \Sigma$ and q_ρ satisfies the following conditions:

$$\Re \left(\frac{\xi^2 q_\rho'''(\xi)}{q_\rho'(\xi)} \right) \geq 0 \quad \left| \frac{I_{s,a,\mu}^{\lambda-1} f(z)}{q_\rho'(\xi)} \right| \leq k^2, \quad (z \in U; k \geq 2; \xi \in \partial U / E(q_\rho)), \quad (2.11)$$

and

$$\phi(I_{s,a,\mu}^{\lambda+1} f(z), I_{s,a,\mu}^\lambda f(z), I_{s,a,\mu}^{\lambda-1} f(z), I_{s,a,\mu}^{\lambda-2} f(z), I_{s,a,\mu}^{\lambda-3} f(z); z) < h(z), \quad (2.12)$$

then

$$I_{s,a,\mu}^{\lambda+1} f(z) < q(z) \quad (z \in U).$$

Proof: By using Theorem (2.2), yield

$$I_{s,a,\mu}^{\lambda+1} f(z) < q_\rho(z) \quad (z \in U),$$

then, we obtain the result from

$$q_\rho(z) < q(z) \quad (z \in U).$$

This completes the proof of Corollary (1).

If $\Omega \neq \mathbb{C}$ is simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case, the class $B_I[h(U), q]$ is written as $B_I[h, q]$. The following two results are immediate consequence of Theorem (2.2) and corollary (2.3).

Theorem (2.4): Let $\phi \in B_I[h, q]$. If the function $f \in \Sigma$ and $q \in \mathbb{Q}_0$ satisfy the following conditions (2.1) and

$$\phi(I_{s,a,\mu}^{\lambda+1}f(z), I_{s,a,\mu}^{\lambda}f(z), I_{s,a,\mu}^{\lambda-1}f(z), I_{s,a,\mu}^{\lambda-2}f(z), I_{s,a,\mu}^{\lambda-3}f(z); z) < h(z), \quad (2.13)$$

then

$$I_{s,a,\mu}^{\lambda+1}f(z) < q(z) \quad (z \in U).$$

Corollary (2.5): Let $\Omega \subset \mathbb{C}$ and let function q be univalent in U with $q(0) = 0$. Also Let $\phi \in B_I[h, q_p]$ for some $p \in (0,1)$, where $q_p(z) = q(\rho z)$. If the function $f \in \Sigma$ and q_p satisfies the conditions (2.11), and

$$\phi(I_{s,a,\mu}^{\lambda+1}f(z), I_{s,a,\mu}^{\lambda}f(z), I_{s,a,\mu}^{\lambda-1}f(z), I_{s,a,\mu}^{\lambda-2}f(z), I_{s,a,\mu}^{\lambda-3}f(z); z) < h(z), \quad (2.14)$$

then

$$I_{s,a,\mu}^{\lambda+1}f(z) < q(z) \quad (z \in U).$$

The following result yield the best dominant of differential subordination (2.12).

Theorem (2.6): Let the function h be univalent in U . Also let $\phi: \mathbb{C}^5 \times U \rightarrow \mathbb{C}$ and Suppose that following differential equation:

$$\phi \left(\begin{array}{c} q(z), \frac{zq'(z) + (\mu-1)q(z)}{\mu}, \frac{z^2q''(z) + (2\mu-1)zq'(z) + (\mu-1)^2q(z)}{\mu^2}, \\ \frac{z^3q'''(z) + 3\mu z^2q''(z) + (3\mu^2 - 3\mu + 1)zq'(z) + (\mu-1)^3q(z)}{\mu^3}, \\ \frac{z^4q''''(z) + 2(2\mu+1)z^3q'''(z) + (6\mu^2+1)z^2q''(z) + (4\mu^4 - 3\mu^2 + \mu - 1)zq'(z) + (\mu-1)^4q(z)}{\mu^4}; z \end{array} \right) = h(z), \quad (2.15)$$

has a solution $q(z)$ with $q(0) = 0$, which satisfies the condition (2.1). If $f \in \Sigma$ satisfies the condition (2.12) and if

$$\phi(I_{s,a,\mu}^{\lambda+1}f(z), I_{s,a,\mu}^{\lambda}f(z), I_{s,a,\mu}^{\lambda-1}f(z), I_{s,a,\mu}^{\lambda-2}f(z), I_{s,a,\mu}^{\lambda-3}f(z); z),$$

is analytic in U , then

$$I_{s,a,\mu}^{\lambda+1}f(z) < q(z) \quad (z \in U)$$

and $q(z)$ is the best dominant.

Proof: Using Theorem(2.2), that $q(z)$ is a dominant of (2.12). Since $q(z)$ satisfies (2.14), it is also a solution of (2.12). Therefore, $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

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