1. Introduction

Harmonic functions are a classic title in the class of geometric functions. Many researchers have studied these function classes from past to present, and since it has a wide range of applications, it is still a popular class. In this study, we will examine the class of harmonic univalent functions, which is a subclass of harmonic functions. Let's show the open unit disk as $\mathbb{U}$. Let's show the family of continuous complex-valued harmonic functions that are harmonic in $\mathbb{U}$ as $\mathcal{H}$. $\mathcal{A}$ denotes the subclass of $\mathcal{H}$, including functions that are analytic in the open unit disk. $f$ be a harmonic function in $\mathbb{U}$ and $f$ may be written as $f = h + \overline{g}$; where $h$ and $g$ are in $\mathcal{A}$. We denominate $h$ is the analytic part of $f$, and $g$ is the co-analytic part of $f$. $S$ denotes normalized analytic univalent functions in the open unit disk.
Clunie and Sheil-Small [5] proved that $f$ is sense-preserving and locally univalent in $\mathbb{U}$ if and only if $|h'(z)| > |g'(z)|$. So no loss of generality we can write $h(0) = 0$ and $h'(0) = 1$ since $h'(z) \neq 0$.

In this way, writing

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k$$

(1)

does not break generality.

Let $SH$ indicate the family of functions $f = h + \bar{g}$ which are univalent, harmonic and sense-preserving in $\mathbb{U}$ for $f$ functions that provide $f(0) = f_z(0) - 1 = 0$.

Consequently, $SH$ includes the class of $S$, which is normalized analytic univalent functions. Clearly, it will become apparent that although the analytic part $h$ of a function $f \in SH$ is locally univalent, it need not be univalent. It can be easily demonstrated that the inequality of $|b_1| < 1$ must be achieved in order to have sense-preserving property. The subclass $SH^0$ of $SH$ covers all functions in $SH$ with $f_z(0) = 0$ property. Clunie and Sheil-Small [5] examined $SH$ as well as its subclasses and discovered some coefficient bounds. Thenceforward, there have been various associated articles on $SH$ and its subclasses. For more details see ([1], [3], [4], [6], [7], [8], [10], [11], [13], [14] and [15] etc.). Furthermore, notice that $SH$ reduces to the class $S$, if the co-analytic part of $f$ is identically zero.

For $f \in A, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the Salagean differential operator $D^n$ is defined by Salagean [9] $D^n : A \rightarrow A$,

$$D^0 f(z) = f(z)$$

$$\ldots$$

$$D^{n+1} f(z) = z(D^n f(z))'.$$

For

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$
and

\[ D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k. \]

Salagean [9] defined a linear operator denoted by \( I^n \) given by

\[ I^0 f(z) = f(z), \]
\[ I^1 f(z) = \frac{1}{z} \int_0^z \frac{f(t)}{t} dt \]
\[ ... \]
\[ I^n f(z) = I(I^{n-1} f(z)), \]

where \( f(z) \in \mathcal{A}, z \in \mathbb{U}, n \in \mathbb{N}_0 \) and

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \]

then

\[ I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k. \]

Then Pall-Szabo [2] defined a linear operator for \( z \in \mathbb{U}, \lambda \geq 0, n \in \mathbb{N}_0 \) denoted by \( L^n \) given by \( L^n : \mathcal{A} \rightarrow \mathcal{A} \),

\[ L^n f(z) = (1 - \lambda)D^n f(z) + \lambda I^n f(z). \]

Therefore, if \( f(z) \in \mathcal{A} \) and

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \]

then

\[ L^n f(z) = z + \sum_{k=2}^{\infty} \left[ k^n (1 - \lambda) + \lambda \frac{1}{k^n} \right] a_k z^k. \]

(2)

Now we define the Salagean integro differential operator for functions \( f = h + \bar{g} \) as stated in (1),

\[ L^0 f(z) = (1 - \lambda)D^0 f(z) + \lambda I^0 f(z) \]
\[ ... \]
\[ L^n f(z) = (1 - \lambda)D^n f(z) + \lambda I^n f(z) \]
\[ = (1 - \lambda) \left( D^n h(z) + (-1)^n \bar{D}^n g(z) \right) + \lambda \left( I^n h(z) + (-1)^n \bar{I}^n g(z) \right), \]

where \( n \in \mathbb{N}_0, 0 \leq \lambda \leq 1 \). Therefore
\[\mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty} \left( k^n(1 - \lambda) + \lambda \frac{1}{k^n} \right) a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left( k^n(1 + \lambda) + \lambda \frac{1}{k^n} \right) b_k z^k. \quad (3)\]

Let show the subclass of \(\mathcal{SH}\) containing functions \(f\) as stated in (1) which provide the following condition as \(\mathcal{SH}(\lambda, n, \mu)\)

\[\text{Re}\left(\frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)}\right) \geq \mu, \quad 0 \leq \mu < 1 \quad (4)\]

where \(\mathcal{L}^n f(z)\), as stated in (2).

We denote the subclass \(\overline{\mathcal{SH}}(\lambda, n, \mu)\) containing harmonic functions \(f_n = h + \overline{g_n}\) in \(\mathcal{SH}\) so that \(h\) and \(g_n\) be defined as follows

\[h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad (5)\]

\(a_k, b_k \geq 0\).

If the parameters are chosen specially, \(\mathcal{SH}(\lambda, n, \mu)\) classes are reduced to different subclasses of harmonic univalent functions (For example Jahangiri [7], Jahangiri et al. [8], Silverman [10], Silverman and Silvia [11], Uralgeaddi and Somanatha [12], Cho and Srivastava [4], Bayram and Yalcin [13], Atshan and Wanas [16]).

2. Main Results

In Theorem 2.1, we present a sufficient coefficient condition for harmonic univalent functions that are the members of \(\mathcal{SH}^0(\lambda, n, \mu)\).

2.1 Theorem

Let \(f = h + \overline{g}\). In here let \(h\) and \(g\) are as stated in (1) with \(b_1 = 0\). Let

\[\sum_{k=2}^{\infty} \left( k^n(1 - \lambda) + \lambda \frac{1}{k^n} \right) \left( k^n(1 - \lambda) + \lambda \frac{1}{k^n} - \mu \right) |a_k| + \sum_{k=2}^{\infty} \left( k^n(1 + \lambda) + \lambda \frac{1}{k^n} \right) \left( k^n(1 + \lambda) + \lambda \frac{1}{k^n} + \mu \right) |b_k| \leq 1 - \mu, \quad (6)\]

where \(n \in \mathbb{N}_0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1\). Then \(f\) is sense-preserving and harmonic univalent in \(\mathbb{U}\) so we can say that \(f \in \mathcal{SH}^0(\lambda, n, \mu)\).

As a special notation for convenience, sometimes, we write

\[A_n = \left[ k^n(1 - \lambda) + \lambda \frac{1}{k^n} \right] \left( k^n(1 - \lambda) + \lambda \frac{1}{k^n} - \mu \right) \]

and

\[B_n = \left[ k^n(1 + \lambda) + \lambda \frac{1}{k^n} \right] \left( k^n(1 + \lambda) + \lambda \frac{1}{k^n} + \mu \right) \]

and

\[C_n = k^n(1 - \lambda) + \lambda \frac{1}{k^n}. \]
in this article.

**Proof.** If we accept that \( z_1 \neq z_2 \), we obtain

\[
\frac{f(z_1) - f(z_2)}{b(z_1) - b(z_2)} \geq 1 - \left| \frac{g(z_1) - g(z_2)}{b(z_1) - b(z_2)} \right| = 1 - \frac{\sum_{k=2}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)}
\]

\[
> 1 - \frac{\sum_{k=2}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|}
\]

\[
\geq 1 - \frac{\sum_{k=2}^{\infty} b_n 1 - \mu |b_k|}{1 - \sum_{k=2}^{\infty} a_n 1 - \mu |a_k|} \geq 0
\]

that proving univalence. Pay attention \( f \) is sense−preserving in \( \mathbb{U} \). To show this feature;

\[
|b'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k| |z|^{k-1}
\]

\[
> 1 - \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} |a_k|
\]

\[
\geq \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} |b_k|
\]

\[
> \sum_{k=2}^{\infty} k|b_k| |z|^{k-1}
\]

\[
\geq |g'(z)|.
\]

Using the fact that \( \text{Re} w > \alpha \iff |1 - \alpha + w| \geq |1 + \alpha - w| \), it suffices to show the following inequality

\[
|(1 - \mu) L^n f(z) + L^{n+1} f(z)| - |(1 + \mu) L^n f(z) - L^{n+1} f(z)| \geq 0. \quad (7)
\]

Replacing for \( L^n f(z) \) and \( L^{n+1} f(z) \) in (7), we obtain

\[
|(1 - \mu) L^n f(z) + L^{n+1} f(z)| - |(1 + \mu) L^n f(z) - L^{n+1} f(z)|
\]

\[
\geq 2(1 - \mu)|z| - \sum_{k=2}^{\infty} C_n (C_n + 1 - \mu) |a_k| |z|^k - \sum_{k=2}^{\infty} C_n (C_n - 1 + \mu) |b_k| |z|^k
\]

\[
- \sum_{k=2}^{\infty} C_n (C_n - 1 - \mu) |a_k| |z|^k - \sum_{k=2}^{\infty} C_n (C_n + 1 + \mu) |b_k| |z|^k
\]

\[
> 2(1 - \mu)|z| \left( \frac{\sum_{k=2}^{\infty} A_n |a_k|}{1 - \mu} - \sum_{k=2}^{\infty} B_n |b_k| \right).
\]

This last phrase is not negative with (6), and therefore the proof is complete.
2.2 Theorem

Let \( f_n = b + \overline{a_n} \) be as stated in (4) with \( b_1 = 0 \). Then \( f_n \in \mathcal{SH}^0(\lambda, n, \mu) \), if and only if

\[
\sum_{k=2}^{\infty} A_n|a_k| + \sum_{k=2}^{\infty} B_n|b_k| \leq 1 - \mu, \tag{8}
\]

where \( n \in \mathbb{N}_0 \), \( 0 \leq \lambda \leq 1, 0 \leq \mu < 1 \).

**Proof.** Notice that \( \mathcal{SH}^0(\lambda, n, \mu) \) is a subclass of \( \mathcal{SH}^0(\lambda, n, \mu) \). Then the “if” part can be easily proved from 2.1 Theorem. For the “only if” part, we must demonstrate that \( f_n \notin \mathcal{SH}^0(\lambda, n, \mu) \) if the inequality (8) is not valid. Here is an issue to be considered; a necessary and sufficient condition for \( f_n = b + \overline{a_n} \) be as stated in (5), to be in \( \mathcal{SH}^0(\lambda, n, \mu) \) is that the condition (4) must be provided.

The above stipulation must be valid for all \( z \) complex numbers where, \( |z| = r < 1 \). After selecting the \( z \) values from the positive real axis where \( 0 \leq |z| = r < 1 \), we obtain

\[
\text{Re}\left\{ \frac{(1 - \mu)z - \sum_{k=2}^{\infty} A_n a_k z^k - \sum_{k=2}^{\infty} B_n b_k \overline{z}^k}{z - \sum_{k=2}^{\infty} C_n a_k z^k + \sum_{k=2}^{\infty} C_n b_k \overline{z}^k} \right\} \geq 0.
\]

This is equivalent to

\[
\left\{ \frac{(1 - \mu) - \sum_{k=2}^{\infty} A_n a_k r^{k-1} - \sum_{k=2}^{\infty} B_n b_k r^{k-1}}{1 - \sum_{k=2}^{\infty} C_n a_k r^{k-1} + \sum_{k=2}^{\infty} C_n b_k r^{k-1}} \right\} \geq 0. \tag{9}
\]

If the inequality (8) is not valid, then the expression in (9) is negative for \( r \) values approaching 1. Hence there exists \( z_0 = r_0 \) in \((0, 1)\) for which the quotient in (9) is negative. This shows the required condition for \( f_n \in \mathcal{SH}^0(\lambda, n, \mu) \) and so the proof is complete.

2.3 Theorem

Let \( f_n = b + \overline{a_n} \) be as stated in (5). At that case a necessary and sufficient condition for \( f_n \in \mathcal{SH}^0(\lambda, n, \mu) \) is

\[
f_n(z) = \sum_{k=1}^{\infty} \left( X_k h_k(z) + Y_k g_{n_k}(z) \right),
\]

where

\[
b_1(z) = z, \quad b_k(z) = z - \frac{1 - \mu}{A_n} z^k
\]

and

\[
g_{n_1}(z) = z, \quad g_{n_k}(z) = z + (-1)^n \frac{1 - \mu}{B_n} \overline{z}^k
\]

for \( X_k \geq 0, Y_k \geq 0, \sum_{k=1}^{\infty} (X_k + Y_k) = 1, k = 2, 3, \ldots, n \in \mathbb{N}_0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1 \).

Notably, the extreme points of \( \mathcal{SH}^0(\lambda, n, \mu) \) are \( \{b_k\} \) and \( \{g_{n_k}\} \).

**Proof.** For \( f_n \) functions which are as stated in (5), we obtain
\[ f_n(z) = \sum_{k=1}^{\infty} (X_kb_k(z) + Y_k g_{nk}(z)) \]

\[ = \sum_{k=1}^{\infty} (X_k + Y_k)z - \sum_{k=2}^{\infty} \frac{1 - \mu}{A_n} X_k z^k + (-1)^n \sum_{k=2}^{\infty} \frac{1 - \mu}{B_n} Y_k z^k. \]

Then

\[ \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} \left( \frac{1 - \mu}{A_n} X_k \right) + \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} \left( \frac{1 - \mu}{B_n} Y_k \right) \]

\[ = \sum_{k=2}^{\infty} X_k + \sum_{k=2}^{\infty} Y_k = 1 - X_1 - Y_1 \leq 1 \]

and so \( f_n \in \mathcal{SH}^0(\lambda, n, \mu) \). Moreover, if \( f_n \in \mathcal{SH}^0(\lambda, n, \mu) \), then

\[ a_k \leq \frac{1 - \mu}{A_n} \]

and

\[ b_k \leq \frac{1 - \mu}{B_n} \]

Set

\[ X_k = \frac{A_n}{1 - \mu} a_k, \]

\[ Y_k = \frac{B_n}{1 - \mu} b_k, \]

for \( k = 2, 3, \ldots \) and

\[ X_1 + Y_1 = 1 - \left( \sum_{k=2}^{\infty} X_k + Y_k \right) \]

where \( X_k \geq 0, Y_k \geq 0 \). Therefore, in accordance with, we have

\[ f_n(z) = (X_1 + Y_1)z + \sum_{k=2}^{\infty} (X_k b_k(z) + Y_k g_{nk}(z)) \]

\[ = \sum_{k=1}^{\infty} (X_k b_k(z) + Y_k g_{nk}(z)). \]

### 2.4 Theorem

Let \( f_n \in \mathcal{SH}^0(\lambda, n, \mu) \). Then for \( |z| = r < 1 \) and \( n \in \mathbb{N}_0, 0 \leq \lambda \leq 1, 0 \leq \mu < 1 \), we have
\[ r - \frac{1 - \mu}{A_n} r^2 \leq |f_n(z)| \leq r + \frac{1 - \mu}{A_n} r^2. \]

**Proof.** Here, we will only prove the right side since the proving of the left side, and the right side of the inequality is very similar. Let \( f_n \in \mathcal{SH}_n^0(\lambda, n, \mu) \). Taking the absolute value of \( f_n \) we can easily see

\[ |f_n(z)| \leq r + \sum_{k=2}^{\infty} (a_k + b_k)r^k \leq r + \frac{(1 - \mu)r^2}{A_n} \sum_{k=2}^{\infty} A_n |a_k| + \frac{(1 - \mu)r^2}{A_n} \sum_{k=2}^{\infty} B_n |b_k| \leq r + \frac{1 - \mu}{A_n} r^2. \]

The left side can be shown in a similar way.

2.5 **Theorem**

The class \( \mathcal{SH}_n^0(\lambda, n, \mu) \) is closed under convex combination.

**Proof.** Let \( f_{n_j} \in \mathcal{SH}_n^0(\lambda, n, \mu) \) for \( j = 1, 2, \ldots \), where \( f_{n_j} \) is given by

\[ f_{n_j}(z) = z - \sum_{k=2}^{\infty} a_{k_j} z^k + (-1)^n \sum_{k=2}^{\infty} b_{k_j} \bar{z}^k. \]

Then by (7),

\[ \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} a_{k_j} + \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} b_{k_j} \leq 1. \] (10)

For \( \sum_{j=1}^{\infty} p_j = 1, 0 \leq p_j \leq 1 \), the convex combination of \( f_{n_j} \) can be written as

\[ \sum_{j=1}^{\infty} p_j f_{n_j}(z) = z - \sum_{k=2}^{\infty} \left( \sum_{j=1}^{\infty} p_j a_{k_j} \right) z^k + (-1)^n \sum_{k=2}^{\infty} \left( \sum_{j=1}^{\infty} p_j b_{k_j} \right) \bar{z}^k. \]

Then by (10),

\[ \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} \left( \sum_{j=1}^{\infty} p_j a_{k_j} \right) + \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} \left( \sum_{j=1}^{\infty} p_j b_{k_j} \right) \]

\[ = \sum_{j=1}^{\infty} p_j \sum_{k=2}^{\infty} \frac{A_n}{1 - \mu} a_{k_j} + \sum_{j=1}^{\infty} p_j \sum_{k=2}^{\infty} \frac{B_n}{1 - \mu} b_{k_j} \]

\[ \leq \sum_{j=1}^{\infty} p_j = 1. \]

This is the situation required by inequality (8). In this way \( \sum_{j=1}^{\infty} p_j f_{n_j}(z) \in \mathcal{SH}_n^0(\lambda, n, \mu) \).

**References**


