A Subclass of Harmonic Univalent Functions Defined by Salagean Integro Differential Operator

<p>Authors Names a. Hasan BAYRAM</p> <p>Article History Received on: 8/5/2020 Revised on: 4/6/2020 Accepted on: 18/6/2020</p> <p>Keywords: Harmonic Linear Operator Univalent Functions</p> <p>DOI:https://doi.org/10.29350/jops.2020.25.3.1138</p>	<p>ABSTRACT</p> <p>In this paper, we scrutinize some fundamental features of a subclass of harmonic functions defined by a new operator. Like coefficient inequalities, convex combinations, distortion bounds.</p> <p>MSC: 30C45, 30C50</p>
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1. Introduction

Harmonic functions are a classic title in the class of geometric functions. Many researchers have studied these function classes from past to present, and since it has a wide range of applications, it is still a popular class. In this study, we will examine the class of harmonic univalent functions, which is a subclass of harmonic functions. Let's show the open unit disk as $\mathbb{U}$. Let's show the family of continuous complex-valued harmonic functions that are harmonic in $\mathbb{U}$ as $\mathcal{H}$. $\mathcal{A}$ denotes the subclass of $\mathcal{H}$, including functions that are analytic in the open unit disk. f be a harmonic function in $\mathbb{U}$ and f may be written as $f = h + \bar{g}$; where h and g are in $\mathcal{A}$. We denominate h is the analytic part of f , and g is the co-analytic part of f . $\mathcal{S}$ denotes normalized analytic univalent functions in the open unit disk.

^a Department of Mathematics

Clunie and Sheil-Small [5] proved that f is sense-preserving and locally univalent in $\mathbb{U}$ if and only if $|\mathfrak{h}'(z)| > |\mathfrak{g}'(z)|$. So no loss of generality we can write $\mathfrak{h}(0) = 0$ and $\mathfrak{h}'(0) = 1$ since $\mathfrak{h}'(z) \neq 0$.

In this way, writing

$$\mathfrak{h}(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } \mathfrak{g}(z) = \sum_{k=1}^{\infty} b_k z^k \quad (1)$$

does not break generality.

Let $\mathcal{SH}$ indicate the family of functions $f = \mathfrak{h} + \bar{\mathfrak{g}}$ which are univalent, harmonic and sense-preserving in $\mathbb{U}$ for f functions that provide $f(0) = f_z(0) - 1 = 0$.

Consequently, $\mathcal{SH}$ includes the class of $\mathcal{S}$, which is normalized analytic univalent functions. Clearly, it will become apparent that although the analytic part $\mathfrak{h}$ of a function $f \in \mathcal{SH}$ is locally univalent, it need not be univalent. It can be easily demonstrated that the inequality $|b_1| < 1$ must be achieved in order to have sense-preserving property. The subclass $\mathcal{SH}^0$ of $\mathcal{SH}$ covers all functions in $\mathcal{SH}$ with $f_z(0) = 0$ property. Clunie and Sheil-Small [5] examined $\mathcal{SH}$ as well as its subclasses and discovered some coefficient bounds. Thenceforward, there have been various associated articles on $\mathcal{SH}$ and its subclasses. For more details see ([1], [3], [4], [6], [7], [8], [10], [11], [13], [14] and [15] etc.). Furthermore, notice that $\mathcal{SH}$ reduces to the class $\mathcal{S}$, if the co-analytic part of f is identically zero.

For $f \in \mathcal{A}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the Salagean differential operator D^n is defined by Salagean [9] $D^n: \mathcal{A} \rightarrow \mathcal{A}$,

$$D^0 f(z) = f(z)$$

...

$$D^{n+1} f(z) = z(D^n f(z))'.$$

For

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

This operator was developed and modified by many researchers over time. As a simple example for $f = \mathfrak{h} + \bar{\mathfrak{g}}$ as stated in (1), Jahangiri et al. [8] described the modified Salagean operator of f like as

$$D^0 f(z) = f(z),$$

...

$$D^n f(z) = D^n \mathfrak{h}(z) + (-1)^n \overline{D^n \mathfrak{g}(z)},$$

where

$$D^n \mathfrak{h}(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$$

and

$$D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

Salagean [9] defined a linear operator denoted by I^n given by

$$\begin{aligned} I^0 f(z) &= f(z), \\ I^1 f(z) &= \frac{1}{z} \int_0^z \frac{f(t)}{t} dt \\ &\dots \\ I^n f(z) &= I(I^{n-1} f(z)), \end{aligned}$$

where $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$, $n \in \mathbb{N}_0$ and

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

then

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k.$$

Then Pall-Szabo [2] defined a linear operator for $z \in \mathbb{U}$, $\lambda \geq 0$, $n \in \mathbb{N}_0$ denoted by $\mathcal{L}^n$ given by $\mathcal{L}^n: \mathcal{A} \rightarrow \mathcal{A}$,

$$\mathcal{L}^n f(z) = (1 - \lambda) D^n f(z) + \lambda I^n f(z).$$

Therefore, if $f(z) \in \mathcal{A}$ and

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

then

$$\mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1 - \lambda) + \lambda \frac{1}{k^n} \right] a_k z^k. \quad (2)$$

Now we define the Salagean integro differential operator for functions $f = h + \bar{g}$ as stated in (1),

$$\begin{aligned} \mathcal{L}^0 f(z) &= (1 - \lambda) D^0 f(z) + \lambda I^0 f(z) \\ &\dots \\ \mathcal{L}^n f(z) &= (1 - \lambda) D^n f(z) + \lambda I^n f(z) \\ &= (1 - \lambda) (D^n h(z) + (-1)^n \overline{D^n g(z)}) + \lambda (I^n h(z) + (-1)^n \overline{I^n g(z)}), \end{aligned}$$

where $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$. Therefore

$$\mathcal{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n(1-\lambda) + \lambda \frac{1}{k^n} \right] a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left[k^n(1+\lambda) + \lambda \frac{1}{k^n} \right] \overline{b_k z^k}. \quad (3)$$

Let show the subclass of $\mathcal{SH}$ containing functions f as stated in (1) which provide the following condition as $\mathcal{SH}(\lambda, n, \mu)$

$$\operatorname{Re} \left(\frac{\mathcal{L}^{n+1} f(z)}{\mathcal{L}^n f(z)} \right) \geq \mu, \quad 0 \leq \mu < 1 \quad (4)$$

where $\mathcal{L}^n f(z)$, as stated in (2).

We denote the subclass $\overline{\mathcal{SH}}(\lambda, n, \mu)$ containing harmonic functions $f_n = h + \overline{g_n}$ in $\mathcal{SH}$ so that h and g_n be defined as follows

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad (5)$$

$a_k, b_k \geq 0$.

If the parameters are chosen specially, $\mathcal{SH}(\lambda, n, \mu)$ classes are reduced to different subclasses of harmonic univalent functions (For example Jahangiri [7], Jahangiri et al. [8], Silverman [10], Silverman and Silvia [11], Uralegaddi and Somanatha [12], Cho and Srivastava [4], Bayram and Yalcin [13], Atshan and Wanas [16]).

2. Main Results

In Theorem 2.1, we present a sufficient coefficient condition for harmonic univalent functions that are the members of $\mathcal{SH}^0(\lambda, n, \mu)$.

2.1 Theorem

Let $f = h + \overline{g}$. In here let h and g are as stated in (1) with $b_1 = 0$. Let

$$\sum_{k=2}^{\infty} \left[k^n(1-\lambda) + \lambda \frac{1}{k^n} \right] \left(k^n(1-\lambda) + \lambda \frac{1}{k^n} - \mu \right) |a_k| + \sum_{k=2}^{\infty} \left[k^n(1+\lambda) + \lambda \frac{1}{k^n} \right] \left(k^n(1+\lambda) + \lambda \frac{1}{k^n} + \mu \right) |b_k| \leq 1 - \mu, \quad (6)$$

where $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$. Then f is sense-preserving and harmonic univalent in $\mathbb{U}$ so we can say that $f \in \mathcal{SH}^0(\lambda, n, \mu)$.

As a special notation for convenience, sometimes, we write

$$A_n = \left[k^n(1-\lambda) + \lambda \frac{1}{k^n} \right] \left(k^n(1-\lambda) + \lambda \frac{1}{k^n} - \mu \right)$$

and

$$B_n = \left[k^n(1+\lambda) + \lambda \frac{1}{k^n} \right] \left(k^n(1+\lambda) + \lambda \frac{1}{k^n} + \mu \right)$$

and

$$C_n = k^n(1-\lambda) + \lambda \frac{1}{k^n}.$$

in this article.

Proof. If we accept that $z_1 \neq z_2$, we obtain

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=2}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \\ &> 1 - \frac{\sum_{k=2}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|} \\ &\geq 1 - \frac{\sum_{k=2}^{\infty} \frac{B_n}{1-\mu} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{A_n}{1-\mu} |a_k|} \geq 0 \end{aligned}$$

that proving univalence. Pay attention f is sense-preserving in $\mathbb{U}$. To show this feature;

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} \\ &> 1 - \sum_{k=2}^{\infty} \frac{A_n}{1-\mu} |a_k| \\ &\geq \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} |b_k| \\ &> \sum_{k=2}^{\infty} k |b_k| |z|^{k-1} \\ &\geq |g'(z)|. \end{aligned}$$

Using the fact that $\operatorname{Re} w \geq \alpha \Leftrightarrow |1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show the following inequality

$$|(1 - \mu)\mathcal{L}^n f(z) + \mathcal{L}^{n+1} f(z)| - |(1 + \mu)\mathcal{L}^n f(z) - \mathcal{L}^{n+1} f(z)| \geq 0. \quad (7)$$

Replacing for $\mathcal{L}^n f(z)$ and $\mathcal{L}^{n+1} f(z)$ in (7), we obtain

$$\begin{aligned} &|(1 - \mu)\mathcal{L}^n f(z) + \mathcal{L}^{n+1} f(z)| - |(1 + \mu)\mathcal{L}^n f(z) - \mathcal{L}^{n+1} f(z)| \\ &\geq 2(1 - \mu)|z| - \sum_{k=2}^{\infty} C_n (C_n + 1 - \mu) |a_k| |z|^k - \sum_{k=2}^{\infty} C_n (C_n - 1 + \mu) |b_k| |z|^k \\ &\quad - \sum_{k=2}^{\infty} C_n (C_n - 1 - \mu) |a_k| |z|^k - \sum_{k=2}^{\infty} C_n (C_n + 1 + \mu) |b_k| |z|^k \\ &> 2(1 - \mu)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{A_n |a_k|}{1 - \mu} - \sum_{k=2}^{\infty} \frac{B_n |b_k|}{1 - \mu} \right\}. \end{aligned}$$

This last phrase is not negative with (6), and therefore the proof is complete.

2.2 Theorem

Let $f_n = h + \overline{g_n}$ be as stated in (4) with $b_1 = 0$. Then $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$, if and only if

$$\sum_{k=2}^{\infty} A_n |a_k| + \sum_{k=2}^{\infty} B_n |b_k| \leq 1 - \mu, \quad (8)$$

where $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$.

Proof. Notice that $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is a subclass of $\mathcal{SH}^0(\lambda, n, \mu)$. Then the “if” part can be easily proved from 2.1 Theorem. For the “only if” part, we must demonstrate that $f_n \notin \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ if the inequality (8) is not valid. Here is an issue to be considered; a necessary and sufficient condition for $f_n = h + \overline{g_n}$ be as stated in (5), to be in $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is that the condition (4) must be provided.

The above stipulation must be valid for all z complex numbers where, $|z| = r < 1$. After selecting the z values from the positive real axis where $0 \leq |z| = r < 1$, we obtain

$$\operatorname{Re} \left\{ \frac{(1 - \mu)z - \sum_{k=2}^{\infty} A_n a_k z^k - \sum_{k=2}^{\infty} B_n b_k \bar{z}^k}{z - \sum_{k=2}^{\infty} C_n a_k z^k + \sum_{k=2}^{\infty} C_n b_k \bar{z}^k} \right\} \geq 0.$$

This is equivalent to

$$\left\{ \frac{(1 - \mu) - \sum_{k=2}^{\infty} A_n a_k r^{k-1} - \sum_{k=2}^{\infty} B_n b_k r^{k-1}}{1 - \sum_{k=2}^{\infty} C_n a_k r^{k-1} + \sum_{k=2}^{\infty} C_n b_k r^{k-1}} \right\} \geq 0. \quad (9)$$

If the inequality (8) is not valid, then the expression in (9) is negative for r values approaching 1. Hence there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (9) is negative. This shows the required condition for $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ and so the proof is complete.

2.3 Theorem

Let $f_n = h + \overline{g_n}$ be as stated in (5). At that case a necessary and sufficient condition for $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is

$$f_n(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)),$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{1 - \mu}{A_n} z^k$$

and

$$g_{n_1}(z) = z, \quad g_{n_k}(z) = z + (-1)^n \frac{1 - \mu}{B_n} \bar{z}^k$$

for $X_k \geq 0$, $Y_k \geq 0$, $\sum_{k=1}^{\infty} (X_k + Y_k) = 1$, $k = 2, 3, \dots, n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$.

Notably, the extreme points of $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For f_n functions which are as stated in (5), we obtain

$$f_n(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z))$$

$$= \sum_{k=1}^{\infty} (X_k + Y_k)z - \sum_{k=2}^{\infty} \frac{1-\mu}{A_n} X_k z^k + (-1)^n \sum_{k=2}^{\infty} \frac{1-\mu}{B_n} Y_k \bar{z}^k.$$

Then

$$\sum_{k=2}^{\infty} \frac{A_n}{1-\mu} \left(\frac{1-\mu}{A_n} X_k \right) + \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} \left(\frac{1-\mu}{B_n} Y_k \right)$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=2}^{\infty} Y_k = 1 - X_1 - Y_1 \leq 1$$

and so $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$. Moreover, if $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$, then

$$a_k \leq \frac{1-\mu}{A_n}$$

and

$$b_k \leq \frac{1-\mu}{B_n}.$$

Set

$$X_k = \frac{A_n}{1-\mu} a_k,$$

$$Y_k = \frac{B_n}{1-\mu} b_k,$$

for $k = 2, 3, \dots$ and

$$X_1 + Y_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + Y_k \right)$$

where $X_k \geq 0$, $Y_k \geq 0$. Therefore, in accordance with, we have

$$f_n(z) = (X_1 + Y_1)z + \sum_{k=2}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z))$$

$$= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)).$$

2.4 Theorem

Let $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$. Then for $|z| = r < 1$ and $n \in \mathbb{N}_0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$, we have

$$r - \frac{1-\mu}{A_n} r^2 \leq |f_n(z)| \leq r + \frac{1-\mu}{A_n} r^2.$$

Proof. Here, we will only prove the right side since the proving of the left side, and the right side of the inequality is very similar. Let $f_n \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$. Taking the absolute value of f_n we can easily see

$$|f_n(z)| \leq r + \sum_{k=2}^{\infty} (a_k + b_k) r^k \leq r + \frac{(1-\mu)r^2}{A_n} \sum_{k=2}^{\infty} \frac{A_n}{1-\mu} |a_k| + \frac{(1-\mu)r^2}{A_n} \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} |b_k| \leq r + \frac{1-\mu}{A_n} r^2.$$

The left side can be shown in a similar way.

2.5 Theorem

The class $\overline{\mathcal{SH}}^0(\lambda, n, \mu)$ is closed under convex combination.

Proof. Let $f_{n_j} \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$ for Let $j = 1, 2, \dots$, where f_{n_j} is given by

$$f_{n_j}(z) = z - \sum_{k=2}^{\infty} a_{k_j} z^k + (-1)^n \sum_{k=2}^{\infty} b_{k_j} \bar{z}^k.$$

Then by (7),

$$\sum_{k=2}^{\infty} \frac{A_n}{1-\mu} a_{k_j} + \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} b_{k_j} \leq 1. \quad (10)$$

For $\sum_{j=1}^{\infty} p_j = 1$, $0 \leq p_j \leq 1$, the convex combination of f_{n_j} can be written as

$$\sum_{j=1}^{\infty} p_j f_{n_j}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{\infty} p_j a_{k_j} \right) z^k + (-1)^n \sum_{k=2}^{\infty} \left(\sum_{j=1}^{\infty} p_j b_{k_j} \right) \bar{z}^k.$$

Then by (10),

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{A_n}{1-\mu} \left(\sum_{j=1}^{\infty} p_j a_{k_j} \right) + \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} \left(\sum_{j=1}^{\infty} p_j b_{k_j} \right) \\ &= \sum_{j=1}^{\infty} p_j \sum_{k=2}^{\infty} \frac{A_n}{1-\mu} a_{k_j} + \sum_{j=1}^{\infty} p_j \sum_{k=2}^{\infty} \frac{B_n}{1-\mu} b_{k_j} \\ &\leq \sum_{j=1}^{\infty} p_j = 1. \end{aligned}$$

This is the situation required by inequality (8). In this way $\sum_{j=1}^{\infty} p_j f_{n_j}(z) \in \overline{\mathcal{SH}}^0(\lambda, n, \mu)$.

References

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- [1] A. Ebadian, S. Azizi, S. Yalçın, "Univalent harmonic mappings and Hardy Spaces", Turkish J. Math. 43(1), 284-292, 2019.
 - [2] A. O. Pall-Szabo, "On a Class of Univalent Functions Defined by Salagean Integro Differential Operator," Miskolc Mathematical Notes, 19(2), 1095-1106, 2018.

- [3] Y. Avcı and E. Zlotkiewicz, "On harmonic univalent mappings," *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, 44, 1-7, 1990.
- [4] N. E. Cho and H. M. Srivastava, "Argument estimates of certain analytic functions defined by a class of multiplier transformations," *Math. Comput. Modelling*, 37, 39-49, 2003.
- [5] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 9, 3-25, 1984.
- [6] T. M. Flett, "The dual of an inequality of Hardy and Littlewood and some related inequalities," *J. Math. Anal. Appl.*, 38, 746-765, 1972.
- [7] J. M. Jahangiri, "Harmonic functions starlike in the unit disk," *J. Math. Anal. Appl.*, 235, 470-477, 1999.
- [8] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, "Salagean-type harmonic univalent functions," *South J. Pure Appl. Math.*, 2, 77-82, 2002.
- [9] G. S. Salagean, "Subclasses of univalent functions," *Lecture Notes in Math. Springer-Verlag Heidelberg*, 1013, 362-372, 1983.
- [10] H. Silverman, "Harmonic univalent functions with negative coefficients," *J. Math. Anal. Appl.*, 220, 283-289, 1998.
- [11] H. Silverman and E. M. Silvia, "Subclasses of harmonic univalent functions," *N. Z. J. Math.*, 28, 275-284, 1999.
- [12] B.A Uralegaddi and C. Somanatha, "Certain classes of univalent functions," *Current topics in analytic function theory*, World Sci. Publishing, Singapore, (Edited by H.M. Srivastava and S. Owa), 371-374, 1992.
- [13] H. Bayram and S. Yalcin, "A subclass of harmonic univalent functions defined by a linear operator," *Palestine Journal of Mathematics*, 6(2):320-326, 2017.
- [14] S. Yalçın, "A new class of Salagean-Type harmonic univalent functions", *Appl. Math. Lett.* 18(2); 191-198, 2005.
- [15] H. Bayram, S. Yalçın, "A New Subclass of Harmonic Univalent Functions Defined By A Linear Operator", *Trans. J. Math. Mech.* 10(2), 63-70, 2018.
- [16] W. G. Atshan and A. K. Wanas, "On a New Class of Harmonic Univalent Functions", *Mathematicki Vesnik*, 65(4), 555-564, 2013.