Meromorphically ς -valent functions defined by integral operator involving $\mathfrak{L}$ -Function in new subclasses

<p>Authors Names a. Ahmed khalaf Radhi. b. Thamer Khalil Al-Khafaji.</p> <p>Article History Received on: 15/5/2020 Revised on: 17/6/2020 Accepted on: 7/7/2020</p> <p>Keywords: integral operator, coefficient inequality $\mathfrak{L}_p^{\alpha,\beta}$, Partial sums, radii of starlikeness and radii convexity. DOI:https://doi.org/10.29350/jops.2020.25.3.1148</p>	<p>ABSTRACT Some relations using in new subclass of meromorphically ς-valent functions $TK(\delta, \gamma, \mu, \lambda)$ defined by integral operator involving $\mathfrak{L}$-function. We get some thing properties, like, coefficient inequality $\mathfrak{L}_\varsigma^{\alpha,\beta}$, growth and distortion bounds, Partial sums, convex set, radii of starlikeness and radii convexity. MSC: 30C45, 30C50</p>
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Introduction.

$$Q(z) = \frac{1}{z^\varsigma} + \sum_{t=1}^{\infty} a_{t-\varsigma} z^{t-\varsigma} ; (t > \varsigma, \varsigma \in \mathbb{N} = \{1, 2, \dots\}) \quad (1)$$

be class of functions denote by TK , including analytic and ς -valent in the punctured unit disk $\Delta^* = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$

Also, denoted by $TK^+ \subseteq TK$ of the form

$$Q(z) = \frac{1}{z^\varsigma} - \sum_{t=1}^{\infty} a_{t-\varsigma} z^{t-\varsigma} \quad (a_{t-\varsigma} \geq 0, \varsigma \in \mathbb{N}) \quad (2)$$

Definition (1): Saxena [12] introduced the $\mathfrak{L}$ - function as follows:

$$\mathfrak{L}(z) = \mathfrak{L}_{\varsigma_i, q_i; r}^{m, t}[Z] = \mathfrak{L}_{\varsigma_i, q_i; r}^{m, t} \left[Z \left| \begin{matrix} (a_j, \alpha_j)_{1, t} (a_j, \alpha_j)_{t+1, \varsigma_i} \\ (b_j, \beta_j)_{1, m} (b_j, \beta_j)_{m+1, q_i} \end{matrix} \right. \right] \\ = \frac{1}{2\pi i} \int_c \sigma(s) z^s ds, \quad (3)$$

where

$$\sigma(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^t \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^r \left\{ \prod_{j=n+1}^{q_i} \Gamma(1 - b_j + \beta_j s) \prod_{j=t+1}^{p_j} \Gamma(a_{ji} - \alpha_{ji} s) \right\}}$$

$\varsigma_i (i = 1, 2, 3, \dots, r)$, $q_i (i = 1, 2, 3, \dots, r)$, m, t are integers satisfying $0 \leq t \leq \varsigma_i$, $0 \leq m \leq q_i$ ($i = 1, 2, 3, \dots, r$), r is finite $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$ are real and positive and a_i, b_i, a_{ij}, b_{ij} are complex numbers such that

$$\alpha_j(b_h + v) \neq \beta_j(a_h - 1 - w).$$

We define the integral operator $\mathfrak{L}_{\varsigma}^{\alpha, \beta}$ of $Q \in TK^+$ for $\alpha, \beta > 0$

$$\mathfrak{L}_{\varsigma}^{\alpha, \beta} Q(z) = \frac{z^{\beta - \varsigma}}{\Gamma(\alpha - \beta) I_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t+1}[Z]} \int_0^z \sigma^{\varsigma - \alpha} (z - \sigma)^{\alpha - \beta - 1} \mathfrak{L}_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t}[\sigma] h(\sigma) d\sigma, \quad (4)$$

where

$$\mathfrak{L}_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t+1}[Z] = \mathfrak{L}_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t+1} \left[Z \left| \begin{matrix} (\alpha, 1) (a_j, \alpha_j) & (a_{ji}, \alpha_{ji}) \\ (b_j, \beta_j) & (b_{ji}, \beta_{ji}) (\beta, 1) \end{matrix} \right. \right], \text{ when}$$

$$\text{Real}(b_j) < \text{Real}(a_j) < 1 + \min_{1 \leq j \leq m} \text{Re}\left(\frac{b_j}{\beta_j}\right),$$

then $\mathfrak{L}_{\varsigma}^{\alpha, \beta} Q(z)$ can be written by

$$Q(z) = \frac{1}{z^{\varsigma}} + \sum_{t=1}^{\infty} a_{t-\varsigma} z^{t-\varsigma}.$$

So

$$\begin{aligned} \mathfrak{b}_{\varsigma}^{\alpha, \beta} Q(z) &= \frac{1}{z^{\varsigma}} + \sum_{t=1}^{\infty} \left[\frac{\mathfrak{b}_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t+1} \left[Z \left| \begin{matrix} (\alpha - \varsigma - t, 1) & (a_j, \alpha_j)(a_{ji}, \alpha_{ji}) \\ (b_j, \beta_j)(b_{ji}, \beta_{ji})(\beta - \varsigma - k, 1) \end{matrix} \right. \right]}{\mathfrak{b}_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t+1} \left[Z \left| \begin{matrix} (\alpha, 1)(a_j, \alpha_j)(a_{ji}, \alpha_{ji}) \\ (b_j, \beta_j)(b_{ji}, \beta_{ji})(\beta, 1) \end{matrix} \right. \right]} \right] a_{t-\varsigma} z^{t-\varsigma} \\ &= \frac{1}{z^{\varsigma}} + \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) a_{t-\varsigma} z^{t-\varsigma}, \end{aligned} \quad (5)$$

where

$$\psi(m, t, \alpha, \beta) = \frac{\mathfrak{b}_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t+1} \left[Z \left| \begin{matrix} (\alpha - \varsigma - t, 1) & (a_j, \alpha_j)(a_{ji}, \alpha_{ji}) \\ (b_j, \beta_j)(b_{ji}, \beta_{ji})(\beta - \varsigma - t, 1) \end{matrix} \right. \right]}{\mathfrak{b}_{\varsigma_i^{+1}, q_i^{+1}; r}^{m, t+1} \left[Z \left| \begin{matrix} (\alpha, 1)(a_j, \alpha_j)(a_{ji}, \alpha_{ji}) \\ (b_j, \beta_j)(b_{ji}, \beta_{ji})(\beta, 1) \end{matrix} \right. \right]}.$$

The I-function extension of Fox's H- function

Now, we study a subclass of a function (2) define below.

Definition (2): Let $Q \in TK^+$ given by (2). Then Q be in the class $TK(\delta, \gamma, \mu, \lambda)$ if it satisfy the condition:

$$\left| \frac{\frac{(-\varsigma-1)z\lambda(\mathfrak{b}_{\varsigma}^{\alpha, \beta} Q(z))'}{z^{-\varsigma}} - (\lambda\varsigma^2 + \lambda\varsigma)}{(-\varsigma^2 - \varsigma) + \frac{z^2(\mathfrak{b}_{\varsigma}^{\alpha, \beta} Q(z))''}{z^{-\varsigma}} - \mu\gamma(\varsigma + \lambda)z^{\varsigma}} \right| < \delta, \quad (7)$$

where $z \in \Delta^*$; $0 \leq \delta < \varsigma$, $0 < \lambda, \gamma < 1$, $0 < \mu < \varsigma$ and $\varsigma \in \mathbb{N}$.

Several authors studied geometric properties of this function subclass for other classes, like, M. K. Aouf [2],[3], W. G. Atshan [4],[5],[6]and[7] and another authors [1],[8],[9],[10] and [11].

Theorem (1): Let $Q \in TK^+$. Then Q is in the class $TK(\delta, \gamma, \mu, \lambda)$ iff

$$\sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1) a_{t-\varsigma} z^{t-\varsigma} \leq \delta\mu\gamma(\varsigma + \lambda), \quad (8)$$

where $z \in \Delta^*$; $0 \leq \delta < \varsigma$, $0 < \lambda, \gamma < 1$, $0 < \mu < \varsigma$ and $\varsigma \in \mathbb{N}$.

For the function the result is sharp

$$Q(z) = \frac{1}{z^\varsigma} + \frac{\delta\mu\gamma(\varsigma + \lambda)}{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)} z^{t-\varsigma}. \quad (9)$$

Proof: By $|z| = 1$, and let the inequality (8) true .Then, we get

$$\begin{aligned} & \left| (-\varsigma - 1)z\lambda \left(\mathfrak{I}_\varsigma^{\alpha, \beta} Q(z) \right)' - (\lambda\varsigma^2 + \lambda\varsigma)z^\varsigma \right| - \delta \left| (-\varsigma^2 - \varsigma)z^\varsigma - z^2 \left(\mathfrak{I}_\varsigma^{\alpha, \beta} Q(z) \right)'' - \mu\gamma(\varsigma + \lambda) \right| \\ &= \left| \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) \lambda (-\varsigma - 1)(t - \varsigma) a_{t-\varsigma} z^{t-\varsigma} \right| \\ & \quad - \delta \left| - \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) (t - \varsigma)(t - \varsigma - 1) a_{t-\varsigma} z^{t-\varsigma} - \mu\gamma(\varsigma + \lambda) \right| \\ &= \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) (\lambda + \delta)(t - \varsigma - 1)(t - \varsigma) a_{t-\varsigma} z^{t-\varsigma} - \mu\gamma(\varsigma + \lambda) \leq o, \end{aligned}$$

by hypothesis. Thus by maximum modulus principle, the function

$$Q(z) \in \text{TK}(\delta, \gamma, \mu, \lambda).$$

Conversely. Let the function $Q(z) \in \text{TK}(\delta, \gamma, \mu, \lambda)$. Then by condition (7), we have

$$\begin{aligned} & \left| \frac{\frac{(-\varsigma-1)z\lambda \left(\mathfrak{I}_\varsigma^{\alpha, \beta} Q(z) \right)'}{z^{-\varsigma}} - (\lambda\varsigma^2 + \lambda\varsigma)}{\frac{z^2 \left(\mathfrak{I}_\varsigma^{\alpha, \beta} Q(z) \right)''}{z^{-\varsigma}} - \mu\gamma(\varsigma + \lambda)z^\varsigma + (-\varsigma^2 - \varsigma)} \right| \\ &= \left| \frac{\sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) \lambda (-\varsigma - 1)(t - \varsigma) a_{t-\varsigma} z^{t-\varsigma}}{\sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) (t - \varsigma)(t - \varsigma - 1) a_{t-\varsigma} z^{t-\varsigma} - \mu\gamma(\varsigma + \lambda)} \right| < \delta. \end{aligned}$$

Since $\text{Real}(z) \leq |z|$ for all z ($z \in \Delta^*$), we get

$$\text{Re} \left\{ \frac{\sum_{k=1}^{\infty} \psi(m, t, \alpha, \beta) \lambda (-\varsigma - 1)(t - \varsigma) a_{t-\varsigma} z^{t-\varsigma}}{\sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) (t - \varsigma)(t - \varsigma - 1) a_{t-\varsigma} z^{t-\varsigma} - \mu\gamma(\varsigma + \lambda)} \right\} < \delta. \quad (10)$$

by choose z on the real axis so that $z \left(\mathfrak{I}_\varsigma^{\alpha, \beta} Q(z) \right)'$ is real.

by clearing the denominator of inequality (10) and let $z \rightarrow 1^-$, through real values

now we can write (10) as,

$$\sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) (\lambda + \delta)(t - \varsigma)(t - \varsigma - 1) a_{t-\varsigma} z^{t-\varsigma} \leq \delta\mu\gamma(\varsigma + \lambda).$$

Sharpness of the result follows by setting

$$Q(z) = z^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma + \lambda)}{\psi(m, y, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)} z^{t-\varsigma}, (y \geq \varsigma). \quad (11)$$

Corollary (1): Let $Q(z) \in TK(\delta, \gamma, \mu, \lambda)$. Then

$$a_{t-\varsigma} \leq \frac{\delta\mu\gamma(\varsigma + \lambda)}{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}. \quad (12)$$

Theorem (2): If $b_n \geq b_1 (n \geq 1)$ and $Q \in TK(\delta, \gamma, \mu, \lambda)$ then

$$\begin{aligned} \frac{1}{r^\varsigma} - \frac{\delta\mu\gamma(\varsigma + \lambda)}{(\lambda + \delta)(1 - \varsigma)(-\varsigma)} r^{-\varsigma} &\leq \left| \mathfrak{b}_\varsigma^{\alpha, \beta} h(z) \right| \\ &\leq \frac{1}{r^\varsigma} + \frac{\delta\mu\gamma(\varsigma + \lambda)}{(\lambda + \delta)(1 - \varsigma)(-\varsigma)} r^{-\varsigma}, (|z| = r < 1). \end{aligned} \quad (13)$$

The result is sharp:

$$Q(z) = z^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma + \lambda)}{\psi(m, y, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)} z^{t-\varsigma}. \quad (14)$$

Proof: Let $Q \in TK(\delta, \gamma, \mu, \lambda)$, then by Theorem (1), we get

$$\sum_{t=1}^{\infty} a_{t-\varsigma} \leq \frac{\delta\mu\gamma(\varsigma + \lambda)}{\psi(m, 1, \alpha, \beta)(\lambda + \delta)(1 - \varsigma)(-\varsigma)}. \quad (15)$$

then

$$\begin{aligned} \left| \mathfrak{b}_\varsigma^{\alpha, \beta} Q(z) \right| &\leq |z|^{-\varsigma} + \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) a_{t-\varsigma} |z|^{t-\varsigma} \\ &\leq |z|^{-\varsigma} + \psi(m, 1, \alpha, \beta) |z|^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma} \\ &= r^{-\varsigma} + \psi(m, t, \alpha, \beta) r^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma} \\ &\leq r^{-\varsigma} + \frac{\delta\mu\gamma(\varsigma + \lambda)}{(\lambda + \delta)(t - \varsigma)(-\varsigma)} r^{-\varsigma}. \end{aligned} \quad (16)$$

By the same way

$$\begin{aligned}
 \left| \mathfrak{b}_\varsigma^{\alpha, \beta} h(z) \right| &\geq \frac{1}{|z|^\varsigma} - \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta) a_{t-\varsigma} |z|^{t-\varsigma} \\
 &\geq \frac{1}{|z|^\varsigma} - \psi(m, 1, \alpha, \beta) |z|^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma} \\
 &= r^{-\varsigma} - \psi(m, 1, \alpha, \beta) r^{-\varsigma} \sum_{t=1}^{\infty} a_{t-\varsigma} \\
 &\leq r^{-\varsigma} + \frac{\delta \mu \gamma (\varsigma + \lambda)}{(\lambda + \delta)(t - \varsigma)(-\varsigma)} r^{-\varsigma}
 \end{aligned} \tag{17}$$

we get (13) by (16) and (17). ■

Theorem (3): Let $b_n \geq b_1 (n \geq 1)$ and $Q \in TK(\delta, \gamma, \mu, \lambda)$,

$$\begin{aligned}
 \frac{1}{\varsigma r^{\varsigma+1}} - \frac{\delta \mu \gamma (\varsigma + \lambda)}{(\lambda + \delta)(t - \varsigma)} r^{-\varsigma-1} &\leq \left| \mathfrak{b}_\varsigma^{\alpha, \beta} Q(z)' \right| \\
 &\leq \frac{1}{\varsigma r^{\varsigma+1}} + \frac{\delta \mu \gamma (\varsigma + \lambda)}{(\lambda + \delta)(t - \varsigma)} r^{-\varsigma-1}, (|z| = r < 1).
 \end{aligned} \tag{18}$$

For the function the result is sharp

$$Q(z) = z^{-\varsigma} + \frac{\delta \mu \gamma (\varsigma + \lambda)}{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)} z^{t-\varsigma}. \tag{19}$$

Proof: By Theorem (1) and let $Q \in TK(\delta, \gamma, \mu, \lambda)$, get

$$\sum_{t=1}^{\infty} a_{t-\varsigma} \leq \frac{\delta \mu \gamma (\varsigma + \lambda)}{\psi(m, 1, \alpha, \beta)(\lambda + \delta)(1 - \varsigma)(-\varsigma)}. \tag{20}$$

Hence

$$\left| \mathfrak{b}_\varsigma^{\alpha, \beta} Q(z)' \right| \leq \frac{1}{|-\varsigma z|^{\varsigma+1}} + \sum_{t=1}^{\infty} (t - \varsigma) a_{t-\varsigma} |z|^{t-\varsigma-1}$$

$$\begin{aligned}
&\leq \frac{1}{|\zeta z|^{\zeta+1}} + \psi(m, 1, \alpha, \beta)(-\zeta)|z|^{-\zeta-1} \sum_{t=1}^{\infty} a_{t-\zeta} \\
&= \frac{1}{\zeta r^{\zeta+1}} + \psi(m, 1, \alpha, \beta)(-\zeta)r^{-\zeta-1} \sum_{t=1}^{\infty} a_{t-\zeta} \\
&\leq \frac{1}{\zeta r^{\zeta+1}} + \frac{\delta\mu\gamma(\zeta + \lambda)}{(\lambda + \delta)(t - \zeta)} r^{-\zeta-1} .
\end{aligned} \tag{21}$$

By the same way

$$\begin{aligned}
\left| \mathfrak{b}_{\zeta}^{\alpha, \beta} Q(z)' \right| &\geq \frac{1}{|-z\zeta|^{\zeta+1}} - \sum_{y=1}^{\infty} (t - \zeta) a_{t-\zeta} |z|^{t-\zeta-1} \\
&\geq \frac{1}{|z\zeta|^{\zeta+1}} - \psi(m, 1, \alpha, \beta)(-\zeta)|z|^{-\zeta-1} \sum_{t=1}^{\infty} a_{t-\zeta} \\
&= \frac{1}{\zeta r^{\zeta+1}} - \psi(m, 1, \alpha, \beta)(-\zeta)r^{-\zeta-1} \sum_{y=1}^{\infty} a_{t-\zeta} \\
&\geq \frac{1}{\zeta r^{\zeta+1}} - \frac{\delta\mu\gamma(\zeta + \lambda)}{(\lambda + \delta)(t - \zeta)} r^{-\zeta-1}
\end{aligned} \tag{22}$$

we get (18), from (22) and (21) ■

Theorem (4): The partial sums is

$S_1(z)$ and $S_w(z)$ as follows $S_1(z) = z^{-\zeta}$ and let $Q \in TK^+$ be given by (2) and

$$S_w(z) = z^{-\zeta} + \sum_{t=1}^{w-1} a_{t-\zeta} z^{t-\zeta} \quad (w \in \mathbb{N} \setminus \{1\}). \tag{23}$$

and suppose that

$$\sum_{t=1}^{\infty} c_{t-\zeta} a_{t-\zeta} \leq 1, \quad \left(c_{k-\zeta} = \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \zeta)(t - \zeta - 1)}{\delta\mu\gamma(\zeta + \lambda)} \right). \tag{24}$$

we have

$$\operatorname{Re} \left(\frac{Q(z)}{s_w(z)} \right) > 1 - \frac{1}{c_w}, (w \in \mathbb{N}) \quad (25)$$

and

$$\operatorname{Re} \left(\frac{s_w(z)}{Q(z)} \right) > \frac{c_w}{1 + c_w} \quad (k \in \mathbb{N}). \quad (26)$$

The bounds in (25) and (26), the good possible for $t \in \mathbb{N}$

Proof: By (24), that

$$c_{t-\zeta+1} > c_{t-\zeta} > 1, (t \in \mathbb{N})$$

then, we have

$$\sum_{t=1}^{w-1} a_{t-\zeta} + c_w \sum_{t=w}^{\infty} a_{t-\zeta} \leq \sum_{y=1}^{\infty} c_{t-\zeta} a_{t-\zeta} \leq 1. \quad (27)$$

By

$$g_1(z) = c_w \left(\frac{h(z)}{s_w(z)} - \left(1 - \frac{1}{c_w} \right) \right) = 1 + \frac{c_w \sum_{n=t}^{\infty} a_{t-\zeta} z^{t-\zeta+1}}{1 + \sum_{t=1}^{w-1} a_{t-\zeta} z^{t-\zeta+1}}, \quad (28)$$

then show that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq 1, \quad (z \in \Delta^*) \quad (29)$$

by applying (28)

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{c_w \sum_{n=t}^{\infty} a_{t-\zeta}}{2 - 2 \sum_{y=1}^{w-1} a_{t-\zeta} - c_w \sum_{t=w}^{\infty} a_{t-\zeta}}, \quad (30)$$

which yields (25), if we take

$$Q(z) = z^{-\zeta} - \frac{z^{w-\zeta}}{c_w}, \quad (31)$$

So, (25) is the good possible for each $k \in \mathbb{N}$.

By the same way

$$g_2(z) = (1 + c_k) \left(\frac{s_w(z)}{h(z)} - \frac{c_w}{1 + c_w} \right) = 1 - \frac{(1 + c_w) \sum_{w=t}^{\infty} a_{t-\zeta} z^{t-\zeta+1}}{1 + \sum_{t=1}^{\infty} a_{t-\zeta} z^{t-\zeta+1}}, \quad (32)$$

and use of (27), we obtain

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + c_k) \sum_{k=w}^{\infty} a_{t-\zeta}}{2 - 2 \sum_{t=1}^{w-1} a_{t-\zeta} - (1 - c_w) \sum_{n=t}^{\infty} a_{t-\zeta}} \leq 1, \quad (33)$$

which leads us to (26). The bound in (26) is sharp for each $w \in \mathbb{N}$, when function is given by (31). ■

Theorem (5): Convex set is the class $TK(\delta, \gamma, \mu, \lambda)$.

Proof: For all Q_1 and $Q_2 \in TK(\delta, \gamma, \mu, \lambda)$, and for every $\mathfrak{z}$ ($0 \leq \mathfrak{z} \leq 1$).

To prove $(1 - \mathfrak{z})Q_1 + \mathfrak{z}Q_2 \in TK(\delta, \gamma, \mu, \lambda)$.

Thus,

$$(1 - \mathfrak{z})Q_1 + \mathfrak{z}Q_2 = z^{-\zeta} + \sum_{t=1}^{\infty} [(1 - \mathfrak{z})a_{t-\zeta} + \mathfrak{z}b_{t-\zeta}] z^{t-\zeta}.$$

then,

$$\begin{aligned} & \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \zeta)(t - \zeta - 1) [(1 - \mathfrak{z})a_{t-\zeta} + \mathfrak{z}b_{t-\zeta}] \\ &= (1 - \mathfrak{z}) \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \zeta)(t - \zeta - 1)a_{t-\zeta} \\ & \quad + \mathfrak{z} \sum_{t=1}^{\infty} \psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \zeta)(t - \zeta - 1)b_{t-\zeta} \\ & \leq (1 - \mathfrak{z})\delta\mu\gamma(\zeta + \lambda) + \mathfrak{z}\delta\mu\gamma(\zeta + \lambda) = \delta\mu\gamma(\zeta + \lambda). \quad \blacksquare \end{aligned}$$

Theorem (6): If the function $Q(z) \in TK(\delta, \gamma, \mu, \lambda)$, then function $Q(z)$ is ζ -valent meromorphic starlike of order φ ($0 \leq \varphi < \zeta$) in the disk $|z| < r$, if

$$r = \inf_n \left\{ \frac{(\varsigma - \varphi)}{(n + \varsigma - \varphi)} \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)} a_{t-\varsigma} \right\}^{\frac{1}{n}}, n \geq \varsigma.$$

For the function $Q(z)$ the result is sharp.

Proof: We need to show that

$$\left| \frac{zQ'(z) + \varsigma Q(z)}{Q(z)} \right| \leq \varsigma - \varphi \quad \text{for } |z| < r. \quad (34)$$

Therefore

$$\left| \frac{\sum_{t=1}^{\infty} (t) a_{t-\varsigma} z^n}{1 + \sum_{t=1}^{\infty} a_{t-\varsigma} z^n} \right| \leq \frac{\sum_{t=1}^{\infty} (t) a_{t-\varsigma} |z|^n}{1 - \sum_{t=1}^{\infty} a_{t-\varsigma} |z|^n}$$

So, (34) satisfied if

$$\frac{\sum_{t=1}^{\infty} (t) a_{t-\varsigma} |z|^n}{1 - \sum_{t=1}^{\infty} a_{t-\varsigma} |z|^n} \leq \varsigma - \varphi,$$

or satisfied if

$$\sum_{t=1}^{\infty} \frac{(t + \varsigma - \varphi) a_{t-\varsigma}}{\varsigma - \varphi} |z|^n \leq 1. \quad (35)$$

Since $Q(z)$ is in the class $TK(\delta, \gamma, \mu, \lambda)$, then

$$\sum_{t=1}^{\infty} \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)} a_{t-\varsigma} \leq 1.$$

So, (35) true if

$$\frac{(n + \varsigma - \varphi)}{\varsigma - \varphi} |z|^n \leq \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)}$$

or equivalent

$$|z| \leq \left\{ \frac{(\varsigma - \varphi)}{(n + \varsigma - \varphi)} \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)} a_{t-\varsigma} \right\}^{\frac{1}{n}}, n \geq \varsigma,$$

Setting $|z| = r$. ■

Theorem (7): If $Q(z) \in TK(\delta, \gamma, \mu, \lambda)$, then $Q(z)$ is ς -valent meromorphic convex of order φ ($0 \leq \varphi < \varsigma$) in the disk $|z| < r$, where

$$r = \inf_n \left\{ \frac{\varsigma(\varsigma - \varphi)\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{(t + \varsigma - \varphi)\delta\mu\gamma(\varsigma + \lambda)} \right\}^{\frac{1}{n+\varsigma}}, n \geq \varsigma.$$

For the function $Q(z)$ the result is sharp.

Proof: We need to prove

$$\left| (\varsigma + 1) + \frac{zQ''(z)}{Q'(z)} \right| \leq \varsigma - \varphi \quad \text{for } |z| < r. \quad (36)$$

So,

$$\left| \frac{zQ''(z) + (1 + \varsigma)Q'(z)}{Q'(z)} \right| \leq \frac{\sum_{t=1}^{\infty} t(t - \varsigma)a_{t-\varsigma}|z|^{t-\varsigma}}{\varsigma - \sum_{t=1}^{\infty} (t - \varsigma)a_{t-\varsigma}|z|^{t-\varsigma}}.$$

(36) satisfied if

$$\frac{\sum_{t=1}^{\infty} t(t - \varsigma)a_{t-\varsigma}|z|^{t-\varsigma}}{\varsigma - \sum_{t=1}^{\infty} (t - \varsigma)a_{t-\varsigma}|z|^{t-\varsigma}} \leq \varsigma - \varphi,$$

or if

$$\sum_{t=1}^{\infty} \frac{(t + \varsigma - \varphi)a_{t-\varsigma}}{\varsigma(\varsigma - \varphi)} |z|^{t-\varsigma} \leq 1. \quad (37)$$

Since $(z) \in TK(\delta, \gamma, \mu, \lambda)$, , we have

$$\sum_{t=1}^{\infty} \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)} a_{t-\varsigma} \leq 1.$$

Hence, (37) will be true if

$$\frac{(t + \varsigma - \varphi)}{\varsigma(\varsigma - \varphi)} |z|^{t-\varsigma} \leq \frac{\psi(m, t, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{\delta\mu\gamma(\varsigma + \lambda)},$$

or equivalently $a_{t-\varsigma}$

$$|z| \leq \left\{ \frac{\varsigma(\varsigma - \varphi)\psi(m, \gamma, \alpha, \beta)(\lambda + \delta)(t - \varsigma)(t - \varsigma - 1)}{(t + \varsigma - \varphi)\delta\mu\gamma(\varsigma + \lambda)} \right\}^{\frac{1}{n}}, n \geq \varsigma,$$

put $|z| = r$. ■

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