Constrained Spline approximation

Received : 18/2/2018  Accepted: 29/5/2018

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Abstract:
In this paper, we find the relationships between the order of constrained approximation by a polynomials which is copositive with a function \( f \), by modulus of smoothness \( \tau_{k-1} \), to the function \( \hat{f} \), which is multiply by \( J_c \), and between the best approximation to the pairs of "intertwining splines of a polynomials" in \( I = [-b, b] \), and the same of relationships but by the modulus of smoothness \( \omega_{k-1}^\varphi \) to the function \( \hat{f} \) multiply by \( n^{-1}|J_c| \). Also we find the order of constrained approximation to the pairs of intertwining polynomials in \( (I_i \cup I_{i+1} ) \).

Key words: approximation, intertwining, spline, modulus of smoothness.

1. Introduction and definitions:
In this paper we study how well can find the order of a constrained approximations by a pair of "intertwining spline" which is coming from combination of overlapping polynomial pieces enfold contaminated period. The polynomial \( P_i \in \Pi_r \) which is copositive with a function
$f \in L_{\psi,p}(I) \cap \Delta^0(I), I \subseteq R, 0 < p < 1$. Let $f$ be a function which is a positive finitely many time say $s \geq 1$, on $I$ say $-b < j_0 < \ldots < j_1 < b = j_0$ for $j_s \in \mathbb{J}_s$.

Recall that the order "Ditzain-Totik modulus of smoothness" is given by [5]:

$$\omega^k_\psi(f, \delta, I)_{\psi,p} = \sup_{0 < h \leq \delta} \|\Delta^k(f, \cdot, \cdot)\|_{L_{\psi,p}(I)}.$$  

Where $\| \cdot \|_{L_{\psi,p}(I)}$ symbolize the weighted quasi normed space [5] on an interval $[-b, b] \subseteq I \subseteq R$. "The weighted quasi normed space" $L_{\psi,p}(I), 0 < p < 1$. have form:

$$L_{\psi,p}(I) = \left\{ f \exists f : I \subseteq R \rightarrow R : \left( \int_I \left| f(x) \psi(x) \right|^p dx \right)^{1/p} < \infty, 0 < p < 1 \right\}$$

and the quasi normed $\| f \|_{L_{\psi,p}(I)} < \infty$, and

$$\Delta^k_\psi(f, x, I)_{\psi} = \Delta^k_\psi(f, x)_{\psi} = \left\{ \sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i} \frac{f(x+\frac{kh}{2}) - f(x-\frac{kh}{2})}{\psi(x+\frac{kh}{2})} \right\}_{x \pm \frac{kh}{2} \in I} o.w$$

is the symmetric difference [5] and in this paper we used new "chebyshev partition $X_j = a \cos\frac{j\pi}{n}[5].$

Let $(\delta = \min|j_{i+1} - j_i|, 0 \leq i \leq s)$ where $j_0 = -b$ and $j_{s+1} = b$, [5], and let $\mathbb{I}_i = \left[ i^{(v)}_i , i^{(v-1)}_i \right], i = 1, \ldots, s$, such that $i_{1} < i_{2} < \ldots < i_{j_{i}^{(\psi-1)}}, v = 1, \ldots, k - 2.$

$$\mathbb{J}_s^{*} = \left\{ i^{(v)}_i , v = 1, \ldots, (k - 1) \right\} , X_0 = \left\{ i^{(v)}_i , 0 < i \leq s, j = 1, \ldots, n \right\} .$$

The simple $\Delta^0(\mathbb{J}_s^{*})$, denoted to the set of all functions $(f), \exists (-1)^{s-i}f(x) \geq 0$.

We call the period $J_c = [b - \mu |l|, b], J_k = [-b + \mu |l|, b]$, contaminated period if $(b - \mu |l| < i_1 \leq b), (-b + \mu |l| < i_1 \leq b)$, respectively for some point $j_i \in \mathbb{J}_s$, and $\mu > 0$ be affixed.

2. Auxiliary Result:

In the following theorems we referred to the relationships when $\left\{ \mathcal{S}, \mathcal{S} \right\},$ appear a pair of "intertwining
spline" has an order \( (r) \), on the knot sequence \( \{x_i\}_{i=0}^n \), and \( \{p_i\}, i = 1, 2, \) a pair of "intertwining polynomials" has an order \( < r \), and a constant \( C \), in all relation in prove of the following theorems are dependent on \( (p, \kappa) \).

**Theorem (1):** Let \( f \in L_{\psi, p}(I) \cap \Delta^0(j_s), 0 < p < 1, j_s = \{i_1, ..., i_s : -b < i_s < ... < i_1 < b = j_0 \}, s \geq 0. \) Let \( \{x_i\}_{i=0}^n \), be a knot sequence, then there an intertwining pair of spline \( \{s, s\}, \) of order \( (r) \), on \( \{x_i\}_{i=0}^n \), and \( s - f, s - f \in \Delta^0(j_s) \) \( \exists \) \( ||s - s||_{L_{\psi, p}(l)} \leq C||f||_{L_{\psi, p}(j_s)}k^{-1}(f, |I|)_{\psi, p}. \)

**Proof:**

By (Theorem (2.1.3) [5]) \( \exists \) a polynomials \( P_1 \in \Pi_r \cap \Delta^0(j_s) \) \( \exists \)
\[ ||f - p_1||_{L_{\psi, p}(j_c)} \leq C\tau^{k-1}(f, |J_c|)_{\psi, p}. \]

Also \( \exists \) a polynomials \( Q_1 \in \Pi_r \cap \Delta^0(j_s) \) \( \exists \)
\[ ||f - Q_1||_{L_{\psi, p}(j_c)} \leq C\tau^{k-1}(f, |J_c|)_{\psi, p}. \]

Now, for one sided polynomial approximation on an period \( J_c, \) \( \exists \) two polynomials \( P_i, Q_i, i = 1, ..., s \) of degree \( < r, \exists \)
\[ P_i \geq f \geq Q_i, \forall x \in J_c, \] which is satisfies:
\[ ||P_i - Q_i||_{L_{\psi, p}(j_c)} \leq C\tau^{k-1}(f, |J_c|)_{\psi, p}. \]

Let \( p_i, q_i, \) on \( J_c, \) are two polynomials, define form \( p_i = P_i, q_i = Q_i, \) if \( (-1)^{s-i} > 0, \) and \( p_i = Q_i, q_i = P_i, \) if \( (-1)^{s-i} < 0, \) hence \( \leq \)
\[ (1)^{s-i}(p_i(x) - f(x)) \geq 0, (1)^{s-i}(q_i(x) - f(x)) \leq 0, x \in J_c. \]

And
\[ ||p_i - q_i||_{L_{\psi, p}(j_c)} = ||P_i - Q_i||_{L_{\psi, p}(j_c)} \leq C\tau^{k-1}(f, |J_c|)_{\psi, p}. \]

By ([3]) we have the contrast:
\[ \tau^{k}(f, t)_p \leq C t \tau^{k-1}(f, |J_c|)_{\psi, p} , \] hence we get
\[ ||p_i - q_i||_{L_{\psi, p}(j_c)} \leq C|Jc|\tau^{k-1}(f, |J_c|)_{\psi, p} \ldots \ldots (1) \]

After we find a polynomials which are approximation with the function \( f, \) now we are blend them for smoothness spline approximation on \( \{x_i\}_{i=0}^n \). Let \( j_i \in j_s, \) and \( J_c, J_k, \) are non-contaminated, then \( p_i, p_i-1, \) overlap on \( J_k, \) which contains \( (d), \) interior knots from \( \{x_i\}_{i=0}^n \).
By " Beaton’s Lemma [3] ; there must be a combination of polynomials overlapping on a sub interval of $\Xi_i$ .

Furthermore the outline of $\Xi_i$ , located between those of $p_i , p_{i-1}$ , hence

$s_{\text{sgn}} (p_{i-1}(x) - f(x)) = s_{\text{sgn}} (p_i(x) - f(x)) = s_{\text{sgn}} (\Xi_i(x) - f(x))" , x \in J_k .

By the same method $\exists$ a spline $\Xi_i$ , of order $(r)$ , at $J_k , \{X_i\}_{i=0}^n$ , that relate with $q_{i-1}$ and $q_i$ , and the outline of $\Xi_i$ , located between those of $q_i , q_{i-1}$ , hence

$s_{\text{sgn}} (q_{i-1}(x) - f(x)) = s_{\text{sgn}} (q_i(x) - f(x)) = s_{\text{sgn}} (\Xi_i(x) - f(x))" , x \in J_k .

By (1) for $J_c \subset J_k , \{X_i\}_{i=0}^n$ , we get:

$$||\Xi_i - S||_{\mathcal{L}_{\psi,p}(J_k)} \leq C \left(||p_{i-1} - q_{i-1}||_{\mathcal{L}_{\psi,p}(J_k)} + C ||p_i - q_i||_{\mathcal{L}_{\psi,p}(J_k)}\right) ,$$

by (2) for $J_c \subset J_k , \{X_i\}_{i=0}^n$ , we get:

$$||\Xi_i - S||_{\mathcal{L}_{\psi,p}(J_k)} \leq C|J_c|r^{-k-1}(\hat{f} , |J_k|, J_k)_{\psi,p} .$$

Now , if there is only one of a polynomial $p_i$ , over $I$ , set $\Xi_i$ , to this polynomial . If there are two polynomials overlapping on $I$ , then there must be a combination spline $\Xi_i$ , set $\Xi_i$ , to $\Xi_i$ , it is clear

$\Xi - f \in \Delta^0(J_\Xi) , \text{on } I$ , by the same manner $\Xi - f \in \Delta^0(J_\Xi) , \text{on } I$ .

Hence by (1) and (2) , we get on $I$ :

$$||\Xi - S||_{\mathcal{L}_{\psi,p}(I)} \leq C|J_c|r^{-k-1}(\hat{f} , |I|, I)_{\psi,p} .$$

$J_c \subset I = [-b , b]$ .

**Theorem (2):** Let $f \in \mathcal{L}_{\psi,p}(I) \cap \Delta^0(J_\Xi) , 0 < p < 1 , J_\Xi = \{i_1 , \ldots , i_s : -b < i_1 < \ldots < i_1 < b = j_0 \} , s \geq 0$ .

And $\{\Xi_i , \Xi\} , \text{of order } (r)$ , on $\{X_i\}_{i=0}^n$ , is the same as in theorem (1) .

Then :

$$||\Xi - S||_{\mathcal{L}_{\psi,p}(I)} \leq Cn^{-1}|J_c|\omega^{-1}(\hat{f} , n^{-1})_{\psi,p} .$$

**Proof:**

After found the polynomials that intertwining with $f$ , on $\Xi_i$ , and which have a approximate regulation, these multipliers will be concerted for smooth spline approximants $\Xi$ and $\Xi$ , on $\{X_i\}_{i=0}^n$ .

Using the same manner of the proof in theorem (1) we get on $\Xi_\Xi$ , that :

$$||\Xi - S||_{\mathcal{L}_{\psi,p}(\Xi)} \leq C|J_c|r^{-k-1}(\hat{f} , |\Xi_\Xi|, \Xi_\Xi)_{\psi,p} .$$

Where $\bigcup_{\Xi} \Xi_\Xi = I = [-b , b] , \Xi = 1 , \ldots , s$ , and every $x \in I$ , is contained in at most $(k)$ , sub interval of $\Xi_i$ .

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Then the assessment can be on the other hand afflicted for on $l_1$:

$$\sum_{i=1}^{s} \tau_k (\hat{f}, |l_i|, l_i)_{\psi,p} \leq C \sum_{i=1}^{s} |l_i|^p \omega_{\phi}^{k-1}(\hat{f}, |l_i|, l_i)_{\psi,p}.\]

By (Theorem (1.6.3) [5]), we get

$$C \sum_{i=1}^{s} \tau_k (\hat{f}, |l_i|, l_i)_{\psi,p} \leq C \sum_{i=1}^{s} |l_i|^p \omega_{\phi}^{k-1}(\hat{f}, |l_i|, l_i)_{\psi,p}.\]

$$Cn^{-p} \omega_{\phi}^{k-1}(\hat{f}, n^{-1})_{\psi,p}.$$

That is:

$$\|\overline{S} - S\|_{L_{\psi,p}(I)} \leq C|f|_{\psi,p} \sum_{i=1}^{s} \tau_{k-1} (\hat{f}, |l_i|, l_i)_{\psi,p} \leq C|f|_{\psi,p} \sum_{i=1}^{s} |l_i|^p \omega_{\phi}^{k-2}(\hat{f}, |l_i|, l_i)_{\psi,p}.\]

$$\leq Cn^{-p} |f|_{\psi,p} \omega_{\phi}^{k-2}(\hat{f}, n^{-1})_{\psi,p}.$$

Hence

$$\|\overline{S} - S\|_{L_{\psi,p}(I)} \leq Cn^{-1} |f|_{\psi,p} \omega_{\phi}^{k-2}(\hat{f}, n^{-1})_{\psi,p}.\]

**Theorem (3):** Let $I_s = \{l_1, \ldots, l_s : -b < l_s < \cdots < l_1 < b = j_0 \}, s \geq 0$.

And $S(\chi)$, be a spline of an order $r$ on $[X_i]_{l=0}^{n}$. Then there exist "an intertwining of a polynomials" $\{P_1, P_2\},$ for $S(x)$, with respect to $J_s \exists$

$$\|P_1 - P_2\|_{L_{\psi,p}(I_s \cup I_{i+1})} \leq C \omega_{\phi}^{k}(\hat{f}, |l_i| \cup l_{i+1})_{\psi,p}.\]

**Proof:**

Firstly, we show that the polynomial:

$$P_i(x) = P_{i+1}(x) + \sum_{j=0}^{s} \gamma_j \mathcal{T}_{i,j}(x),$$

$i = 1, 2$.

Where $P_{i+1}(x)$, be a polynomial of degree $\leq r$, and

$$\gamma_j = \frac{\delta^{(2m)}(x) - \delta^{(2m)}(x)}{(2m)!}, j = 0, \ldots, s,$$ also

$$\mathcal{T}_{i,j}(x) = \begin{cases} 3_{a,j}(x) & a = i + 1 \\ \chi_{a,j}(x) & a = i + 2 \end{cases},$$

be a polynomial of degree $\leq r$, that pleasure the confirmation $(p_1)$ of theorem (1), and which is pleasure the conclusion in (Lemma (5.22) [1]), and

$$\chi_{a,j}(x) = \begin{cases} 1 & x \geq x_a \\ 0 & o.w \end{cases}, x \in I, \exists$$

$$|X_{a,j}(x) - \chi_{a,j}(x)| \leq C(M)\psi_{\phi}^{M}(x),$$

where $M \in N$, be a fixed. It is comparatively directly transmit to confirm, then:
\[ \| \mathcal{P}_1 - \mathcal{P}_2 \|_{L_{\psi,p}(t_i)} = \left\| \sum_{j=0}^{n} y_j (T_{1,j} - T_{2,j}) \right\|_{L_{\psi,p}(t_i)} \]
\[ \leq C \sum_{j=0}^{n} \| y_j \|_{L_{\psi,p}(t_i)} | T_{1,j}(x) - T_{2,j}(x) | \]
\[ \leq C \sum_{j=0}^{n} \| S_j^{(2m)} \|_{L_{\psi,p}(t_i)} \| \triangle a,j(x) - \chi a,j(x) \| \]
\[ \leq C \sum_{j=0}^{n} \| S_{j+1} - S_j \|_{L_{\psi,p}(t_i)} \| \psi^{M-2m}_{a,j}(x) \| , \]

Where \( S_{j+1}(x) \), denotes for “smooth spline approximation” on \( \{ x_i \}_{i=1}^{n} \),
\[ \mathbb{S}_{j+1}(x) \equiv \mathbb{S} I_{t_i} , \quad and \quad \mathbb{S}_{j}(x) \equiv \mathbb{S} I_{t_i} , \quad x \in I_{t_i} . \]

That is sense \( S_{j+1}, S_{j} \), are pleasure the confirmation of (Theorem (1)), hence
\[ \mathbb{S} - \mathbb{f} , \quad \mathbb{S}_{j} - \mathbb{f} \in \Delta^0(\mathbb{I}_{t_i}), \]
where \( \mathbb{f} \in L_{\psi,p}(t_i) \cap \Delta^0(\mathbb{I}_{t_i}) , \)
\[ \| \mathbb{S}_{j+1} - \mathbb{S}_{j} \|_{L_{\psi,p}(t_i)} = \| \mathbb{S} - \mathbb{f} \|_{L_{\psi,p}(t_i)} \]
\[ \| \mathcal{P}_1 - \mathcal{P}_2 \|_{L_{\psi,p}(t_i)} \]
\[ \leq \sum_{j=0}^{n} C \| \mathbb{S} \| \]
\[ - \mathbb{S} \| L_{\psi,p}(t_i) \| \psi^{M-2m}_{a,j}(x) \quad . \]

Since \( I_{t_1} \subset I_1 \cup I_{t_1+1} \),
\[ \| \mathcal{P}_1 - \mathcal{P}_2 \|_{L_{\psi,p}(1 \cup I_{t_1+1})} \]
\[ \leq \| \mathbb{S} \| \]
\[ - \mathbb{S} \| L_{\psi,p}(1 \cup I_{t_1+1}) \| \left( \sum_{j=0}^{n} C \| \psi^{M-2m}_{a,j}(x) \| \right) , \]

Not that \( \sum_{j=0}^{n} C \| \psi^{M-2m}_{a,j}(x) \| \leq C(p,k,M) , \quad M = 2m \geq 2 , \)
\[ \| \mathcal{P}_1 - \mathcal{P}_2 \|_{L_{\psi,p}(1 \cup I_{t_1+1})} \]
\[ \leq C \| \mathbb{S} \| \]
\[ - \mathbb{S} \| L_{\psi,p}(1 \cup I_{t_1+1}) \| + C \| \mathbb{S} \| \]
\[ - P \| L_{\psi,p}(1 \cup I_{t_1+1}) \quad = \mathbb{L}_1 + \mathbb{L}_2 . \]

\[ \mathbb{L}_1 \leq C \| \mathbb{S} - \mathbb{f} \|_{L_{\psi,p}(1 \cup I_{t_1+1})} + C \| \mathbb{f} \| \]
\[ - P \| L_{\psi,p}(1 \cup I_{t_1+1}) \quad . \]

And
\[ \mathbb{L}_2 \leq C \| \mathbb{S} - \mathbb{f} \|_{L_{\psi,p}(1 \cup I_{t_1+1})} + C \| \mathbb{f} \| \]
\[ - P \| L_{\psi,p}(1 \cup I_{t_1+1}) \quad . \]
By [Lemma (5.3.4), Theorem (1.6.1) and Theorem (2.1.2)] ([5]), we get

\[ L_1 \leq C \omega_p^k(f, |l_1 \cup l_{i+1}|, l_1 \cup l_{i+1})_{\psi,p}, \]

also

\[ L_2 \leq C \omega_p^k(f, |l_1 \cup l_{i+1}|, l_1 \cup l_{i+1})_{\psi,p}. \]

Hence

\[ \|p_1 - p_2\|_{L_p, p(I \cup l_{i+1})} \leq C \omega_p^k(f, |l_1 \cup l_{i+1}|, l_1 \cup l_{i+1})_{\psi,p}. \]

Where \( C \) be a constant which is dependent on \( p, k, M \).

References


