A novel approach on neutrosophic crisp sets

Received :27/ 12 / 2017 Accepted : 7 / 6 /2018

Saied A. Johnny Hashmiya Ibrahim Nasser
AL-Qadisiyah Education - Iraq AL-Qadisiyah Education - Iraq
Saied.jhony@iraqima.org hashimea.ibrahem@iraqima.org

ABSTRACT

The focus of this paper is to introduce a new type of neutrosophic crisp sets as the neutrosophic crisp soft sets and which is the generalization of an ordered triple in the definition of Salama set's [9]. After given the fundamental definitions of generalized neutrosophic crisp set operations, we obtain several properties and we discussed some theorems in this concept. Finally, the concept to the neutrosophic crisp soft lattices.

Keyword: \(nc\)-set, \(ncs\)-set, \(ncs\)-lattices.

1. Introduction:

Soft set theory was firstly introduced by molodtsov in 1999 , [1] , as a general mathematical for dealing tool for dealing with problems that contain uncertainty. The algebraic structure of soft set theory also has been studied in more detail [2] , [3] , [4] and [5]. Smarandache defined the notion of neutrosophic sets, which is a generalization of Zadeh's fuzzy set and Atanassov's intuitionistic fuzzy set. Neutrosophic crisp sets have been investigated by Salama et al. [9], [10], [11] and [13]. In this paper is to introduce and study some new neutrosophic crisp soft notions via neutrosophic crisp soft lattices.

2. Terminologies:

We recollect some relevant basic preliminaries and in particular, the work of Molodtsov D. in [1], Faruk K. [8] and Salama et al. [9], [11] and [13].

Definition (2.1) [1]:

Let \(X\) be the initial universe set and \(A\) be a set of parameters. Let \(P(X)\) denote the power set of \(X\). Consider a non-empty set \(A, A \subseteq X\). A pair \((F,A)\) is called a soft set over \(X\) (for
short s-set), where \( F \) is a mapping given by: \( F: A \to P(X) \).

**Remark (2.2) [2]:**

(i) The s-set can be represented by \( F_A \).

(ii) Every set is s-set.

(iii) If \( A = \{1\} \), a s-set can be considered a crisp set.

(iv) \( S(X) \) is the collection of all s-sets over \( X \).

**Definition (2.3) [9]:**

Let \( X \) be a non-empty fixed set. A neutrosophic crisp set (for short ncs-set) \( A \) is an object having the form \( A = \langle A_1, A_2, A_3 \rangle \), where \( A_1, A_2 \) and \( A_3 \) are subsets of \( X \) satisfying:

\[
A_1 \cap A_2 = \emptyset, \quad A_1 \cap A_3 = \emptyset \quad \text{and} \quad A_2 \cap A_3 = \emptyset.
\]

**Remark (2.4) [11]:**

Every crisp set in \( X \) is obviously a (NC-set) having the form \( \langle A_1, A_2, A_3 \rangle \).

**Definition (2.5) [13]:**

The object having the form \( \langle A_1, A_2, A_3 \rangle \) is called:

(i) ncs-set with type I if it satisfies

\[
A_1 \cap A_2 = \emptyset, \quad A_1 \cap A_3 = \emptyset \quad \text{and} \quad A_2 \cap A_3 = \emptyset \quad \text{(NC-set type I)}.
\]

(ii) ncs-set with Type II if it satisfies:

\[
A_1 \cap A_2 = \emptyset, \quad A_1 \cap A_3 = \emptyset, \quad A_2 \cap A_3 = \emptyset \quad \text{and} \quad A_1 \cup A_2 \cup A_3 = X \quad \text{(NC-set type II)}.
\]

(iii) ncs-set with type I if it satisfies:

\[
A_1 \cap A_2 \cap A_3 = \emptyset, \quad A_1 \cup A_2 \cup A_3 = X \quad \text{(nc-set type III)}.
\]

**Definition (2.6) [8]:**

Let \( \mathcal{L} \subseteq S(X) \), \( \gamma \) and \( \lambda \) be two binary operations on \( \mathcal{L} \). If \( \mathcal{L} \) is equipped with two commutative and associative binary operations \( \gamma \) and \( \lambda \), which are connected by the absorption law, then algebraic structure \( (\mathcal{L}, \gamma, \lambda) \) is called soft lattice.

**3. Main result**

**Definition (3.1):**

Let \( X \) be a universe and \( A \) be a set of parameters that are describe the elements of a set \( X \). A neutrosophic crisp soft set (for short ncs-set) over \( X \) (denoted by \( \widehat{N} \)) is a set defined by:

\[
\widehat{N} = \langle F_A, G_A, H_A \rangle, \quad \text{where} \quad F_A, G_A \quad \text{and} \quad H_A \quad \text{are disjoint s-sets over \( X \).}
\]

**Example (3.2):**

Let \( X = \{x_1, x_2, x_3\} \) be a universe set and \( A = \{e_1, e_2, e_3\} \) a set of parameters \( F: A \to P(X) \) be a mapping such that:

\[
F(e_1) = \{x_1, x_3\}; \quad F(e_2) = \{x_1, x_2\} \quad \text{&} \quad F(e_3) = \emptyset.
\]

It is clear that:

\[
F_A = \{\{x_1, x_3\}, \{x_1, x_2\}, \emptyset\}, \quad G_A = F_A^c \quad \text{and} \quad H_A = \emptyset \quad \text{are disjoint s-sets.}
\]

Then we'll give a ncs-set is \( \widehat{N} = \langle F_A, G_A, H_A \rangle \).

**Remark (3.3):**

(4) Every ncs-set formed by three disjoint s-sets.
(iv) Every \( nc \)-set is a \( ncs \)-set.

The difference between \( nc \)-set and \( ncs \)-set arise from this fact in remark (2.2.ii).

(iii) When \( A = |1| \) a \( ncs \)-set is a \( nc \)-set.

**Definition (3.4):**

Let \( \mathcal{N}_1 = \langle F_A, G_A, H_A \rangle \) & \( \mathcal{N}_2 = \langle F_B, G_B, H_B \rangle \) be two \( ncs \)-sets over a universe set \( X \). Then:

(i) \( \mathcal{N}_1 \) is called a \( ncs \)-subset of \( \mathcal{N}_2 \) (\( \mathcal{N}_1 \subseteq \mathcal{N}_2 \)), if \( F_A \subseteq F_B \), \( G_A \subseteq G_B \) and \( H_A \supseteq H_B \).

(ii) \( \mathcal{N}_1 \) & \( \mathcal{N}_2 \) are called an equal (\( \mathcal{N}_1 = \mathcal{N}_2 \)), if \( \mathcal{N}_1 \subseteq \mathcal{N}_2 \) and \( \mathcal{N}_2 \subseteq \mathcal{N}_1 \).

(iii) The complement of \( \mathcal{N} \) is denoted by \( \mathcal{N}^c \) and may be defined as

\( \mathcal{N}_1^c = \langle F^c_A, G^c_A, H^c_A \rangle \).

(iv) The union of \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) denoted by \( \mathcal{N}_1 \cup \mathcal{N}_2 \) may be defined as:

\( \mathcal{N}_1 \cup \mathcal{N}_2 = \langle F_A \cup F_B, G_A \cup G_B, H_A \cap H_B \rangle \)

or \( \mathcal{N}_1 \cup \mathcal{N}_2 = \langle F_A \cap F_B, G_A \cap G_B, H_A \cap H_B \rangle \).

(v) The intersection of \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), is denoted by \( \mathcal{N}_1 \cap \mathcal{N}_2 \) may be defined as:

\( \mathcal{N}_1 \cap \mathcal{N}_2 = \langle F_A \cap F_B, G_A \cap G_B, H_A \cup H_B \rangle \)

or \( \mathcal{N}_1 \cap \mathcal{N}_2 = \langle F_A \cup F_B, G_A \cup G_B, H_A \cap H_B \rangle \).

(vi) The difference of \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) is denoted by \( \mathcal{N}_1 \setminus \mathcal{N}_2 \) and defined as \( \mathcal{N}_1 \setminus \mathcal{N}_2 = \mathcal{N}_1 \cap \mathcal{N}_2^c \).

**Remark (3.5):**

Let \( \{ \mathcal{N}_j : j \in J \} \) be an arbitrary family of \( ncs \)-sets over \( X \). Then:

(4) \( \bigcup_j \mathcal{N}_j = \langle \bigcup F_{jA}, \bigcup G_{jA}, \bigcap H_{jA} \rangle \) or \( \bigcup_j \mathcal{N}_j = \langle \bigcup F_{jA}, \bigcap G_{jA}, \bigcap H_{jA} \rangle \).

(5) \( \bigcap_j \mathcal{N}_j = \langle \bigcap A_{j1}, \bigcap A_{j2}, \bigcup A_{j3} \rangle \) or \( \bigcap_j \mathcal{N}_j = \langle \bigcap A_{j1}, \bigcup A_{j2}, \bigcup A_{j3} \rangle \).

**Definition (3.6):**

May be define \( \Phi \) and \( \vec{X} \) as follows:

(4) \( \Phi = \langle \emptyset, \vec{0}, X \rangle \) or \( \langle \emptyset, X, \vec{0} \rangle \) or \( \langle X, \vec{X}, \vec{0} \rangle \)

(\( NC \) null \( s \)-set).

(5) \( \vec{X} = \langle X, \vec{X}, \vec{0} \rangle \) or \( \langle \vec{X}, \vec{0}, X \rangle \) or \( \langle \vec{0}, \vec{X}, \vec{X} \rangle \)

(\( NC \) absolute \( s \)-set).

**Definition (3.7):**

A \( NCS \)-set over \( X \), is called a \( NCS \)-point over \( X \), denoted by \( \hat{p} = \langle p^x, p^y, p^z \rangle \), \( p^x \neq p^y \neq p^z \), \( p^x, p^y, p^z \in S_p(X) \), \( S_p(X) \) is the collection of all soft points over a universe \( X \)

**Definition (3.8):**

Let \( \mathcal{N} = \langle F_A, G_A, H_A \rangle \) be a \( ncs \)-sets over \( X \). Then \( \mathcal{N} \) is called a singleton \( ncs \)-set over \( X \), if for each \( F_A, G_A \) & \( H_A \) are singleton \( s \)-sets over \( X \).
Remark (3.9):
Every $nccs$-point is a singleton $nccs$-set. The converse is not true as the following example shows.

Example (3.10):
(i) Given $X = \{x, y, z\}$ is a universe set and $A = \{e_1, e_2, e_3\}$ be a set of parameters. Then:
$\mathcal{N} = \{p^x_{e_1}, p^y_{e_2}, p^z_{e_3}\}$ is a $nccs$-point.

(ii) Let $X = \{\alpha, \beta, \gamma, \delta\}$ be a universe set and $A = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be a set of parameters. Then $\mathcal{N} = \{F_A, G_A, H_A\}$ is a singleton $nccs$-set but it is not $nccs$-point such that:
$F_A = \{(\ell_1, \{\alpha\}), (\ell_2, \{\beta\}), \emptyset, \emptyset\}$
$G_A = \emptyset, \emptyset, (\ell_3, \{\gamma\}), \emptyset.$
$H_A = \emptyset, \emptyset, \emptyset, (\ell_4, \{\delta\}).$

Remark (3.11):
(i) The collection of all $nccs$-sets over a universe $X$ is denoted by $\mathcal{N}(X)$.

(ii) The collection of all $nccs$-points over $X$ is denoted by $\mathcal{N}_p(X)$.

Theorem (3.12):
Let $\mathcal{N}_1 = \{F_A, G_A, H_A\}$ and $\mathcal{N}_2 = \{F_B, G_B, H_B\}$ be two $nccs$-sets over a universe $X$. Then:
(i) $\mathcal{N}_1 \subseteq \mathcal{N}_2$ iff $\mathcal{P} \in \mathcal{N}_2$ for all $\mathcal{P} \in \mathcal{N}_1$.
(ii) $\mathcal{N}_1 = \mathcal{N}_2$ iff $\mathcal{N}_1 \subseteq \mathcal{N}_2$ and $\mathcal{N}_2 \subseteq \mathcal{N}_1$.

Proof: Obvious.

Theorem (3.13):
Let $\mathcal{N} = \{F_A, G_A, H_A\}$ be a $nccs$-sets over $X$. Then:
(i) $\mathcal{N} = \bigcup_{\mathcal{P} \in \mathcal{N}} \{\mathcal{P}\}$.

Proof: Clear.

Theorem (3.14):
Let $\{\mathcal{N}_j : j \in J\}$ be an arbitrary family of $nccs$-sets over $X$. Then:
(i) $\mathcal{N} \in \bigcup_{j \in J} \mathcal{N}_j$ iff $\mathcal{N} \in \mathcal{N}_j$ for some $j \in J$.

Proof: Clear.

Theorem (3.15):
Let $f : X \to Y$ be a mapping. Then:
(i) If $\mathcal{N} = \{F_A, G_A, H_A\}$ be a $nccs$-sets over $X$, then $f(\mathcal{N}) = \{f(F_A), f(G_A), f(H_A)\}$ is a $nccs$-sets over $Y$.

(ii) The image of $nccs$-point $\mathcal{P} = \{p^x_{e_1}, p^y_{e_2}, p^z_{e_3}\}$ over $X$ under a mapping $f$ denoted by $f(\mathcal{P})$ and defined as $f(\mathcal{P}) = \{f(p^x_{e_1}), f(p^y_{e_2}), f(p^z_{e_3})\}$.

Definition (3.16):
Let $\mathcal{N}_1 = \{F_A, G_A, H_A\}$ and $\mathcal{N}_2 = \{F_B, G_B, H_B\}$ be two $NCS$-sets over $X$. Then:
(i) A Cartesian product of $\mathcal{N}_1$ and $\mathcal{N}_2$ is defined as:
If \( (\hat{\varphi}_1, \hat{\varphi}_2), (\hat{\varphi}_2, \hat{\varphi}_3) \in \mathcal{R} \), then \( (\hat{\varphi}_1, \hat{\varphi}_3) \in \mathcal{R} \), for all \( \hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3 \in \hat{\mathcal{N}} \).

(iv) \( \mathcal{R} \) is anti-symmetric:

if \( (\hat{\varphi}_1, \hat{\varphi}_2), (\hat{\varphi}_2, \hat{\varphi}_1) \in \mathcal{R} \), then \( \hat{\varphi}_1 = \hat{\varphi}_2 \) for all \( \hat{\varphi}_1, \hat{\varphi}_2 \in \hat{\mathcal{N}} \).

4. \textit{ncs-Lattice}:

Definition (4.1):

Let \( \mathcal{R} \subseteq \hat{\mathcal{N}}(X) \), \( \land \) and \( \lor \) be two binary operations on \( \mathcal{R} \). If the set \( \mathcal{N} \) is equipped with two commutative and associative binary operations \( \land \) and \( \lor \), which are connected by the absorption law.

Then algebraic structure \( (\mathcal{R}, \land, \lor) \) is called \( \textit{ncs}\ell \).

Theorem (4.2):

Let \( (\mathcal{R}, \land, \lor) \) be a \( \textit{ncs}\ell \) \& \( \hat{\mathcal{N}}_1 = \langle F_A, G_A, H_A \rangle \)

and \( \hat{\mathcal{N}}_2 = \langle F_B, G_B, H_B \rangle \in \mathcal{R} \). Then \( \hat{\mathcal{N}}_1 \land \hat{\mathcal{N}}_2 = \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 = \hat{\mathcal{N}}_2 \).

Proof:

\( \hat{\mathcal{N}}_1 \land \hat{\mathcal{N}}_2 = (\hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2) \lor \hat{\mathcal{N}}_2 \)

\( = \hat{\mathcal{N}}_2 \lor (\hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2) \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \)

\( = (\hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1) \lor (\hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_2) \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \)

Conversely,

\( \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 = (\hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2) \lor (\hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2) \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \)

\( = \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \)

\( = \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \)

\( = \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \lor \hat{\mathcal{N}}_2 \lor \hat{\mathcal{N}}_1 \).

65
Example (4.3):
Let \( X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \) be a universe set and \( A = \{e_1, e_2, e_3\} \) be a set of parameters with \( \mathcal{R} = \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5\} \subseteq \mathcal{N}(X) \) such that:
\[
\begin{align*}
\mathcal{N}_1 &= \langle\{x_1, x_2, x_3\}, \{x_4, x_5\}, \emptyset\rangle, \\
\mathcal{N}_2 &= \langle\emptyset, \emptyset, \emptyset\rangle, \\
\mathcal{N}_3 &= \langle x_2, \emptyset, \emptyset\rangle, \\
\mathcal{N}_4 &= \langle\{x_1, x_3\}, \emptyset, \emptyset\rangle, \\
\mathcal{N}_5 &= \langle\{x_1, x_2, x_3\}, \emptyset, \emptyset\rangle.
\end{align*}
\]
Then \( (\mathcal{R}, \cup, \cap) \) is NCSL. Tables of the operations are as follows, respectively:

<table>
<thead>
<tr>
<th>( \cup )</th>
<th>( \mathcal{N}_1 )</th>
<th>( \mathcal{N}_2 )</th>
<th>( \mathcal{N}_3 )</th>
<th>( \mathcal{N}_4 )</th>
<th>( \mathcal{N}_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
</tr>
<tr>
<td>( \mathcal{N}_2 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_2 )</td>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_5 )</td>
</tr>
<tr>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
</tr>
<tr>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_5 )</td>
</tr>
<tr>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
</tr>
</tbody>
</table>

And

<table>
<thead>
<tr>
<th>( \cap )</th>
<th>( \mathcal{N}_1 )</th>
<th>( \mathcal{N}_2 )</th>
<th>( \mathcal{N}_3 )</th>
<th>( \mathcal{N}_4 )</th>
<th>( \mathcal{N}_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
<td>( \mathcal{N}_1 )</td>
</tr>
<tr>
<td>( \mathcal{N}_2 )</td>
<td>( \mathcal{N}_2 )</td>
<td>( \mathcal{N}_2 )</td>
<td>( \mathcal{N}_2 )</td>
<td>( \mathcal{N}_2 )</td>
<td>( \mathcal{N}_2 )</td>
</tr>
<tr>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_3 )</td>
<td>( \mathcal{N}_3 )</td>
</tr>
<tr>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_4 )</td>
<td>( \mathcal{N}_4 )</td>
</tr>
<tr>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
<td>( \mathcal{N}_5 )</td>
</tr>
</tbody>
</table>

Theorem (4.4):
Let \( (\mathcal{R}, \gamma, \lambda) \) be a ncsl such that:
\[
\mathcal{N}_1 = \langle F_A, G_A, H_A \rangle, \quad \mathcal{N}_2 = \langle F_B, G_B, H_B \rangle \in \mathcal{N}.
\]
Then a relation \( \leq \) that is defined by:
\[
\mathcal{N}_1 \leq \mathcal{N}_2 \iff \mathcal{N}_1 \land \mathcal{N}_2 = \mathcal{N}_1 \lor \mathcal{N}_2.
\]
is an ordering on \( \mathcal{R} \).

Proof:

i. \( \leq \) is reflexive. \( \mathcal{N}_1 \leq \mathcal{N}_1 \iff \mathcal{N}_1 \land \mathcal{N}_1 = \mathcal{N}_1 \).

ii. \( \leq \) is antisymmetric. Let \( \mathcal{N}_1 \leq \mathcal{N}_2 \) & \( \mathcal{N}_2 \leq \mathcal{N}_1 \).

Then \( \mathcal{N}_1 = \mathcal{N}_1 \land \mathcal{N}_2 = \mathcal{N}_2 \land \mathcal{N}_1 = \mathcal{N}_2 \).

iii. \( \leq \) is transitive. Let \( \mathcal{N}_1 \leq \mathcal{N}_2 \) & \( \mathcal{N}_2 \leq \mathcal{N}_3 \).

Then \( \mathcal{N}_1 \land \mathcal{N}_3 = (\mathcal{N}_1 \land \mathcal{N}_2) \land \mathcal{N}_3 \)
\[
= \mathcal{N}_1 \land (\mathcal{N}_2 \land \mathcal{N}_3) = \mathcal{N}_1 \land \mathcal{N}_2 = \mathcal{N}_1.
\]

This implies that \( \mathcal{N}_1 \leq \mathcal{N}_3 \).

Theorem (4.5):
Let \( (\mathcal{R}, \gamma, \lambda) \) be a NCSL such that:
\[
\mathcal{N}_1 = \langle F_A, G_A, H_A \rangle \text{ and } \mathcal{N}_2 = \langle F_B, G_B, H_B \rangle \in \mathcal{N}.
\]
Then:
(i) \( \mathcal{N}_1 \land \mathcal{N}_2 \subseteq \mathcal{N}_1 \land \mathcal{N}_2 \subseteq \mathcal{N}_2 \).

(ii) \( \mathcal{N}_1 \subseteq \mathcal{N}_1 \lor \mathcal{N}_2 \land \mathcal{N}_1 \lor \mathcal{N}_2 \subseteq \mathcal{N}_1 \lor \mathcal{N}_2 \).

Proof:

By definition (4.1) , we have:

\( (\mathcal{N}_1 \land \mathcal{N}_2) \lor \mathcal{N}_1 = \mathcal{N}_1 \lor (\mathcal{N}_1 \land \mathcal{N}_2) = \mathcal{N}_1 \).

From theorem (4.4), we get

\( (\mathcal{N}_1 \land \mathcal{N}_2) \lor \mathcal{N}_1 \subseteq \mathcal{N}_1 \).

It can be show that \( (\mathcal{N}_1 \land \mathcal{N}_2) \subseteq \mathcal{N}_2 \).

The proof (ii) can made similarity.

Theorem (4.6):

Let \( (\mathcal{R}, \lor, \land) \) be a NCSL such that:

\( \mathcal{N}_1 = \langle F_A, G_A, H_A \rangle , \mathcal{N}_2 = \langle F_B, G_B, H_B \rangle \),

\( \mathcal{N}_3 = \langle F_C, G_C, H_C \rangle ) \land \mathcal{N}_4 = \langle F_D, G_D, H_D \rangle \in \mathcal{N} \).

If \( \mathcal{N}_1 \subseteq \mathcal{N}_2 \land \mathcal{N}_3 \subseteq \mathcal{N}_4 \), then \( \mathcal{N}_1 \land \mathcal{N}_3 \subseteq \mathcal{N}_2 \land \mathcal{N}_4 \).

Proof:

From hypothesis and theorem (4.4), we have:

\( \mathcal{N}_1 \land \mathcal{N}_2 = \mathcal{N}_1 \land \mathcal{N}_3 \land \mathcal{N}_4 = \mathcal{N}_3 \land \mathcal{N}_4 \).

Then from theorem (4.4), \( \mathcal{N}_1 \land \mathcal{N}_3 \subseteq \mathcal{N}_2 \land \mathcal{N}_4 \).

Theorem (4.7):

Let \( (\mathcal{R}, \lor, \land) \) be a NCSL such that:

\( \mathcal{N}_1 = \langle F_A, G_A, H_A \rangle , \mathcal{N}_2 = \langle F_B, G_B, H_B \rangle , \mathcal{N}_3 = \langle F_C, G_C, H_C \rangle ) \land \mathcal{N}_4 = \langle F_D, G_D, H_D \rangle \in \mathcal{N} \). If \( \mathcal{N}_2 \subseteq \mathcal{N}_3 \), then \( \mathcal{N}_2 \land \mathcal{N}_3 \subseteq \mathcal{N}_1 \lor \mathcal{N}_3 \).

Proof:

Proof is made similarity to theorem (4.6).

Example (4.8):

From example (4.3), Since \( \mathcal{N}_2 \subseteq \mathcal{N}_3 \& \mathcal{N}_4 \subseteq \mathcal{N}_5 \), we have: \( \mathcal{N}_2 \land \mathcal{N}_3 \subseteq \mathcal{N}_4 \land \mathcal{N}_5 \).

Remark (4.9):

Let \( (\mathcal{R}, \lor, \land) \) be a NCSL such that:

\( \mathcal{N}_1 = \langle F_A, G_A, H_A \rangle , \mathcal{N}_2 = \langle F_B, G_B, H_B \rangle \in \mathcal{R} \). Then:

\( \mathcal{N}_1 \land \mathcal{N}_2 \& \mathcal{N}_1 \lor \mathcal{N}_2 \) are the least upper and the greatest lower bound of \( \mathcal{N}_1 \) and \( \mathcal{N}_1 \) respectively.

Theorem (4.10):

Let \( \mathcal{R} \subseteq \mathcal{N}(\mathcal{X}) \). Then \( (\mathcal{N}, \lor, \land) \) is a NCSL.

Proof:

For all \( \mathcal{N}_1 = \langle F_A, G_A, H_A \rangle , \mathcal{N}_2 = \langle F_B, G_B, H_B \rangle \) and \( \mathcal{N}_3 = \langle F_C, G_C, H_C \rangle \in \mathcal{N} \).

From remark (4.9),
\[ \bar{N}_1 \land \bar{N}_2 \leq \bar{N}_1 \] and \[ \bar{N}_1 \land \bar{N}_2 \leq \bar{N}_2. \]

Then from theorem (4.6), we have \( \bar{N}_1 \land \bar{N}_2 \leq \bar{N}_2 \land \bar{N}_1 \).

Similarly, \( \tilde{N}_2 \land \tilde{N}_1 \leq \tilde{N}_1 \land \tilde{N}_2 \).

Imply that:
\[ \bar{N}_1 \land \bar{N}_2 = \bar{N}_2 \land \bar{N}_1. \]

By the same way, the proof of \( \bar{N}_1 \lor \bar{N}_2 = \bar{N}_2 \lor \bar{N}_1 \) can be made.

Now, from theorem (3.5), we have:
\[ (\bar{N}_1 \land \bar{N}_2) \land \bar{N}_3 \leq \bar{N}_2 \] and \( (\bar{N}_1 \land \bar{N}_2) \land \bar{N}_3 \leq \bar{N}_3. \)

Also, from theorem (3.6), implies that:
\[ (\bar{N}_1 \land \bar{N}_2) \land \bar{N}_3 \leq \bar{N}_1 \land \bar{N}_2 \leq \bar{N}_1. \]

Similarly, \( \tilde{N}_1 \land (\tilde{N}_2 \land \tilde{N}_3) \leq (\tilde{N}_1 \land \tilde{N}_2) \land \tilde{N}_3 \).

Then \( \tilde{N}_1 \land (\tilde{N}_2 \land \tilde{N}_3) = (\tilde{N}_1 \land \tilde{N}_2) \land \tilde{N}_3. \)

By the same way, the proof of \( \tilde{N}_1 \lor (\tilde{N}_2 \lor \tilde{N}_3) = (\tilde{N}_1 \lor \tilde{N}_2) \lor \tilde{N}_3 \) can be made.

Finally, can be made:
\[ \tilde{N}_1 \land (\tilde{N}_1 \lor \tilde{N}_2) = \tilde{N}_1 \] and \( \tilde{N}_1 \lor (\tilde{N}_1 \land \tilde{N}_2) = \tilde{N}_1. \)

\[ \bar{N}_1 \land \bar{N}_2 \leq \bar{N}_1 \] and \[ \bar{N}_1 \land \bar{N}_2 \leq \bar{N}_2. \]

References:


