On a New Certain Subclass of Meromorphically p-valent Functions Defined by a Linear Operator

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Abstract: In the present paper, we have introduced a new class of meromorphically p-valent functions \( \sum_{\lambda, \mu, \eta, q_s, s} \) defined by a linear operator \( T_{p,q,s}(\alpha) \). We discuss some interesting properties, like, coefficient inequality, convex set, distortion bounds, neighborhoods of a function \( f \in \sum_{\lambda, \mu, \eta, q_s, s} \), and integral operator.

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1. Introduction

Let \( \sum_{\lambda, \mu, \eta, q_s, s} \) denote the class of functions of the form:

\[
f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n ; \quad (a_n \geq 0; \quad p \in \mathbb{N} = \{1,2,\ldots\}) , (1)
\]

which are analytic and p-valent in the punctured unit disk

\[U^* = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{C}\backslash\{0\}.\]

We define the Hadamard product (or Convolution) of \( f \) and \( g \) by

\[
(f \ast g)(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n b_n z^n = (g \ast f)(z), (2)
\]

where \( f \) is given by (1) and \( g \) is defined as follows:

\[
g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} b_n z^n;
\]

For positive real values of \( \alpha_1, \ldots, \alpha_q \) and \( \beta_1, \ldots, \beta_s \) (\( \beta_j \neq 0, -1, \ldots; \quad f = 1,2,\ldots,s \)), we now define the generalized hypergeometric function

\[
q_F(s; \alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \quad \text{by}
\]

\[
q_F(s; \alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_s)_n} \frac{z^n}{n!} , (3)
\]

(\( q \leq s + 1; \quad q, s \in \mathbb{N} = \mathbb{N} \cup \{0\}; \quad z \in U^* \)),

where \((\theta)_n\) is the Pochhammer symbol defined

\[
(\theta)_n = \begin{cases} \theta(\theta + 1)(\theta + 2) \ldots (\theta + n - 1), & n \in \mathbb{N} \setminus \{0\} \\ 1, & n = 0 \end{cases}, (4)
\]

Corresponding to the function \( h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \), defined by

\[
h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p} q_F(s; \alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z), (5)
\]

we consider a linear operator

\[
T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \sum_{\lambda, \mu, \eta, q_s, s} \rightarrow \sum_{\lambda, \mu, \eta, q_s, s}
\]

which is defined by means of the following Hadamard product (or convolution):

\[
T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \ast f(z), (6)
\]

We observe that, for a function \( f(z) \) of the form (1), we have

\[
T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} \left( \frac{\alpha_1}_n \ldots (\alpha_q)_n a_n z^n \right), (7)
\]

If, for convenience, we write

\[
T_{p,q,s}(\alpha) = T_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s), (8)
\]

then one can easily verify from the definition (6) that

\[
T_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 + p) T_{p,q,s}(\alpha_1) f(z). (9)
\]

The linear operator \( T_{p,q,s}(\alpha_1) \) was investigated recently by Liu and Srivastava [8].

Some interesting Subclasses of analytic functions associated with the generalized hypergeometric function were considered recently by (for example) Dziok and Srivastava [3] and [4], Gangadharan et al. [5] and Liu [7].

Definition 1: Let \( \sum_{\lambda, \mu, \eta, q_s, s} \) be denote the new class of functions \( f(z) \) contained in:

\[
f(z) \in \sum_{\lambda, \mu, \eta, q_s, s}, \quad \text{which satisfy the condition:}
\]

\[
\lambda(\frac{r+2}{r+1} + \frac{r+1}{r}) f'(z) + \beta z^2 f''(z) - \gamma z f''(z) = \eta, (10)
\]

where \( z \in U^* ; \quad 0 < \eta < p; \quad p \in \mathbb{N} \) and for some suitably restricted real parameters \( \lambda, \eta \).

Such type of study was carried out by several different authors for another classes, like, Nunokawa and Ahuja [9], Aouf and Hossen [1] and Cho et al. [2].

2. Coefficient Inequality

First, we derive the coefficient inequality for the class \( \sum_{\lambda, \mu, \eta, q_s, s} \) contained in:

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Theorem 1: Let \( f \in \sum_p \). Then \( f \) is in the class 
\[
\sum \left( \lambda, \mu, \eta, \alpha, \eta, \alpha \right) \text{ if and only if }
\]
\[
\left( \lambda \eta \right) = \frac{(\lambda \eta)(\lambda - 1)(\lambda + \lambda) - \eta (\lambda(\lambda - 2) + 1)}{(\beta_1 \eta) \cdots (\beta_n \eta) \ n!} \leq \eta \mu \rho(\lambda + 1) - 1, \text{(11)}
\]
where \( 0 < \eta < \lambda \); \( \lambda, \mu, \eta, \alpha \in N \).

The result is sharp for the function
\[
\sigma(z) = \frac{1}{C_n + n!} \frac{(\lambda \eta)(\lambda(\lambda + 1) - 1)}{(\beta_1 \eta) \cdots (\beta_n \eta) \ n!} \leq \eta \mu \rho(\lambda + 1) - 1. \text{ (12)}
\]

Proof: Suppose that the inequality (11) holds true and \( |z| = 1 \). Then, we have

\[
\xi(\lambda \eta) = \frac{(\lambda \eta)(\lambda(\lambda - 1)(\lambda + \lambda) - \eta (\lambda(\lambda - 2) + 1))}{(\beta_1 \eta) \cdots (\beta_n \eta) \ n!} \leq \eta \mu \rho(\lambda + 1) - 1, \text{ (11)}
\]

by hypothesis. Thus by maximum modulus principle, \( f(z) \in \sum_p \). To show the converse, suppose that \( f(z) \in \sum_p \). Then (10), we have

\[
\sigma(z) \leq \eta \mu \rho(\lambda + 1) - 1. \text{ (13)}
\]

We choose the value of \( z \) on the real axis so that \( \sigma(T_{p,q,s}(\alpha_1) f(z)) \) is real.

Upon clearing the denominator of (13) and letting \( z \to 1^- \), through real values so we can write (13) as

\[
\sum_{n=p}^\infty n(\lambda(n-1)(n+p) - \eta(n-2) \leq \sum_{n=p}^\infty \frac{(\alpha_1 \cdots \alpha_q \eta)(\alpha_a \eta)n}{(\beta_1 \eta) \cdots (\beta_n \eta) \ n!} |z|^n
\]

Corollary 1: Let \( f(z) \in \sum_p \). Then

\[
\xi(\lambda \eta) = \frac{(\lambda \eta)(\lambda(\lambda - 1)(\lambda + \lambda) - \eta (\lambda(\lambda - 2) + 1))}{(\beta_1 \eta) \cdots (\beta_n \eta) \ n!} \leq \eta \mu \rho(\lambda + 1) - 1.
\]

We choose the value of \( z \) on the real axis so that \( \sigma(T_{p,q,s}(\alpha_1) f(z)) \) is real.

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\]

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(n ≥ p).

3. Convex Set

In the following theorem, we will prove the class \( \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \) is convex set.

**Theorem 2:** The class \( \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \) is a convex set.

**Proof:** Let \( f_1 \) and \( f_2 \) be the arbitrary elements of \( \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \).
Then for every \( t (0 ≤ t ≤ 1) \), we show that \((1 - t)f_1 + tf_2 \in \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \).

Thus, we have
\[
(1 - t)f_1 + tf_2 = \frac{1}{2^p} \sum_{n=p}^{\infty} \left[ (1 - t)a_n + ta_n \right] z^n.
\]

Hence,
\[
\sum_{n=p}^{\infty} n(\lambda(n - 1)(n + p) - \eta(\mu(n - 2)) + 1) \left( \frac{a_n}{(\beta_1)_n \ldots (\beta_p)_n} \right) \leq \eta(\mu(p + 1) - 1),
\]

This completes the proof.

4. Distortion Bounds

In the following theorems, we obtain the growth and distortion bounds for the linear operator \( T_{p,q,s}(\alpha) \).

**Theorem 3:** If \( f(z) \in \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \), then
\[
\frac{1}{2^p} \left( 2\lambda p - \eta(\mu(p - 2) + 1) \right) \leq \left| T_{p,q,s}(\alpha) \right| \leq \frac{1}{2^p} \left( 2\lambda p - \eta(\mu(p - 2) + 1) \right).
\]

The result is sharp for the function \( f(z) \).

**Proof:** Let \( f(z) \in \Sigma_p (\lambda, \mu, \eta, \alpha, q, s) \). Then by Theorem 1, we get
\[
p(\lambda p(2\lambda p - \eta(\mu(p - 2) + 1)) \left( \frac{a_n}{(\beta_1)_n \ldots (\beta_p)_n} \right),
\]

\[
\sum_{n=p}^{\infty} a_n \leq \eta(\mu(p + 1) - 1).
\]

The result is sharp for the function \( f(z) \).

5. 8-Neighborhood of a function \( f \in \Sigma_p \):

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruscheweyh [10], we begin by introducing here the \( \delta \)-Neighborhood of a function \( f \in \Sigma_p \) of the form (1) by means of the definition below:

\[
N_\delta(f) = \left\{ g \in \Sigma_p : g(z) = \frac{1}{2^p} + \sum_{n=p}^{\infty} b_n z^n \right\},
\]

\[
\sum_{n=p}^{\infty} n|a_n - b_n| ≤ \delta, 0 ≤ \delta < 1.
\]
Particularly for the identity function \( f(z) = \frac{1}{z^p} \), we have

\[
N_\delta(e) = \left\{ g \in \sum_p : g(z) = \frac{1}{z^p} + \sum_{n=p}^\infty b_n z^n \text{ and } \sum_{n=p}^\infty \left| n \cdot b_n \right| \leq \delta \right\}. \tag{21}
\]

**Theorem 5:** If \( g(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \) and

\[
y = 1 - \frac{\delta (\lambda (p-1)2p - \eta (\mu(p-2) - 1))}{p (\mu(p-1)2p - \eta (\mu(p-2) + 1))} \frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p}, \tag{22}
\]

the coefficient inequality

\[
\sum_{n=p}^\infty \left| a_n - b_n \right| \leq \frac{\delta}{p} (n \geq p).
\]

Since \( g(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), then by using Theorem (1), we get

\[
\sum_{n=p}^\infty b_n \leq \frac{\eta (\mu(p+1) - 1) p!}{(\lambda(p-1)2p - \eta (\mu(p-2) + 1))} \frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p}.
\]

so that

\[
\left| \frac{g(z)}{f(z)} - 1 \right| < \frac{\delta (\lambda(p-1)2p - \eta (\mu(p-2) - 1))}{p (\lambda(p-1)2p - \eta (\mu(p-2) + 1))} \frac{(\alpha_1)_p \cdots (\alpha_q)_p}{(\beta_1)_p \cdots (\beta_s)_p} \leq 1 - y.
\]

Hence, by Definition 2, \( f(z) \in \sum_{p,y} (\lambda, \mu, \eta, \alpha, q, s) \) for \( y \) given by (22). This complete the proof.

6. Radii of starlikeness and convexity:

In the following Theorems, we discuss the radii starlikeness and convexity.

**Theorem 6:** If \( f(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), then \( f(z) \) is multivalent meromorphic starlike of order \( \theta (0 \leq \theta < p) \) in the disk \( |z| < r_1 \), where

\[
r_1 = \inf \left\{ (p+\theta)n(\lambda (n-1)(n+p) - \eta (\mu(n-2) + 1)) \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \right\}^{1/p}.
\]

The result is sharp for the function \( f(z) \) is given by (12).

**Proof:** It is sufficient to show that

\[
\left| f(z) \right| + p \leq p - \theta \text{ for } |z| < r_1, \tag{23}
\]

But

\[
\left| f(z) \right| \leq \sum_{n=p}^\infty (n+\theta) a_n |z|^{n+p} \leq \frac{p - \theta}{\sum_{n=p}^\infty a_n} |z|^{n+p} \leq 1. \tag{24}
\]

Since \( f(z) \in \sum_p (\lambda, \mu, \eta, \alpha, g, s) \), we have

Thus, (23) will be satisfied if
\[
\sum_{n=p}^{\infty} \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \left( \alpha_1 \right)_n \ldots \left( \alpha_q \right)_n}{\eta p(\mu(p+1) - 1)n!} \leq 1.
\]

Hence, (24) will be true if
\[
\frac{(n + 2p - \theta)}{p - \theta} |z|^{n+p} \leq \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \left( \alpha_1 \right)_n \ldots \left( \alpha_q \right)_n}{\eta p(\mu(p+1) - 1)n!},
\]
or equivalently
\[
|z| \leq \left\{ \frac{(p - \theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \left( \alpha_1 \right)_n \ldots \left( \alpha_q \right)_n}{n(n + 2p - \theta) - \eta(\mu(p+1) - 1)n!} \right\}^{\frac{1}{n+p}}, \quad n \geq p
\]
which follows the result.

**Theorem 7:** If \( f(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), then \( f(z) \) is multivalent meromorphic convex of order \( \theta (0 \leq \theta < p) \) in the disk \( |z| < r_2 \), where
\[
r_2 = \inf_{n} \left\{ \frac{p(\theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \left( \alpha_1 \right)_n \ldots \left( \alpha_q \right)_n}{n(n + 2p - \theta) - \eta(\mu(p+1) - 1)n!} \right\}^{\frac{1}{n+p}}, \quad n \geq p.
\]
The result is sharp for the function \( f(z) \) given by (12).

**Proof:** It is sufficient to show that
\[
\left| \frac{zf''(z)}{f'(z)} + 1 + p \right| \leq p - \theta \quad \text{for} \quad |z| < r_2. \tag{25}
\]
But
\[
\left| \frac{zf''(z)}{f'(z)} + 1 + p \right| = \left| \frac{zf''(z) + (1 + p)f'(z)}{f'(z)} \right| \leq \frac{\sum_{n=p}^{\infty} n(n+p)a_n|z|^{n+p}}{p - \sum_{n=p}^{\infty} n a_n|z|^{n+p}}.
\]
Since \( f(z) \in \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), we have
\[
\sum_{n=p}^{\infty} \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \left( \alpha_1 \right)_n \ldots \left( \alpha_q \right)_n}{\eta p(\mu(p+1) - 1)n!} \leq 1.
\]
Hence, (26) will be true if
\[
\frac{n(n + 2p - \theta)}{p(\theta)} |z|^{n+p} \leq \frac{n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \left( \alpha_1 \right)_n \ldots \left( \alpha_q \right)_n}{\eta p(\mu(p+1) - 1)n!},
\]
or equivalently
\[
|z| \leq \left\{ \frac{(p - \theta)n(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) \left( \alpha_1 \right)_n \ldots \left( \alpha_q \right)_n}{n(n + 2p - \theta) - \eta(\mu(p+1) - 1)n!} \right\}^{\frac{1}{n+p}}, \quad n \geq p,
\]
which follows the result.

**Theorem 8:** Let the function \( f(z) \) be given by (1) in the class \( \sum_p (\lambda, \mu, \eta, \alpha, q, s) \). Then, the integral operator
\[
\omega(z) = \epsilon \int_{0}^{1} u^z f(uz)\,du, \quad (0 < u \leq 1, 0 < \epsilon < \infty), \tag{27}
\]
is in the class \( \sum_p (\lambda, \mu, \eta, \alpha, q, s) \), where
\[
\tau = \frac{\epsilon(\lambda(p-1)2p - \eta(\mu(p+1) - 1) + (e + p + 1)(\lambda(p-1)2p - \eta(\mu(p+1) - 1)) \eta \epsilon(p-2)(\mu(p+1) - 1)}{(e + p + 1)(p + 1)(\lambda(p-1)2p - \eta(\mu(p+1) - 1)) \eta \epsilon(p-2)(\mu(p+1) - 1)}.
\]
The result is sharp for the function \( f(z) \) given by (15).
Proof: Let
\[ f(z) = \frac{1}{2^n} + \sum_{n=p}^\infty a_n z^n \]
is in the class \( \sum \) \( (\lambda, \mu, \eta, \alpha, s) \). Then
\[
\omega(z) = \varepsilon \int_0^1 u^\tau f(uz) du = \int_0^1 \left( \frac{1}{2^n} - \sum_{n=p}^\infty u^{n+\varepsilon} a_n z^n \right) d\varepsilon
\]
\[
= \frac{1}{2^n} + \sum_{n=p}^\infty \frac{\varepsilon}{\varepsilon + n + 1} a_n z^n.
\]
It is enough to show that
\[
\sum_{n=p}^\infty \frac{\varepsilon(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)}{(\varepsilon + n + 1)\mu(\mu(p+1) - 1)n!}
\]
\[
\leq 1. \tag{28}
\]
Since \( f(z) \in \sum \) \( (\lambda, \mu, \eta, \alpha, s) \), then by Theorem 1, we get or equivalently
\[
\tau \leq \frac{\varepsilon(\lambda(n-1)(n+p) - \eta)(\mu(p+1) - 1) + (\varepsilon + n + 1)(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1))}{(\varepsilon + n + 1)(\lambda(n-1)(n+p) - \eta(\mu(n-2) + 1)) + \varepsilon(n-2)(\mu(p+1) - 1)} = \omega(n).
\]
A simple computation will show that \( \omega(n) \) is increasing function of \( n \).
This means that \( \omega(n) \geq \omega(p) \). Using this, we obtain the result.

References


