Coefficient Bounds and Fekete-Szegő Problem for a New Subclasses of Holomorphic Bi-Univalent Functions Defined by Horadam polynomials

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ABSTRACT

In the present paper, by making use the Horadam polynomials, we introduce and investigate two new subclasses $\mathcal{H}_{\Sigma}(\alpha, \mu, r)$ and $\mathcal{R}_{\Sigma}(\gamma, \delta, \lambda, r)$ of the function class Σ of holomorphic bi-univalent functions in the open unit disk Δ . For functions belonging to these subclasses, we obtain upper bounds for the second and third coefficients and discuss Fekete-Szegő problem. Furthermore, we point out several new special cases of our results.

1. Introduction

Let $\mathcal{A}$ symbolize the class of function $f(z)$ which are holomorphic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized under the conditions $f(0) = 0$ and $f'(0) = 1$ and having the following shape:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let $\mathcal{S}$ symbolize the subclass of functions in $\mathcal{A}$ which are univalent in Δ . Since univalent functions are one-to-one, they are invertible and inverse functions need not be defined on the

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entire unit disk Δ . However, the famous Koebe one-quarter theorem [5] ensure that the image of the unit disk Δ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, each function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \quad (1.2)$$

Let D be the class of functions F which is holomorphic in Δ with

$$F(0) = 0 \text{ and } |F(z)| < 1 \quad (z \in \Delta).$$

Let $\mathcal{N}(z)$ and $\mathcal{M}(z)$ be holomorphic in Δ then the function $\mathcal{N}(z)$ is said to subordinate to $\mathcal{M}(z)$ in Δ written by

$$\mathcal{N}(z) < \mathcal{M}(z) \quad (z \in \Delta), \quad (1.3)$$

such that $\mathcal{N}(z) = \mathcal{M}(F(z)) \quad (z \in \Delta)$. From the definition of the subordination, it is easy to show that the subordination (1.3) implies that

$$\mathcal{N}(0) = \mathcal{M}(0) \text{ and } \mathcal{N}(\Delta) \subset \mathcal{M}(\Delta). \quad (1.4)$$

In particular, if $\mathcal{M}(z)$ is univalent in Δ , then the subordination (1.3) is equivalent to the condition (1.4).

The function $f \in \mathcal{A}$ is considered bi-univalent in Δ if together f^{-1} and f are univalent in Δ . Indicated by the Taylor-Maclaurin series expansion (1.1), the class of all bi-univalent functions in Δ can be symbolized by Σ . In the year 2010, Srivastava et al. [10] refreshed the study of various classes of bi-univalent functions. Moreover, many penmans explored bounds for different subclasses of bi-univalent functions (see, for example [3,4,8,11,12]). The coefficient estimate problem involving the bound of $|a_n|$ ($n \in \mathbb{N} - \{1,2\}, \mathbb{N} = \{1,2,3, \dots\}$) is still an open problem.

The Horadam polynomials $h_k(r)$ are defined by the following recurrence relation (see [7])

$$h_k(r) = prh_{k-1}(r) + qh_{k-2}(r), \quad (r \in \mathbb{R}, k \in \mathbb{N} - \{1,2\}, \mathbb{N} = \{1,2,3, \dots\}), \quad (1.5)$$

with $h_1(r) = e, h_2(r) = br$, where e, b, p , and q are some real constants. It is very clear from (1.5) that $h_3(r) = pbr^2 + eq$. The generating function of the Horadam polynomials $h_k(r)$ is given by (see [6])

$$\Psi(r, z) = \sum_{k=1}^{\infty} h_k(r) z^{k-1} = \frac{e + (b - ep)rz}{1 - prz - qz^2}.$$

2. Coefficient bounds and Fekete–Szegő inequality for the class $\mathcal{H}_\Sigma(\alpha, \mu, r)$

Definition (2.1): A function $f \in \Sigma$ is said to be in the class $\mathcal{H}_\Sigma(\alpha, \mu, r)$ for $0 \leq \alpha \leq 1$, $0 \leq \mu < 1$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$(1 - \alpha) \left(\frac{zf'(z)}{(1 - \mu)f(z) + \mu zf'(z)} \right) + \alpha \left(\frac{f'(z) + zf''(z)}{f'(z) + \mu zf''(z)} \right) < \Psi(r, z) + 1 - e \quad (2.1)$$

and

$$(1 - \alpha) \left(\frac{wg'(w)}{(1 - \mu)g(w) + \mu wg'(w)} \right) + \alpha \left(\frac{g'(w) + wg''(w)}{g'(w) + \mu wg''(w)} \right) < \Psi(r, w) + 1 - e, \quad (2.2)$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

We note that for $\alpha = 0$ in Definition (2.1), we have the following definition:

Definition (2.2): A function $f \in \Sigma$ is said to be in the class $T_\Sigma(\mu, r)$ for $0 \leq \mu < 1$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$\left(\frac{zf'(z)}{(1 - \mu)f(z) + \mu zf'(z)} \right) < \Psi(r, z) + 1 - e \quad (2.3)$$

and

$$\left(\frac{wg'(w)}{(1 - \mu)g(w) + \mu wg'(w)} \right) < \Psi(r, w) + 1 - e, \quad (2.4)$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

We note that for $\alpha = 1$ in Definition (2.1), we have the following definition:

Definition (2.3): A function $f \in \Sigma$ is said to be in the class $\mathcal{C}_\Sigma(\mu, r)$ for $0 \leq \mu < 1$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$\left(\frac{f'(z) + zf''(z)}{f'(z) + \mu zf''(z)} \right) < \Psi(r, z) + 1 - e \quad (2.5)$$

and

$$\left(\frac{g'(w) + wg''(w)}{g'(w) + \mu wg''(w)} \right) < \Psi(r, w) + 1 - e, \quad (2.6)$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

Remark (2.1)

- 1) For $\mu = 0$, the function class $\mathcal{H}_\Sigma(\alpha, \mu, r)$ shortens to the function class $\mathcal{M}_\sigma(\alpha, r)$ presented and investigated by Abirami et al. [1].
- 2) For $\alpha = 0$ and $\mu = 0$, the function class $\mathcal{H}_\Sigma(\alpha, \mu, r)$ shortens to the function class $\mathcal{W}_\Sigma(r)$ presented and investigated by Srivastava et al. [9].
- 3) For $\alpha = 1$ and $\mu = 0$, the function class $\mathcal{H}_\Sigma(\alpha, \mu, r)$ shortens to the function class $\mathcal{K}_\Sigma(r)$ presented and investigated by Abirami et al. [1].

Theorem (2.1): For $0 \leq \alpha \leq 1$, $0 \leq \mu < 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{H}_\Sigma(\alpha, \mu, r)$. Then

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{|[(\mu^2 - 2\mu + 1) + \alpha(3\mu^2 - 4\mu + 1))b - (1 + \alpha)^2(1 - \mu)^2p]br^2 - (1 + \alpha)^2(1 - \mu)^2eq|}} \quad (2.7)$$

and

$$|a_3| \leq \frac{b^2r^2}{(1 + \alpha)^2(1 - \mu)^2} + \frac{|br|}{2(1 + 2\alpha)(1 - \mu)}, \quad (2.8)$$

and for some $v \in \mathbb{R}$,

$$|a_3 - va_2^2| \leq \begin{cases} \frac{|br|}{2(1+2\alpha)(1-\mu)} & \text{if} \\ \left(|v-1| \leq \frac{|[(\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1))b-(1+\alpha)^2(1-\mu)^2p]br^2-(1+\alpha)^2(1-\mu)^2eq|}{2(1+2\alpha)(1-\mu)b^2r^2} \right) \\ \frac{|br|^3|v-1|}{|[(\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1))b-(1+\alpha)^2(1-\mu)^2p]br^2-(1+\alpha)^2(1-\mu)^2eq|} & \text{if} \\ \left(|v-1| \geq \frac{|[(\mu^2-2\mu+1)+\alpha(3\mu^2-4\mu+1))b-(1+\alpha)^2(1-\mu)^2p]br^2-(1+\alpha)^2(1-\mu)^2eq|}{2(1+2\alpha)(1-\mu)b^2r^2} \right). \end{cases} \quad (2.9)$$

Proof: Let $f \in \mathcal{H}_\Sigma(\alpha, \mu, r)$. Then there are two holomorphic functions $k, y: \Delta \rightarrow \Delta$ given by

$$|k(z)| = k_1z + k_2z^2 + k_3z^3 + \dots \quad (z \in \Delta) \quad (2.10)$$

and

$$|y(w)| = y_1w + y_2w^2 + y_3w^3 + \dots \quad (w \in \Delta), \quad (2.11)$$

with $k(0) = y(0) = 0$, $|k(z)| < 1$, $|y(w)| < 1$ and $z, w \in \Delta$ such that

$$(1 - \alpha) \left(\frac{zf'(z)}{(1 - \mu)f(z) + \mu zf'(z)} \right) + \alpha \left(\frac{f'(z) + zf''(z)}{f'(z) + \mu zf''(z)} \right) < \Psi(r, k(z)) + 1 - e$$

and

$$(1 - \alpha) \left(\frac{wg'(w)}{(1 - \mu)g(w) + \mu wg'(w)} \right) + \alpha \left(\frac{g'(w) + wg''(w)}{g'(w) + \mu wg''(w)} \right) < \Psi(r, y(w)) + 1 - e.$$

Or, equivalently,

$$(1 - \alpha) \left(\frac{zf'(z)}{(1 - \mu)f(z) + \mu zf'(z)} \right) + \alpha \left(\frac{f'(z) + zf''(z)}{f'(z) + \mu zf''(z)} \right) = 1 + h_1(r) - e + h_2(r)k(z) + h_3(r)[k(z)]^2 + \dots \quad (2.12)$$

and

$$(1 - \alpha) \left(\frac{wg'(w)}{(1 - \mu)g(w) + \mu wg'(w)} \right) + \alpha \left(\frac{g'(w) + wg''(w)}{g'(w) + \mu wg''(w)} \right) = 1 + h_1(r) - e + h_2(r)y(w) + h_3(r)[y(w)]^2 + \dots \quad (2.13)$$

Combining (2.10), (2.11), (2.12) and (2.13) yields

$$(1 - \alpha) \left(\frac{zf'(z)}{(1 - \mu)f(z) + \mu zf'(z)} \right) + \alpha \left(\frac{f'(z) + zf''(z)}{f'(z) + \mu zf''(z)} \right) = 1 + h_2(r)k_1z + [h_2(r)k_2 + h_3(r)k_1^2]z^2 + \dots \quad (2.14)$$

and

$$(1 - \alpha) \left(\frac{wg'(w)}{(1 - \mu)g(w) + \mu wg'(w)} \right) + \alpha \left(\frac{g'(w) + wg''(w)}{g'(w) + \mu wg''(w)} \right) = 1 + h_2(r)y_1w + [h_2(r)y_2 + h_3(r)y_1^2]w^2 + \dots \quad (2.15)$$

It is clear that if $|k(z)| < 1$ and $|y(w)| < 1$, $z, w \in \Delta$, then

$$|k_i| \leq 1 \text{ and } |y_i| \leq 1 \quad (i \in \mathbb{N}). \quad (2.16)$$

From (2.14) and (2.15), it follows that

$$(1 + \alpha)(1 - \mu)a_2 = h_2(r)k_1, \quad (2.17)$$

$$(1 + 3\alpha)(\mu^2 - 1)a_2^2 + 2(1 + 2\alpha)(1 - \mu)a_3 = h_2(r)k_2 + h_3(r)k_1^2, \quad (2.18)$$

$$-(1 + \alpha)(1 - \mu)a_2 = h_2(r)y_1 \quad (2.19)$$

and

$$[(\mu^2 - 4\mu + 3) + \alpha(3\mu^2 - 8\mu + 5)]a_2^2 - 2(1 + 2\alpha)(1 - \mu)a_3 = h_2(r)y_2 + h_3(r)y_1^2. \quad (2.20)$$

From (2.17) and (2.19), we get

$$k_1 = -y_1 \quad (2.21)$$

and

$$2(1 + \alpha)^2(1 - \mu)^2 a_2^2 = [h_2(r)]^2(k_1^2 + y_1^2). \quad (2.22)$$

If we add (2.18) to (2.20), we attain

$$[2(\mu^2 - 2\mu + 1) + 2\alpha(3\mu^2 - 4\mu + 1)]a_2^2 = h_2(r)(k_2 + y_2) + h_3(r)(k_1^2 + y_1^2). \quad (2.23)$$

By using (2.22) in (2.23), we deduce that

$$\left[2(\mu^2 - 2\mu + 1) + 2\alpha(3\mu^2 - 4\mu + 1) - \frac{2(1 + \alpha)^2(1 - \mu)^2 h_3(r)}{[h_2(r)]^2} \right] a_2^2 = h_2(r)(k_2 + y_2), \quad (2.24)$$

which yields

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{|[(\mu^2 - 2\mu + 1) + \alpha(3\mu^2 - 4\mu + 1))b - (1 + \alpha)^2(1 - \mu)^2 p]br^2 - (1 + \alpha)^2(1 - \mu)^2 eq|}}.$$

Next, by subtracting (2.20) from (2.18), we have

$$4(1 + 2\alpha)(1 - \mu)(a_3 - a_2^2) = h_2(r)(k_2 - y_2) + h_3(r)(k_1^2 - y_1^2). \quad (2.25)$$

In view of (2.21) and (2.22), we obtain from (2.25)

$$a_3 = \frac{[h_2(r)]^2(k_1^2 + y_1^2)}{2(1 + \alpha)^2(1 - \mu)^2} + \frac{h_2(r)(k_2 - y_2)}{4(1 + 2\alpha)(1 - \mu)}.$$

Hence using (1.5), we deduce that

$$|a_3| \leq \frac{b^2 r^2}{(1 + \alpha)^2(1 - \mu)^2} + \frac{|br|}{2(1 + 2\alpha)(1 - \mu)}.$$

Finally, by using (2.24) and (2.25) for some $v \in \mathbb{R}$, we obtain

$$\begin{aligned} a_3 - va_2^2 &= \frac{h_2(r)(k_2 - y_2)}{4(1 + 2\alpha)(1 - \mu)} \\ &\quad + \frac{[h_2(r)]^3(1 - v)(k_2 + y_2)}{[2(\mu^2 - 2\mu + 1) + 2\alpha(3\mu^2 - 4\mu + 1)][h_2(r)]^2 - 2(1 + \alpha)^2(1 - \mu)^2 h_3(r)} \end{aligned}$$

$$= \frac{h_2(r)}{2} \left[\left(\Omega(v, r) + \frac{1}{2(1 + 2\alpha)(1 - \mu)} \right) k_2 + \left(\Omega(v, r) - \frac{1}{2(1 + 2\alpha)(1 - \mu)} \right) y_2 \right],$$

where

$$\Omega(v, r) = \frac{[h_2(r)]^2(1 - v)}{[(\mu^2 - 2\mu + 1) + \alpha(3\mu^2 - 4\mu + 1)][h_2(r)]^2 - (1 + \alpha)^2(1 - \mu)^2 h_3(r)}.$$

According to (1.5), we deduce that

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|br|}{2(1+2\alpha)(1-\mu)} & \text{if } \left(0 \leq |\Omega(\nu, r)| \leq \frac{1}{2(1+2\alpha)(1-\mu)}\right) \\ |br||\Omega(\nu, r)| & \text{if } \left(|\Omega(\nu, r)| \geq \frac{1}{2(1+2\alpha)(1-\mu)}\right). \end{cases}$$

After some computations, we get

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|br|}{2(1+2\alpha)(1-\mu)} & \text{if} \\ \left(|\nu - 1| \leq \frac{\left| \left[((\mu^2 - 2\mu + 1) + \alpha(3\mu^2 - 4\mu + 1))b - (1+\alpha)^2(1-\mu)^2p \right] br^2 - (1+\alpha)^2(1-\mu)^2eq \right|}{2(1+2\alpha)(1-\mu)b^2r^2} \right) \\ \frac{|br|^3|\nu - 1|}{\left| \left[((\mu^2 - 2\mu + 1) + \alpha(3\mu^2 - 4\mu + 1))b - (1+\alpha)^2(1-\mu)^2p \right] br^2 - (1+\alpha)^2(1-\mu)^2eq \right|} & \text{if} \\ \left(|\nu - 1| \geq \frac{\left| \left[((\mu^2 - 2\mu + 1) + \alpha(3\mu^2 - 4\mu + 1))b - (1+\alpha)^2(1-\mu)^2p \right] br^2 - (1+\alpha)^2(1-\mu)^2eq \right|}{2(1+2\alpha)(1-\mu)b^2r^2} \right). \end{cases}$$

By putting $\alpha = 0$ in Theorem (2.1), we attain the corollary in the below:

Corollary (2.1): Let the function $f(z)$ indicated by (1.1) be in the class $T_{\Sigma}(\mu, r)$ ($0 \leq \mu < 1, r \in \mathbb{R}$). Then

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{\left| \left[(\mu^2 - 2\mu + 1)b - (1-\mu)^2p \right] br^2 - (1-\mu)^2eq \right|}}$$

and

$$|a_3| \leq \frac{b^2r^2}{(1-\mu)^2} + \frac{|br|}{2(1-\mu)},$$

and for some $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|br|}{2(1-\mu)} & \text{if} \\ \left(|\nu - 1| \leq \frac{\left| \left[(\mu^2 - 2\mu + 1)b - (1-\mu)^2p \right] br^2 - (1-\mu)^2eq \right|}{2(1-\mu)b^2r^2} \right) \\ \frac{|br|^3|\nu - 1|}{\left| \left[(\mu^2 - 2\mu + 1)b - (1-\mu)^2p \right] br^2 - (1-\mu)^2eq \right|} & \text{if} \\ \left(|\nu - 1| \geq \frac{\left| \left[(\mu^2 - 2\mu + 1)b - (1-\mu)^2p \right] br^2 - (1-\mu)^2eq \right|}{2(1-\mu)b^2r^2} \right). \end{cases}$$

By putting $\alpha = 1$ in Theorem (2.1), we attain the corollary in the below:

Corollary (2.2): Let $f(z)$ indicated by (1.1) be in the class $\mathcal{C}_{\Sigma}(\mu, r)$ ($0 \leq \mu < 1, r \in \mathbb{R}$). Then

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{|[(4\mu^2 - 6\mu + 2)b - 4(1 - \mu)^2p]br^2 - 4(1 - \mu)^2eq|}}$$

and

$$|a_3| \leq \frac{b^2r^2}{4(1 - \mu)^2} + \frac{|br|}{6(1 - \mu)},$$

and for some $v \in \mathbb{R}$,

$$|a_3 - va_2^2| \leq \begin{cases} \frac{|br|}{6(1 - \mu)} & \text{if} \\ \left(|v - 1| \leq \frac{|[(2\mu^2 - 3\mu + 1)b - 2(1 - \mu)^2p]br^2 - 2(1 - \mu)^2eq|}{3(1 - \mu)b^2r^2} \right) \\ \frac{|br|^3|v - 1|}{|[(4\mu^2 - 6\mu + 2)b - 4(1 - \mu)^2p]br^2 - 4(1 - \mu)^2eq|} & \text{if} \\ \left(|v - 1| \geq \frac{|[(2\mu^2 - 3\mu + 1)b - 2(1 - \mu)^2p]br^2 - 2(1 - \mu)^2eq|}{3(1 - \mu)b^2r^2} \right). \end{cases}$$

Remark (2.2): In Theorem (2.1), if we choose

- 1) $\mu = 0$, then we get the outcomes which was proven by Abirami et al. [1].
- 2) $\alpha = 0$ and $\mu = 0$, then we get the outcomes which was proven by Srivastava et al. [9].
- 3) $\alpha = 1$ and $\mu = 0$, then we attain the outcomes which was proven by Abirami et al. [1].

3. Coefficient bounds and Fekete–Szegő inequality for the class $\mathcal{R}_\Sigma(Y, \delta, \lambda, r)$

Definition (3.1): A function $f \in \Sigma$ is said to be in the class $\mathcal{R}_\Sigma(Y, \delta, \lambda, r)$ for $Y \geq 0, 0 \leq \delta \leq 1, \lambda \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$1 + \frac{1}{\lambda} \left[Y\delta(zf''(z) - 2) + (\delta(Y + 1) + Y)f'(z) + (1 - Y)(1 - \delta)\frac{f(z)}{z} - 1 \right] < \Psi(r, z) + 1 - e \quad (3.1)$$

and

$$1 + \frac{1}{\lambda} \left[Y\delta(wg''(w) - 2) + (\delta(Y + 1) + Y)g'(w) + (1 - Y)(1 - \delta)\frac{g(w)}{w} - 1 \right] < \Psi(r, w) + 1 - e, \quad (3.2)$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

We note that for $\delta = 0$ in Definition (3.1), we have the following definition:

Definition (3.2): A function $f \in \Sigma$ is said to be in the class $\mathcal{B}_\Sigma(Y, \lambda, r)$ for $Y \geq 0, \lambda \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{R}$, if the following conditions of subordination are fulfilled:

$$1 + \frac{1}{\lambda} \left[Y f'(z) + (1 - Y) \frac{f(z)}{z} - 1 \right] < \Psi(r, z) + 1 - e \quad (3.3)$$

and

$$1 + \frac{1}{\lambda} \left[Y g'(w) + (1 - Y) \frac{g(w)}{w} - 1 \right] < \Psi(r, w) + 1 - e, \quad (3.4)$$

where the function $g = f^{-1}$ is indicated by (1.2) and e is real constant.

Remark (3.1)

For $Y = \lambda = 1$ and $\delta = 0$, the function class $\mathcal{R}_\Sigma(Y, \delta, \lambda, r)$ shortens to the function class Σ' presented and investigated by Alamoush [2].

Theorem (3.1): For $Y \geq 0, 0 \leq \delta \leq 1, \lambda \in \mathbb{C} \setminus \{0\}$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{R}_\Sigma(Y, \delta, \lambda, r)$. Then

$$|a_2| \leq \frac{|\lambda b r| \sqrt{|b r|}}{\sqrt{[\lambda(2\delta(5Y+1) + 2Y+1)b - (\delta(5Y+1) + Y+1)^2 p] b r^2 - (\delta(5Y+1) + Y+1)^2 e q}}} \quad (3.5)$$

and

$$|a_3| \leq \frac{|\lambda|^2 b^2 r^2}{(\delta(5Y+1) + Y+1)^2} + \frac{|\lambda b r|}{(2\delta(5Y+1) + 2Y+1)}, \quad (3.6)$$

and for some $v \in \mathbb{R}$,

$$|a_3 - v a_2^2| \leq \begin{cases} \frac{|\lambda b r|}{2\delta(5Y+1) + 2Y+1} \text{ if } \\ \left(|v - 1| \leq \frac{[\lambda(2\delta(5Y+1) + 2Y+1)b - (\delta(5Y+1) + Y+1)^2 p] b r^2 - (\delta(5Y+1) + Y+1)^2 e q}{|\lambda| (b r)^2 (2\delta(5Y+1) + 2Y+1)} \right) \\ \frac{|\lambda|^2 |b r|^3 |v - 1|}{[\lambda(2\delta(5Y+1) + 2Y+1)b - (\delta(5Y+1) + Y+1)^2 p] b r^2 - (\delta(5Y+1) + Y+1)^2 e q} \text{ if } \\ \left(|v - 1| \geq \frac{[\lambda(2\delta(5Y+1) + 2Y+1)b - (\delta(5Y+1) + Y+1)^2 p] b r^2 - (\delta(5Y+1) + Y+1)^2 e q}{|\lambda| (b r)^2 (2\delta(5Y+1) + 2Y+1)} \right). \end{cases} \quad (3.7)$$

Proof: Let $f \in \mathcal{R}_\Sigma(Y, \delta, \lambda, r)$. Then there are two holomorphic functions $k, y: \Delta \rightarrow \Delta$ given by

$$|k(z)| = k_1 z + k_2 z^2 + k_3 z^3 + \dots \quad (z \in \Delta) \quad (3.8)$$

and

$$|y(w)| = y_1 w + y_2 w^2 + y_3 w^3 + \dots \quad (w \in \Delta), \quad (3.9)$$

with $k(0) = y(0) = 0, |k(z)| < 1, |y(w)| < 1$ and $z, w \in \Delta$ such that

$$1 + \frac{1}{\lambda} \left[Y\delta(zf''(z) - 2) + (\delta(Y+1) + Y)f'(z) + (1-Y)(1-\delta)\frac{f(z)}{z} - 1 \right] < \Psi(r, k(z)) + 1 - e$$

and

$$1 + \frac{1}{\lambda} \left[Y\delta(wg''(w) - 2) + (\delta(Y+1) + Y)g'(w) + (1-Y)(1-\delta)\frac{g(w)}{w} - 1 \right] < \Psi(r, y(w)) + 1 - e.$$

Or, equivalently,

$$1 + \frac{1}{\lambda} \left[Y\delta(zf''(z) - 2) + (\delta(Y+1) + Y)f'(z) + (1-Y)(1-\delta)\frac{f(z)}{z} - 1 \right] \\ = 1 + h_1(r) - e + h_2(r)k(z) + h_3(r)[k(z)]^2 + \dots \quad (3.10)$$

and

$$1 + \frac{1}{\lambda} \left[Y\delta(wg''(w) - 2) + (\delta(Y+1) + Y)g'(w) + (1-Y)(1-\delta)\frac{g(w)}{w} - 1 \right] \\ = 1 + h_1(r) - e + h_2(r)y(w) + h_3(r)[y(w)]^2 + \dots. \quad (3.11)$$

Combining (3.8), (3.9), (3.10) and (3.11) yields

$$1 + \frac{1}{\lambda} \left[Y\delta(zf''(z) - 2) + (\delta(Y+1) + Y)f'(z) + (1-Y)(1-\delta)\frac{f(z)}{z} - 1 \right] \\ = 1 + h_2(r)k_1z + [h_2(r)k_2 + h_3(r)k_1^2]z^2 + \dots \quad (3.12)$$

and

$$1 + \frac{1}{\lambda} \left[Y\delta(wg''(w) - 2) + (\delta(Y+1) + Y)g'(w) + (1-Y)(1-\delta)\frac{g(w)}{w} - 1 \right] \\ = 1 + h_2(r)y_1w + [h_2(r)y_2 + h_3(r)y_1^2]w^2 + \dots. \quad (3.13)$$

It is clear that if $|k(z)| < 1$ and $|y(w)| < 1, z, w \in \Delta$, then

$$|k_i| \leq 1 \text{ and } |y_i| \leq 1 \quad (i \in \mathbb{N}). \quad (3.14)$$

Equating the coefficients in (3.12) and (3.13), we find that

$$\frac{\delta(5Y+1) + Y + 1}{\lambda} a_2 = h_2(r)k_1, \quad (3.15)$$

$$\frac{2\delta(5Y+1) + 2Y + 1}{\lambda} a_3 = h_2(r)k_2 + h_3(r)k_1^2, \quad (3.16)$$

$$-\frac{\delta(5Y+1)+Y+1}{\lambda}a_2 = h_2(r)y_1 \quad (3.17)$$

and

$$\frac{2\delta(5Y+1)+2Y+1}{\lambda}(2a_2^2 - a_3) = h_2(r)y_2 + h_3(r)y_1^2. \quad (3.18)$$

From (3.15) and (3.17), we get

$$k_1 = -y_1 \quad (3.19)$$

and

$$\frac{2(\delta(5Y+1)+Y+1)^2}{\lambda^2}a_2^2 = [h_2(r)]^2(k_1^2 + y_1^2). \quad (3.20)$$

If we add (3.16) to (3.18), we attain

$$\frac{2(2\delta(5Y+1)+2Y+1)}{\lambda}a_2^2 = h_2(r)(k_2 + y_2) + h_3(r)(k_1^2 + y_1^2). \quad (3.21)$$

By using (3.20) in (3.21), we deduce that

$$\left[\frac{2(2\delta(5Y+1)+2Y+1)}{\lambda} - \frac{2(\delta(5Y+1)+Y+1)^2 h_3(r)}{\lambda^2 [h_2(r)]^2} \right] a_2^2 = h_2(r)(k_2 + y_2), \quad (3.22)$$

which yields

$$|a_2| \leq \frac{|\lambda br| \sqrt{|br|}}{\sqrt{|[\lambda(2\delta(5Y+1)+2Y+1)b - (\delta(5Y+1)+Y+1)^2 p]br^2 - (\delta(5Y+1)+Y+1)^2 eq|}}.$$

Next, by subtracting (3.18) from (3.16), we have

$$\left(\frac{2(2\delta(5Y+1)+2Y+1)}{\lambda} \right) (a_3 - a_2^2) = h_2(r)(k_2 - y_2) + h_3(r)(k_1^2 - y_1^2). \quad (3.23)$$

In view of (3.19) and (3.20), we obtain from (3.23)

$$a_3 = \frac{\lambda^2 [h_2(r)]^2 (k_1^2 + y_1^2)}{2(\delta(5Y+1)+Y+1)^2} + \frac{\lambda h_2(r)(k_2 - y_2)}{2(2\delta(5Y+1)+2Y+1)}.$$

Hence using (1.5), we deduce that

$$|a_3| \leq \frac{|\lambda|^2 b^2 r^2}{(\delta(5Y+1)+Y+1)^2} + \frac{|\lambda br|}{(2\delta(5Y+1)+2Y+1)}.$$

Finally, by using (3.22) and (3.23) for some $v \in \mathbb{R}$, we obtain

$$\begin{aligned}
a_3 - \nu a_2^2 &= \frac{\lambda h_2(r)(k_2 - y_2)}{2(2\delta(5Y+1) + 2Y+1)} \\
&\quad + \frac{\lambda^2 [h_2(r)]^3 (1 - \nu)(k_2 + y_2)}{2\lambda(2\delta(5Y+1) + 2Y+1)[h_2(r)]^2 - 2(\delta(5Y+1) + Y+1)^2 h_3(r)} \\
&= \frac{\lambda h_2(r)}{2} \left[\left(\Omega(\nu, r) + \frac{1}{2\delta(5Y+1) + 2Y+1} \right) k_2 + \left(\Omega(\nu, r) - \frac{1}{2\delta(5Y+1) + 2Y+1} \right) y_2 \right],
\end{aligned}$$

where

$$\Omega(\nu, r) = \frac{\lambda [h_2(r)]^2 (1 - \nu)}{\lambda [2\delta(5Y+1) + 2Y+1][h_2(r)]^2 - (\delta(5Y+1) + Y+1)^2 h_3(r)}.$$

According to (1.5), we deduce that

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\lambda br|}{2\delta(5Y+1) + 2Y+1} & \text{if } \left(0 \leq |\Omega(\nu, r)| \leq \frac{1}{2\delta(5Y+1) + 2Y+1} \right) \\ |\lambda br| |\Omega(\nu, r)| & \text{if } \left(|\Omega(\nu, r)| \geq \frac{1}{2\delta(5Y+1) + 2Y+1} \right). \end{cases}$$

After some computations, we get

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\lambda br|}{2\delta(5Y+1) + 2Y+1} & \text{if} \\ \left(|\nu - 1| \leq \frac{|\lambda(2\delta(5Y+1) + 2Y+1)b - (\delta(5Y+1) + Y+1)^2 p| br^2 - (\delta(5Y+1) + Y+1)^2 eq|}{|\lambda|(br)^2(2\delta(5Y+1) + 2Y+1)} \right) \\ \frac{|\lambda|^2 |br|^3 |\nu - 1|}{|\lambda(2\delta(5Y+1) + 2Y+1)b - (\delta(5Y+1) + Y+1)^2 p| br^2 - (\delta(5Y+1) + Y+1)^2 eq|} & \text{if} \\ \left(|\nu - 1| \geq \frac{|\lambda(2\delta(5Y+1) + 2Y+1)b - (\delta(5Y+1) + Y+1)^2 p| br^2 - (\delta(5Y+1) + Y+1)^2 eq|}{|\lambda|(br)^2(2\delta(5Y+1) + 2Y+1)} \right). \end{cases}$$

By putting $\delta = 0$ in Theorem (3.1), we attain the corollary in the below:

Corollary (3.1): Let the function $f(z)$ indicated by (1.1) be in the class $\mathcal{B}_\Sigma(Y, \lambda, r)$ ($Y \geq 0$, $\lambda \in \mathbb{C} \setminus \{0\}$, $r \in \mathbb{R}$). Then

$$|a_2| \leq \frac{|\lambda br| \sqrt{|br|}}{\sqrt{|\lambda(2Y+1)b - (Y+1)^2 p| br^2 - (Y+1)^2 eq|}}$$

and

$$|a_3| \leq \frac{|\lambda|^2 b^2 r^2}{(Y+1)^2} + \frac{|\lambda br|}{(2Y+1)},$$

and for some $\nu \in \mathbb{R}$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\lambda br|}{(2Y+1)} \text{ if} \\ \left(|v-1| \leq \frac{|[\lambda(2Y+1)b - (Y+1)^2p]br^2 - (Y+1)^2eq|}{|\lambda|(br)^2(2Y+1)} \right) \\ \frac{|\lambda|^2|br|^3|v-1|}{|[\lambda(2Y+1)b - (Y+1)^2p]br^2 - (Y+1)^2eq|} \text{ if} \\ \left(|v-1| \geq \frac{|[\lambda(2Y+1)b - (Y+1)^2p]br^2 - (Y+1)^2eq|}{|\lambda|(br)^2(2Y+1)} \right). \end{cases}$$

Remark (3.2)

If we put $Y = \lambda = 1$ and $\delta = 0$ in Theorem (3.1), we get the results which were given by Alamoush [2].

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