The Performance of Some Restricted Estimators In Restricted Linear Regression Model

Authors Names	ABSTRACT
<p>a. Bader Aboud Mohammed b. Mustafa Ismaeel Naif</p> <p>Received on: 19/2/2021 Revised on: 7/4/2021 Accepted on: 8/4/2021</p> <p>Keywords: <i>Linear regression model Restricted biased estimators Mean square error Multicollinearity Simulation study</i></p> <p>DOI:https://doi.org/10.29350/jops.2021.26.2.1287</p>	<p>In the linear regression model, the restricted biased estimation as one of important methods to address the high variance and multicollinearity problems. In this paper, we make the simulation study of some restricted biased estimators. The mean square error (MSE) criterion are used to make a comparison among them. According to the simulation study we observe that, the performance of the restricted modified unbiased ridge regression estimator (RMUR) was proposed by Bader and Mustafa Ismaeel Naif [16] is better than of these estimators. Numerical example has been considered to illustrate the performance of the estimators.</p>

1. Introduction

Consider the standard linear regression model

$$Y = X\beta + e, \quad (1.1)$$

where X is an $n \times p$ matrix of the explanatory variables, Y is an $n \times 1$ vector of the response, β is a $1 \times p$ vector of the unknown parameters and e is an $n \times 1$ vector of the random errors with the mean $E(e) = 0$ and the variance $Var(e) = \sigma^2 I_n$, I_n is identity matrix of order n . sometimes, we have a linear restriction on β as:

$$R\beta = r \quad (1.2)$$

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where R is an $m \times p$ non zero matrix with $\text{rank}(R) = m < p$ and r is an $m \times 1$ vector. The ordinary restricted least square estimator (RLS) is given by

$$\hat{\beta}_{RLS} = \hat{\beta} + S^{-1}R'(RS^{-1}R')^{-1}(r - R\hat{\beta}), \quad (1.3)$$

where $S^{-1} = (X'X)^{-1}$ and $\hat{\beta} = S^{-1}X'y$ is the least square estimator (OLSE). The RLS estimator is widely used because it improves the variance value, but most researchers often encounter of the multicollinearity problem. That is, $X'X$ is always ill-conditioned due to the linear relationship among the regressors of X matrix. Therefore, the unknown coefficient, estimated by RLS is usually instable and misleading(Najarian [15]). In order to improve the performance of the estimators, the researchers proposed to use the bias estimation technique. Therefore, Sarkar [5], obtained the restricted ridge regression estimator (RRR) by combining in particular way the two approaches underling the RLS and the ORR estimator given as

$$\hat{\beta}_R = (S + kI)^{-1}X'Y. \quad (1.4)$$

The RRR estimator considered as a modify the RLS estimator by using ORR philosophy The RRR is denoted by $\beta^*(k)$ is given as:

$$\beta^*(k) = W\hat{\beta}_{RLS}, \quad (1.5)$$

where $W = (I + kS^{-1})^{-1}$. However the $\beta^*(k)$ dose not satisfy the linear restriction (1.2). Kaciranlar [17] proposed the restricted Liu estimator (RLE) by following the way that Sarkar [18] has done while obtaining the RRR estimator. By replacing, in the Liu estimator, the OLSE estimator by the RLS estimator they obtained the restricted Liu estimator as the follows:

$$\hat{\beta}_{rd} = (X'X + I)^{-1}(X'X + dI)\hat{\beta}_{RLS}. \quad (1.6)$$

Also the RLE estimator dose not satisfy the linear restriction (1.2). Alheety [2], obtained the restricted $(k - d)$ class estimator to improve the restricted Liu estimator, by substituting the ORR for OLSE estimators. The restricted $(k - d)$ estimator is general case and includes the RLS, RRR and RLE estimators. The restricted $(k - d)$ class is denoted by $\hat{\beta}(k, d)$ as the follows

$$\hat{\beta}(k, d) = F_{k,d} \hat{\beta}_{RLS}, \quad (1.7)$$

where $F_{k,d} = (X'X + I)^{-1}(X'X + (k + d)I)(X'X + kI)^{-1}(X'X)$, the special cases of the restricted $(k - d)$ class estimator are given by

$$\hat{\beta}(0,1) = \hat{\beta}_{RLS}. \quad (1.8)$$

$$\hat{\beta}(k, 1 - k) = \beta^*(k). \quad (1.9)$$

$$\hat{\beta}(0, d) = \hat{\beta}(d), \quad (1.10)$$

Where $\hat{\beta}(d)$ is the liu estimator . Bader and Mustafa [4], introduced the restricted modified unbiased ridge regression (RMUR) based on the modified unbiased ridge regression (MURR) was proposed by Batah and Gore [5], which is given as follows

$$\hat{\beta}_J(k) = \left[I - k(X'X + kI_p)^{-1} \right] (X'X + kI_p)^{-1} (X'Y + kJ)$$

The RMUR estimator is denoted by $\hat{\beta}_r^*(k)$ as the following

$$\hat{\beta}_r^*(k) = \hat{\beta}_J(k) + S_k^{-1} R' (R S_k^{-1} R')^{-1} (r - R \hat{\beta}_J(k)), \quad (1.11)$$

Hence the RMUR estimator is hold the linear restriction (2). Therefore, the goal of this paper is to review and compare some restricted estimators .In section 2 ,we study the properties of the RLS, RRR, (k - d) class and the RMUR estimators while in section 3, we make simulation study of these estimators by using the Matlab program . Section 4 contains numerical example, to illustrate the results Finally, the conclusions with some remarks are given in section 5 .

2.Restricted Estimators And Its Properties

In this section , we study the statistical properties of some restricted estimators and then, we show that, the performance of some the restricted estimators by using the mean square error matrix (MSE) criterion . The MSE of any estimator is given by:

$$MSE(\beta^*) = Var(\beta^*) + (bias(\beta^*)).(bias(\beta^*))', \quad (2.1)$$

where

$$Var(\beta^*) = E[(\beta^* - E(\beta^*))((\beta^* - E(\beta^*)))'], \quad (2.2)$$

and

$$Bias(\beta^*) = E(\beta^*) - \beta, \quad (2.3)$$

where $E(\beta^*)$ the expected value of β^*

2.1 Restricted Ridge Regression (RRR)

Sarkar [18] proposed the restricted ridge regression (RRR) estimator . The variance, bias and mean square error of the RRR estimator are given

$$Var(\beta^*(k)) = \sigma^2 W A W' \quad (2.4)$$

$$bias(\beta^*(k)) = -k S_k^{-1} \beta \quad (2.5)$$

$$MSE(\beta^*(k)) = \sigma^2 W A W' + k^2 \beta' S_k^{-2} \beta. \quad (2.6)$$

Thus the scalar men square error of the RRR estimator is given by

$$mse(\beta^*(k)) = \sigma^2 tr(WAW') + k^2 tr(\beta' S_k^{-2} \beta), \quad (2.7)$$

Where tr denote the trace of matrix.

2.2 Restricted (k - d) Class Estimator

Alheety [2] introduced the restricted (k - d) class estimator. The variance, bias and mean square error of the (k - d) class estimator are given

$$Var(\hat{\beta}(k, d)) = \sigma^2 F_{k,d} S^{-1} F_{k,d}' \quad (2.8)$$

$$bias(\hat{\beta}(k, d)) = (S + I)^{-1} (d(S + kI)^{-1} S - I) \beta, \quad (2.9)$$

$$MSE(\hat{\beta}(k, d)) = \sigma^2 F_{k,d} S^{-1} F_{k,d}' + (S + I)^{-1} (d(S + kI)^{-1} S - I) \beta \beta' (S + I)^{-1} (d(S + kI)^{-1} S - I). \quad (2.10)$$

So, the scalar mean square error is denoted by

$$mse(\beta^*) = tr Var(\beta^*) + \|E(\beta^*) - \beta\|^2$$

So that, the scalar mean square error of (k - d) class estimator is given by:

$$mse(\hat{\beta}(k, d)) = \sigma^2 tr(F_{k,d} S^{-1} F_{k,d}') + \beta' (d(S + kI)^{-1} S - I) (S + I)^{-2} (d(S + kI)^{-1} S - I) \beta. \quad (2.11)$$

2.3 Restricted Modified Unbiased Ridge Regression (RMUR) Estimator

We want to show that the properties of the RMUR estimator, so that, the variance, bias and mean square error of the RMUR estimator are given

$$Var(\hat{\beta}_r^*(k)) = \sigma^2 N_k S_k M S_k^{-1} S_k' M' N_k' \quad (2.12)$$

$$bias(\hat{\beta}_r^*(k)) = -k N_k \beta \quad (2.13)$$

$$MSE(\hat{\beta}_r^*(k)) = \sigma^2 N_k S_k M S_k^{-1} S_k' M' N_k' + k^2 N_k \beta \beta' N_k', \quad (2.14)$$

where $N_k = (S_k^{-1} - S_k^{-1} R' (R S_k^{-1} R')^{-1} R S_k^{-1})$, $M = (I - k S_k^{-1}) = S S_k^{-1}$. So that the scalar mean square error is given by

$$mse(\hat{\beta}_r^*(k)) = \sigma^2 tr(N_k S S S_k^{-1} N_k) + k^2 tr(N_k \beta \beta' N_k) \quad (2.15)$$

2.4. Estimated ridge parameter k

To study the performance of the MSE , we using the different values of ridge parameter k to evaluate mean square error. Hoerl and Kennard [8], introduced k is denoted by k_{HK} as the follows:

$$k_{HK} = \frac{\hat{\sigma}^2}{\hat{\varphi}_{max}^2 OLSE}, \quad (2.16)$$

Where $\hat{\varphi}_{max}^2$ is the maximum element of $\hat{\varphi}_{OLSE}$. Hoerl [8], suggested k is denoted by

$$k_{HKB} = \frac{p \hat{\sigma}^2}{\hat{\varphi}_{OLSE}' \hat{\varphi}_{OLSE}}. \quad (2.17)$$

Lawless and Wang [12], suggested k denoted by k_{LW} as the follows:

$$k_{LW} = \frac{p \hat{\sigma}^2}{\hat{\varphi}_{OLSE}' X' X \hat{\varphi}_{OLSE}}. \quad (2.18)$$

Hocking [7], proposed k denoted by k_{HSL} as the follows:

$$k_{HSL} = \hat{\sigma}^2 \frac{(\sum_{i=1}^p (\lambda_i \hat{\varphi}_{OLSE})^2)}{\sum_{i=1}^p (\lambda_i \hat{\varphi}_{OLSE}^2)^2}. \quad (2.19)$$

Nomura [16], suggested k is denoted by k_{HMO} as the follows:

$$k_{HMO} = \frac{p \hat{\sigma}^2}{\sum_{i=1}^p \left(\frac{\hat{\varphi}_{iOLSE}}{1 + \lambda_i \left(\frac{\hat{\varphi}_{iOLSE}^2}{\hat{\sigma}^2} \right)^{\frac{1}{2}}} \right)}. \quad (2.20)$$

Kibria [11], introduced the estimators for k based on arithmetic mean (AM), geometric mean (GM), and median of $\frac{\hat{\sigma}^2}{\hat{\varphi}_i^2}$. These are defined as follows:

The estimator based on AM is denoted by k_{AM} as the follows:

$$k_{AM} = \frac{1}{p} \sum_{i=1}^p \frac{\hat{\sigma}^2}{\hat{\varphi}_{iOLSE}}, \quad (2.21)$$

based on (GM), the estimator k_{GM} as the follows:

$$k_{GM} = \frac{\hat{\sigma}^2}{(\prod_{i=1}^p \hat{\varphi}_{iOLSE}^2)^{1/p}}, \quad (2.22)$$

based on median, the ridge parameter k_{MED} is the follows:

$$k_{MED} = \text{midian} \left\{ \frac{\hat{\sigma}^2}{\hat{\varphi}_{OLSE}^2} \right\}. \quad (2.23)$$

Khalaf and Shukur [9] suggested based on k_{HK} denoted by k_{KS} as

$$k_{KS} = \frac{\lambda_{\max} \hat{\sigma}^2}{(n-p) \hat{\sigma}^2 + \lambda_{\max} \hat{\varphi}_{\max}^2 OLSE}, \quad (2.24)$$

Where $\lambda_{\max}$ the maximum eigenvalues of $X'X$.

Alkhamisi et al. [3] suggested the following estimators of k based on Kibria [11] , Khalaf and Shukur [9] , denoted by k_{sairth} as:

$$k_{sairth} = \frac{1}{p} \sum_{i=1}^p \frac{\lambda_i \hat{\sigma}^2}{(n-p) \hat{\sigma}^2 + \lambda_i \hat{\varphi}_i^2 OLSE}. \quad (2.25)$$

$$k_{smd} = \text{midian} \left(\frac{\lambda_i \hat{\sigma}^2}{(n-p) \hat{\sigma}^2 + \lambda_i \hat{\varphi}_i^2 OLSE} \right). \quad (2.26)$$

M .N. Lattef . et al [14] proposed the following estimators of k

$$k_{MU1} = \frac{\lambda_{med} \sum_{i=1}^p \hat{\varphi}_i^2 OLSE}{\lambda_{\max}}. \quad (2.27)$$

$$k_{MU2} = \left| \frac{p \hat{\sigma}^2}{\hat{\varphi}_{OLSE}' \hat{\varphi}_{OLSE}} - \frac{p \hat{\sigma}^2}{\hat{\varphi}_{OLSE}' X' X \hat{\varphi}_{OLSE}} \right|. \quad (2.28)$$

$$k_{MU3} = \min \left(\sqrt{\frac{\lambda_{\min} \sum_{i=1}^p \hat{\varphi}_i^2 OLSE}{\hat{\sigma}^2}} \right). \quad (2.29)$$

$$k_{MU4} = \max \left(\sqrt{\frac{\lambda_{\min} \sum_{i=1}^p \hat{\varphi}_i^2 OLSE}{\hat{\sigma}^2}} \right). \quad (2.30)$$

3. Simulation study

In this section , In this section, we make simulation study of the some restricted estimators by using the Matlab program. The purpose of this study to find the performance of this estimators. This simulation has been designed depends on specific factors that are expected to influence the properties of the estimators which may be subjected to a statistical investigation . Since the degree of the collinearity among several explanatory variables (X_s) is very essential, Kibria [10], was followed to generate X_s using the following equation :

$$X_{ij} = (1 - \phi^2)^{1/2} Z_{ij} + \phi Z_{ip}, i = 1, 2, \dots, n, j = 1, 2, \dots, p \quad (3.1)$$

where the Z_{ij} independent standard normal pseudo-random numbers and ϕ represents the correlation between any two variables. These variables are standardized so that $X'X$ is being in correlation form. The response variable y is considered by:

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} \dots \dots \dots + \beta_p X_{ip} + e_i, \quad i = 1, 2, \dots, n \quad (3.2)$$

where e_i is independent and identically distributed random variables (i.i.d.) $N(0, \sigma^2)$. Therefore, zero intercept for (3.2) will be assumed. Also the number of explanatory variables $p = 4$ while the values of σ are chosen as (0.1, 1, 5, 15). The correlation ϕ will choose as (0.85, 0.95, 0.99) and sample size n as (50, 100, 150). The coefficients $\beta_1, \beta_2, \dots, \beta_p$ are selected as the eigenvectors corresponding to the largest eigenvalue of the matrix $X'X$ subject to constraint $\beta'\beta = 1$. Thus, for all n, σ, ϕ, p , sets of X_s are created. The experiment was replicated 2000 times by creating new error terms. Estimated mean square error (EMSE) is calculated as follows:

$$EMSE(\beta^*) = \frac{1}{2000} \sum_{i=1}^{2000} (\beta^* - \beta)' (\beta^* - \beta),$$

where β^* would be any estimators (RLS, RRR, (k - d) class or RMUR).

Table1: : Estimated MSE when $n = 50$, $\phi = .85$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k \cdot d_{CLAS}}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k \cdot d_{CLASS}}$	β_{RMUR}
0.1	k_{HK}	1.5206	0.8447	0.8358	0.3429	1	k_{HK}	1.0477	0.9119	0.9104	0.4130
	k_{HKB}	1.5206	0.8446	0.8358	0.3429		k_{HKB}	1.0477	0.9099	0.9104	0.4301
	k_{LW}	1.5206	0.8447	0.8358	0.3429		k_{LW}	1.0477	0.9119	0.9104	0.4128
	k_{HSL}	1.5206	0.8386	0.8353	0.3537		k_{HSL}	1.0477	0.9019	0.8929	0.9975
	k_{HMO}	1.5206	0.8432	0.8357	0.3453		k_{HMO}	1.0477	0.8692	0.9092	0.7367
	k_{AM}	1.5206	0.8394	0.8354	0.3522		k_{AM}	1.0477	0.8753	0.9094	0.7031
	k_{GM}	1.5206	0.8447	0.8358	0.3429		k_{GM}	1.0477	0.9122	0.9104	0.4108
	k_{MED}	1.5206	0.8443	0.8357	0.3435		k_{MED}	1.0477	0.9125	0.9104	0.4088
	k_{KS}	1.5206	0.8440	0.8357	0.3440		k_{KS}	1.0477	0.9122	0.9104	0.4108
	k_{sarith}	1.5206	0.8443	0.8357	0.3435		k_{sarith}	1.0477	0.9125	0.9104	0.4083
	k_{SMD}	1.5206	0.8445	0.8357	0.3432		k_{SMD}	1.0477	0.9125	0.9104	0.4085
	k_{MU1}	1.5206	0.8105	0.8332	0.4126		k_{MU1}	1.0477	0.7932	0.8998	0.9791
	k_{MU2}	1.5206	0.8147	0.8335	0.4032		k_{MU2}	1.0477	0.8732	0.9094	0.7150
	k_{MU3}	1.5206	0.8440	0.8357	0.3440		k_{MU3}	1.0477	0.9122	0.9104	0.4108
	k_{MU4}	1.5206	0.8445	0.8357	0.3432		k_{MU4}	1.0477	0.9125	0.9104	0.4085

Table 2: : Estimated MSE when $n = 50$, $\phi = .85$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k \cdot d_{CLAS}}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k \cdot d_{CLASS}}$	β_{RMUR}
5	k_{HK}	1.0559	0.4920	0.4921	0.7350	15	k_{HK}	0.8984	0.8879	0.8892	0.8085
	k_{HKB}	1.0559	0.4923	0.4922	0.7233		k_{HKB}	0.8984	0.8852	0.8889	0.8293
	k_{LW}	1.0559	0.4921	0.4921	0.7291		k_{LW}	0.8984	0.8865	0.8890	0.8195
	k_{HSL}	1.0559	0.4926	0.4922	0.7179		k_{HSL}	0.8984	0.8769	0.8879	0.8804
	k_{HMO}	1.0559	0.4925	0.4922	0.7205		k_{HMO}	0.8984	0.8819	0.8885	0.8521
	k_{AM}	1.0559	0.4924	0.4922	0.7206		k_{AM}	0.8984	0.8789	0.8881	0.8696
	k_{GM}	1.0559	0.4918	0.4921	0.7475		k_{GM}	0.8984	0.8901	0.8894	0.7938
	k_{MED}	1.0559	0.4917	0.4921	0.7545		k_{MED}	0.8984	0.8905	0.8895	0.7940
	k_{KS}	1.0559	0.4919	0.4921	0.7400		k_{KS}	0.8984	0.8898	0.8894	0.7945
	k_{sarith}	1.0559	0.4917	0.4921	0.7603		k_{sarith}	0.8984	0.8907	0.8895	0.7954
	k_{SMD}	1.0559	0.4917	0.4921	0.7584		k_{SMD}	0.8984	0.8907	0.8895	0.7949
	k_{MU1}	1.0559	0.4952	0.4925	0.7195		k_{MU1}	0.8984	0.8728	0.8873	0.8996
	k_{MU2}	1.0559	0.4946	0.4925	0.7165		k_{MU2}	0.8984	0.8770	0.8879	0.8801
	k_{MU3}	1.0559	0.4919	0.4921	0.7400		k_{MU3}	0.8984	0.8898	0.8894	0.7945
	k_{MU4}	1.0559	0.4917	0.4921	0.7584		k_{MU4}	0.8984	0.8907	0.8895	0.7949

Table 3: Estimated MSE when $n = 50$, $\phi = .95$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}
0.1	k_{HK}	1.8721	1.6500	1.5992	0.3562	1	k_{HK}	2.0332	0.8923	0.8641	0.4159
	k_{HKB}	1.8721	1.6497	1.5992	0.3563		k_{HKB}	2.0332	0.8903	0.8637	0.4158
	k_{LW}	1.8721	1.6500	1.5992	0.3562		k_{LW}	2.0332	0.8937	0.8643	0.4161
	k_{HSL}	1.8721	1.6111	1.5919	0.3669		k_{HSL}	2.0332	0.8883	0.8634	0.4158
	k_{HMO}	1.8721	1.6421	1.5977	0.3585		k_{HMO}	2.0332	0.8895	0.8636	0.4158
	k_{AM}	1.8721	1.6316	1.5957	0.3614		k_{AM}	2.0332	0.8922	0.8641	0.4159
	k_{GM}	1.8721	1.6500	1.5992	0.3562		k_{GM}	2.0332	0.8948	0.8645	0.4163
	k_{MED}	1.8721	1.6493	1.5991	0.3564		k_{MED}	2.0332	0.9005	0.8656	0.4182
	k_{KS}	1.8721	1.6489	1.5990	0.3566		k_{KS}	2.0332	0.8948	0.8645	0.4163
	k_{sarith}	1.8721	1.6491	1.5990	0.3565		k_{sarith}	2.0332	0.9024	0.8659	0.4193
	k_{SMD}	1.8721	1.6495	1.5991	0.3564		k_{SMD}	2.0332	0.9018	0.8658	0.4189
	k_{MU1}	1.8721	1.4121	1.5522	0.4279		k_{MU1}	2.0332	0.7999	0.8467	0.4455
	k_{MU2}	1.8721	1.4438	1.5588	0.4168		k_{MU2}	2.0332	0.7551	0.8379	0.4652
	k_{MU3}	1.8721	1.6489	1.5990	0.3566		k_{MU3}	2.0332	0.8948	0.8645	0.4163
	k_{MU4}	1.8721	1.6495	1.5991	0.3564		k_{MU4}	2.0332	0.9018	0.8658	0.4189

Table 4: Estimated MSE when $n = 50$, $\phi = .95$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}
5	k_{HK}	0.9034	0.8340	0.8343	0.7563	15	k_{HK}	0.9490	0.9131	0.9066	0.7322
	k_{HKB}	0.9034	0.8309	0.8338	0.7738		k_{HKB}	0.9490	0.9082	0.9055	0.7264
	k_{LW}	0.9034	0.8244	0.8326	0.8128		k_{LW}	0.9490	0.8910	0.9016	0.7451
	k_{HSL}	0.9034	0.8251	0.8327	0.8082		k_{HSL}	0.9490	0.8896	0.9013	0.7468
	k_{HMO}	0.9034	0.8283	0.8333	0.7892		k_{HMO}	0.9490	0.9019	0.9041	0.7319
	k_{AM}	0.9034	0.8255	0.8328	0.8056		k_{AM}	0.9490	0.9017	0.9040	0.7321
	k_{GM}	0.9034	0.8355	0.8345	0.7504		k_{GM}	0.9490	0.9139	0.9068	0.7361
	k_{MED}	0.9034	0.8362	0.8346	0.7560		k_{MED}	0.9490	0.9139	0.9068	0.7365
	k_{KS}	0.9034	0.8349	0.8344	0.7520		k_{KS}	0.9490	0.9076	0.9054	0.7267
	k_{sarith}	0.9034	0.8367	0.8347	0.7817		k_{sarith}	0.9490	0.9161	0.9072	0.7630
	k_{SMD}	0.9034	0.8366	0.8347	0.7764		k_{SMD}	0.9490	0.9158	0.9072	0.7583
	k_{MU1}	0.9034	0.8197	0.8316	0.8431		k_{MU1}	0.9490	0.8440	0.8901	0.8043
	k_{MU2}	0.9034	0.8256	0.8328	0.8052		k_{MU2}	0.9490	0.8655	0.8955	0.7770
	k_{MU3}	0.9034	0.8349	0.8344	0.7520		k_{MU3}	0.9490	0.9076	0.9054	0.7267
	k_{MU4}	0.9034	0.8366	0.8347	0.7764		k_{MU4}	0.9490	0.9158	0.9072	0.7583

Table 5: Estimated MSE when $n = 50$, $\phi = .99$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}
0.1	k_{HK}	2.0985	1.8248	1.6943	0.3617	1	k_{HK}	2.3074	1.3764	1.2504	0.4218
	k_{HKB}	2.0985	1.8229	1.6936	0.3619		k_{HKB}	2.3074	1.3624	1.2442	0.3979
	k_{LW}	2.0985	1.8248	1.6943	0.3617		k_{LW}	2.3074	1.3553	1.2411	0.3933
	k_{HSL}	2.0985	0.6762	1.0057	0.7582		k_{HSL}	2.3074	1.3478	1.2378	0.3905
	k_{HMO}	2.0985	1.7152	1.6498	0.3685		k_{HMO}	2.3074	1.3548	1.2409	0.3931
	k_{AM}	2.0985	1.7264	1.6544	0.3678		k_{AM}	2.3074	1.3469	1.2374	0.3903
	k_{GM}	2.0985	1.8248	1.6943	0.3617		k_{GM}	2.3074	1.3769	1.2506	0.4232
	k_{MED}	2.0985	1.8242	1.6941	0.3617		k_{MED}	2.3074	1.3789	1.2515	0.4308
	k_{KS}	2.0985	1.8237	1.6939	0.3618		k_{KS}	2.3074	1.3582	1.2424	0.3949
	k_{sarith}	2.0985	1.8242	1.6941	0.3618		k_{sarith}	2.3074	1.3859	1.2546	0.4745
	k_{SMD}	2.0985	1.8243	1.6941	0.3617		k_{SMD}	2.3074	1.3848	1.2541	0.4652
	k_{MU1}	2.0985	0.6760	1.0045	0.7602		k_{MU1}	2.3074	1.0464	1.0981	0.4079
	k_{MU2}	2.0985	1.2025	1.4196	0.4233		k_{MU2}	2.3074	1.0051	1.0777	0.4125
	k_{MU3}	2.0985	1.8237	1.6939	0.3618		k_{MU3}	2.3074	1.3582	1.2424	0.3949
	k_{MU4}	2.0985	1.8243	1.6941	0.3617		k_{MU4}	2.3074	1.3848	1.2541	0.4652

Table 6: Estimated MSE when $n = 50$, $\phi = .99$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}
5	k_{HK}	2.5383	0.6358	0.5513	0.7725	15	k_{HK}	3.7283	1.2421	1.0998	0.9160
	k_{HKB}	2.5383	0.6217	0.5450	0.6684		k_{HKB}	3.7283	1.2077	1.0841	0.8531
	k_{LW}	2.5383	0.4826	0.4794	0.6105		k_{LW}	3.7283	0.5409	0.7289	0.8303
	k_{HSL}	2.5383	0.5265	0.5009	0.6066		k_{HSL}	3.7283	0.4409	0.6528	0.8346
	k_{HMO}	2.5383	0.5980	0.5342	0.6196		k_{HMO}	3.7283	1.0910	1.0296	0.8244
	k_{AM}	2.5383	0.6075	0.5385	0.6310		k_{AM}	3.7283	1.2138	1.0869	0.8592
	k_{GM}	2.5383	0.6361	0.5514	0.7770		k_{GM}	3.7283	1.2486	1.1027	0.9421
	k_{MED}	2.5383	0.6116	0.5404	0.6384		k_{MED}	3.7283	1.2135	1.0867	0.8588
	k_{KS}	2.5383	0.5309	0.5030	0.6064		k_{KS}	3.7283	1.0186	0.9948	0.8229
	k_{sarith}	2.5383	0.6430	0.5545	0.9008		k_{sarith}	3.7283	1.2909	1.1218	1.8783
	k_{SMD}	2.5383	0.6408	0.5535	0.8523		k_{SMD}	3.7283	1.2873	1.1202	1.6509
	k_{MU1}	2.5383	0.2684	0.3409	0.6678		k_{MU1}	3.7283	0.3518	0.4635	0.8571
	k_{MU2}	2.5383	0.3355	0.3969	0.6372		k_{MU2}	3.7283	0.6712	0.8112	0.8268
	k_{MU3}	2.5383	0.5309	0.5030	0.6064		k_{MU3}	3.7283	1.0186	0.9948	0.8229
	k_{MU4}	2.5383	0.6408	0.5535	0.8523		k_{MU4}	3.7283	1.2873	1.1202	1.6509

Table 7: Estimated MSE when $n = 100$, $\phi = .85$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,d_{CLASS}}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,d_{CLASS}}$	β_{RMUR}
0.1	k_{HK}	1.1833	0.8280	0.8260	0.3415	1	k_{HK}	1.2376	1.2326	1.2298	0.3826
	k_{HKB}	1.1833	0.8280	0.8260	0.3416		k_{HKB}	1.2376	1.2319	1.2298	0.3866
	k_{LW}	1.1833	0.8280	0.8260	0.3415		k_{LW}	1.2376	1.2326	1.2298	0.3826
	k_{HSL}	1.1833	0.8195	0.8258	0.4220		k_{HSL}	1.2376	1.2048	1.2292	0.5474
	k_{HMO}	1.1833	0.8277	0.8260	0.3445		k_{HMO}	1.2376	1.2243	1.2296	0.4324
	k_{AM}	1.1833	0.8275	0.8260	0.3463		k_{AM}	1.2376	1.2038	1.2292	0.5530
	k_{GM}	1.1833	0.8280	0.8260	0.3415		k_{GM}	1.2376	1.2327	1.2298	0.3823
	k_{MED}	1.1833	0.8280	0.8260	0.3419		k_{MED}	1.2376	1.2328	1.2298	0.3818
	k_{KS}	1.1833	0.8280	0.8260	0.3420		k_{KS}	1.2376	1.2327	1.2298	0.3823
	k_{sarith}	1.1833	0.8280	0.8260	0.3420		k_{sarith}	1.2376	1.2328	1.2298	0.3817
	k_{SMD}	1.1833	0.8280	0.8260	0.3417		k_{SMD}	1.2376	1.2328	1.2298	0.3817
	k_{MU1}	1.1833	0.8034	0.8253	0.5725		k_{MU1}	1.2376	1.1937	1.2289	0.6035
	k_{MU2}	1.1833	0.8193	0.8258	0.4242		k_{MU2}	1.2376	1.2022	1.2291	0.5616
	k_{MU3}	1.1833	0.8280	0.8260	0.3420		k_{MU3}	1.2376	1.2327	1.2298	0.3823
	k_{MU4}	1.1833	0.8280	0.8260	0.3417		k_{MU4}	1.2376	1.2328	1.2298	0.3817

Table 8: Estimated MSE when $n = 100$, $\phi = .85$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,d_{CLASS}}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,d_{CLASS}}$	β_{RMUR}
5	k_{HK}	0.9245	1.0811	1.0809	0.6809	15	k_{HK}	1.1521	1.4723	1.4744	1.0914
	k_{HKB}	0.9245	1.0805	1.0809	0.6717		k_{HKB}	1.1521	1.4654	1.4741	1.0605
	k_{LW}	0.9245	1.0812	1.0809	0.6840		k_{LW}	1.1521	1.4688	1.4743	1.0738
	k_{HSL}	0.9245	1.0762	1.0807	0.6855		k_{HSL}	1.1521	1.4418	1.4734	1.0144
	k_{HMO}	0.9245	1.0796	1.0809	0.6683		k_{HMO}	1.1521	1.4556	1.4738	1.0344
	k_{AM}	0.9245	1.0807	1.0809	0.6736		k_{AM}	1.1521	1.4431	1.4734	1.0158
	k_{GM}	0.9245	1.0819	1.0809	0.7060		k_{GM}	1.1521	1.4758	1.4745	1.1150
	k_{MED}	0.9245	1.0820	1.0810	0.7157		k_{MED}	1.1521	1.4761	1.4745	1.1179
	k_{KS}	0.9245	1.0818	1.0809	0.7048		k_{KS}	1.1521	1.4757	1.4745	1.1144
	k_{sarith}	0.9245	1.0821	1.0810	0.7196		k_{sarith}	1.1521	1.4762	1.4745	1.1191
	k_{SMD}	0.9245	1.0821	1.0810	0.7183		k_{SMD}	1.1521	1.4762	1.4745	1.1187
	k_{MU1}	0.9245	1.0674	1.0803	0.7491		k_{MU1}	1.1521	1.4422	1.4734	1.0149
	k_{MU2}	0.9245	1.0713	1.0805	0.7226		k_{MU2}	1.1521	1.4538	1.4738	1.0310
	k_{MU3}	0.9245	1.0818	1.0809	0.7048		k_{MU3}	1.1521	1.4757	1.4745	1.1144
	k_{MU4}	0.9245	1.0821	1.0810	0.7183		k_{MU4}	1.1521	1.4762	1.4745	1.1187

Table 9: Estimated MSE when $n = 100$, $\phi = .95$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}
0.1	k_{HK}	1.8047	1.4752	1.4575	0.3544	1	k_{HK}	0.9323	0.9207	0.9209	0.8966
	k_{HKB}	1.8047	1.4752	1.4575	0.3544		k_{HKB}	0.9323	0.9209	0.9209	0.9013
	k_{LW}	1.8047	1.4752	1.4575	0.3544		k_{LW}	0.9323	0.9212	0.9209	0.9081
	k_{HSL}	1.8047	1.4420	1.4550	0.3839		k_{HSL}	0.9323	0.9237	0.9212	0.9369
	k_{HMO}	1.8047	1.4694	1.4571	0.3597		k_{HMO}	0.9323	0.9213	0.9209	0.9096
	k_{AM}	1.8047	1.4436	1.4551	0.3825		k_{AM}	0.9323	0.9210	0.9209	0.9044
	k_{GM}	1.8047	1.4752	1.4575	0.3544		k_{GM}	0.9323	0.9207	0.9209	0.8928
	k_{MED}	1.8047	1.4751	1.4575	0.3545		k_{MED}	0.9323	0.9206	0.9209	0.8907
	k_{KS}	1.8047	1.4750	1.4575	0.3546		k_{KS}	0.9323	0.9207	0.9209	0.8954
	k_{sarith}	1.8047	1.4750	1.4575	0.3545		k_{sarith}	0.9323	0.9206	0.9209	0.8886
	k_{SMD}	1.8047	1.4751	1.4575	0.3545		k_{SMD}	0.9323	0.9206	0.9209	0.8889
	k_{MU1}	1.8047	1.3675	1.4493	0.4568		k_{MU1}	0.9323	0.9260	0.9214	0.9525
	k_{MU2}	1.8047	1.3617	1.4488	0.4628		k_{MU2}	0.9323	0.9222	0.9210	0.9228
	k_{MU3}	1.8047	1.4750	1.4575	0.3546		k_{MU3}	0.9323	0.9207	0.9209	0.8954
	k_{MU4}	1.8047	1.4751	1.4575	0.3545		k_{MU4}	0.9323	0.9206	0.9209	0.8889

Table 10: Estimated MSE when $n = 100$, $\phi = .95$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	0.7553	σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}
5	k_{HK}	0.8897	0.8054	0.8058	0.7060	15	k_{HK}	0.9323	0.9207	0.9209	0.8966
	k_{HKB}	0.8897	0.8058	0.8058	0.7029		k_{HKB}	0.9323	0.9209	0.9209	0.9013
	k_{LW}	0.8897	0.8059	0.8058	0.7430		k_{LW}	0.9323	0.9212	0.9209	0.9081
	k_{HSL}	0.8897	0.8086	0.8061	0.7089		k_{HSL}	0.9323	0.9237	0.9212	0.9369
	k_{HMO}	0.8897	0.8066	0.8059	0.7101		k_{HMO}	0.9323	0.9213	0.9209	0.9096
	k_{AM}	0.8897	0.8067	0.8059	0.7878		k_{AM}	0.9323	0.9210	0.9209	0.9044
	k_{GM}	0.8897	0.8053	0.8057	0.8159		k_{GM}	0.9323	0.9207	0.9209	0.8928
	k_{MED}	0.8897	0.8052	0.8057	0.7506		k_{MED}	0.9323	0.9206	0.9209	0.8907
	k_{KS}	0.8897	0.8054	0.8058	0.8541		k_{KS}	0.9323	0.9207	0.9209	0.8954
	k_{sarith}	0.8897	0.8052	0.8057	0.8464		k_{sarith}	0.9323	0.9206	0.9209	0.8886
	k_{SMD}	0.8897	0.8052	0.8057	0.8073		k_{SMD}	0.9323	0.9206	0.9209	0.8889
	k_{MU1}	0.8897	0.8129	0.8066	0.7595		k_{MU1}	0.9323	0.9260	0.9214	0.9525
	k_{MU2}	0.8897	0.8095	0.8063	0.7506		k_{MU2}	0.9323	0.9222	0.9210	0.9228
	k_{MU3}	0.8897	0.8054	0.8058	0.8464		k_{MU3}	0.9323	0.9207	0.9209	0.8954
	k_{MU4}	0.8897	0.8052	0.8057	0.7553		k_{MU4}	0.9323	0.9206	0.9209	0.8889

Table 11: Estimated MSE when $n = 100$, $\phi = .99$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}
0.1	k_{HK}	2.3266	2.0074	1.9012	0.3625	1	k_{HK}	1.2975	0.3170	0.3174	0.7353
	k_{HKB}	2.3266	2.0071	1.9011	0.3624		k_{HKB}	1.2975	0.3170	0.3175	0.7173
	k_{LW}	2.3266	2.0074	1.9012	0.3625		k_{LW}	1.2975	0.3205	0.3184	0.7102
	k_{HSL}	2.3266	1.9751	1.8920	0.3648		k_{HSL}	1.2975	0.3189	0.3181	0.7097
	k_{HMO}	2.3266	1.9997	1.8990	0.3620		k_{HMO}	1.2975	0.3175	0.3177	0.7102
	k_{AM}	2.3266	1.9774	1.8926	0.3646		k_{AM}	1.2975	0.3186	0.3181	0.7097
	k_{GM}	2.3266	2.0074	1.9012	0.3625		k_{GM}	1.2975	0.3170	0.3174	0.7377
	k_{MED}	2.3266	2.0073	1.9011	0.3625		k_{MED}	1.2975	0.3170	0.3174	0.7294
	k_{KS}	2.3266	2.0072	1.9011	0.3625		k_{KS}	1.2975	0.3171	0.3175	0.7133
	k_{sarith}	2.3266	2.0073	1.9011	0.3625		k_{sarith}	1.2975	0.3170	0.3174	0.7648
	k_{SMD}	2.3266	2.0073	1.9011	0.3625		k_{SMD}	1.2975	0.3170	0.3174	0.7606
	k_{MU1}	2.3266	1.7199	1.8160	0.3944		k_{MU1}	1.2975	0.3388	0.3210	0.7160
	k_{MU2}	2.3266	1.7236	1.8171	0.3939		k_{MU2}	1.2975	0.3216	0.3186	0.7105
	k_{MU3}	2.3266	2.0072	1.9011	0.3625		k_{MU3}	1.2975	0.3171	0.3175	0.7133
	k_{MU4}	2.3266	2.0073	1.9011	0.3625		k_{MU4}	1.2975	0.3170	0.3174	0.7606

Table 12: Estimated MSE when $n = 100$, $\phi = .99$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}
5	k_{HK}	0.9664	0.5237	0.5288	0.7385	15	k_{HK}	1.3026	0.7994	0.7978	0.3428
	k_{HKB}	0.9664	0.5253	0.5295	0.7424		k_{HKB}	1.3026	0.7994	0.7978	0.3428
	k_{LW}	0.9664	0.5448	0.5360	0.7295		k_{LW}	1.3026	0.7994	0.7978	0.3428
	k_{HSL}	0.9664	0.5371	0.5336	0.7323		k_{HSL}	1.3026	0.7533	0.7969	0.7540
	k_{HMO}	0.9664	0.5296	0.5310	0.7375		k_{HMO}	1.3026	0.7986	0.7977	0.3505
	k_{AM}	0.9664	0.5326	0.5321	0.7349		k_{AM}	1.3026	0.7982	0.7977	0.3555
	k_{GM}	0.9664	0.5236	0.5288	0.7379		k_{GM}	1.3026	0.7994	0.7978	0.3428
	k_{MED}	0.9664	0.5238	0.5289	0.7392		k_{MED}	1.3026	0.7994	0.7978	0.3430
	k_{KS}	0.9664	0.5262	0.5298	0.7417		k_{KS}	1.3026	0.7994	0.7978	0.3431
	k_{sarith}	0.9664	0.5230	0.5286	0.7412		k_{sarith}	1.3026	0.7994	0.7978	0.3430
	k_{SMD}	0.9664	0.5231	0.5286	0.7398		k_{SMD}	1.3026	0.7994	0.7978	0.3429
	k_{MU1}	0.9664	0.5785	0.5455	0.7249		k_{MU1}	1.3026	0.7494	0.7968	0.7730
	k_{MU2}	0.9664	0.5446	0.5360	0.7296		k_{MU2}	1.3026	0.7765	0.7974	0.5944
	k_{MU3}	0.9664	0.5262	0.5298	0.7417		k_{MU3}	1.3026	0.7994	0.7978	0.3431
	k_{MU4}	0.9664	0.5231	0.5286	0.7398		k_{MU4}	1.3026	0.7994	0.7978	0.3429

Table 13: Estimated MSE when $n = 150$, $\phi = .85$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}
0.1	k_{HK}	1.3026	0.7994	0.7978	0.3428	1	k_{HK}	1.2113	1.1169	1.1143	0.3820
	k_{HKB}	1.3026	0.7994	0.7978	0.3428		k_{HKB}	1.2113	1.1164	1.1143	0.3860
	k_{LW}	1.3026	0.7994	0.7978	0.3428		k_{LW}	1.2113	1.1169	1.1143	0.3820
	k_{HSL}	1.3026	0.7533	0.7969	0.7540		k_{HSL}	1.2113	1.0974	1.1138	0.5270
	k_{HMO}	1.3026	0.7986	0.7977	0.3505		k_{HMO}	1.2113	1.1105	1.1141	0.4304
	k_{AM}	1.3026	0.7982	0.7977	0.3555		k_{AM}	1.2113	1.0968	1.1138	0.5308
	k_{GM}	1.3026	0.7994	0.7978	0.3428		k_{GM}	1.2113	1.1170	1.1143	0.3817
	k_{MED}	1.3026	0.7994	0.7978	0.3430		k_{MED}	1.2113	1.1170	1.1143	0.3812
	k_{KS}	1.3026	0.7994	0.7978	0.3431		k_{KS}	1.2113	1.1170	1.1143	0.3817
	k_{sarith}	1.3026	0.7994	0.7978	0.3430		k_{sarith}	1.2113	1.1171	1.1143	0.3811
	k_{SMD}	1.3026	0.7994	0.7978	0.3429		k_{SMD}	1.2113	1.1171	1.1143	0.3812
	k_{MU1}	1.3026	0.7494	0.7968	0.7730		k_{MU1}	1.2113	1.0850	1.1135	0.6045
	k_{MU2}	1.3026	0.7765	0.7974	0.5944		k_{MU2}	1.2113	1.0881	1.1136	0.5866
	k_{MU3}	1.3026	0.7994	0.7978	0.3431		k_{MU3}	1.2113	1.1170	1.1143	0.3817
	k_{MU4}	1.3026	0.7994	0.7978	0.3429		k_{MU4}	1.2113	1.1171	1.1143	0.3812

Table 14: Estimated MSE when $n = 150$, $\phi = .85$, $p =$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k'dCLASS}$	β_{RMUR}
5	k_{HK}	0.9113	0.9529	0.9528	0.6386	15	k_{HK}	0.9610	1.2306	1.2308	0.8371
	k_{HKB}	0.9113	0.9524	0.9528	0.6559		k_{HKB}	0.9610	1.2292	1.2307	0.8413
	k_{LW}	0.9113	0.9530	0.9528	0.6368		k_{LW}	0.9610	1.2294	1.2307	0.8395
	k_{HSL}	0.9113	0.9507	0.9528	0.7164		k_{HSL}	0.9610	1.2273	1.2307	0.8571
	k_{HMO}	0.9113	0.9519	0.9528	0.6754		k_{HMO}	0.9610	1.2283	1.2307	0.8486
	k_{AM}	0.9113	0.9521	0.9528	0.6687		k_{AM}	0.9610	1.2277	1.2307	0.8534
	k_{GM}	0.9113	0.9532	0.9528	0.6300		k_{GM}	0.9610	1.2315	1.2308	0.8684
	k_{MED}	0.9113	0.9533	0.9528	0.6291		k_{MED}	0.9610	1.2316	1.2308	0.8794
	k_{KS}	0.9113	0.9532	0.9528	0.6300		k_{KS}	0.9610	1.2315	1.2308	0.8663
	k_{sarith}	0.9113	0.9533	0.9528	0.6288		k_{sarith}	0.9610	1.2317	1.2308	0.8847
	k_{SMD}	0.9113	0.9533	0.9528	0.6289		k_{SMD}	0.9610	1.2317	1.2308	0.8831
	k_{MU1}	0.9113	0.9478	0.9527	0.7939		k_{MU1}	0.9610	1.2230	1.2306	0.8884
	k_{MU2}	0.9113	0.9490	0.9527	0.7651		k_{MU2}	0.9610	1.2259	1.2307	0.8687
	k_{MU3}	0.9113	0.9532	0.9528	0.6300		k_{MU3}	0.9610	1.2315	1.2308	0.8663
	k_{MU4}	0.9113	0.9533	0.9528	0.6289		k_{MU4}	0.9610	1.2317	1.2308	0.8831

Table 15: Estimated MSE when $n = 150$, $\phi = .95$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}
0.1	k_{HK}	1.8743	1.6685	1.6496	0.3563	1	k_{HK}	1.7040	1.4579	1.4423	0.4036
	k_{HKB}	1.8743	1.6684	1.6496	0.3563		k_{HKB}	1.7040	1.4553	1.4422	0.3965
	k_{LW}	1.8743	1.6685	1.6496	0.3563		k_{LW}	1.7040	1.4579	1.4423	0.4036
	k_{HSL}	1.8743	1.6251	1.6466	0.3916		k_{HSL}	1.7040	1.4102	1.4388	0.4382
	k_{HMO}	1.8743	1.6653	1.6494	0.3587		k_{HMO}	1.7040	1.4420	1.4412	0.4039
	k_{AM}	1.8743	1.6623	1.6492	0.3610		k_{AM}	1.7040	1.4143	1.4391	0.4337
	k_{GM}	1.8743	1.6685	1.6496	0.3563		k_{GM}	1.7040	1.4581	1.4424	0.4046
	k_{MED}	1.8743	1.6684	1.6496	0.3563		k_{MED}	1.7040	1.4587	1.4424	0.4083
	k_{KS}	1.8743	1.6683	1.6496	0.3564		k_{KS}	1.7040	1.4581	1.4424	0.4046
	k_{sarith}	1.8743	1.6683	1.6496	0.3564		k_{sarith}	1.7040	1.4589	1.4424	0.4098
	k_{SMD}	1.8743	1.6684	1.6496	0.3563		k_{SMD}	1.7040	1.4588	1.4424	0.4093
	k_{MU1}	1.8743	1.5033	1.6379	0.5050		k_{MU1}	1.7040	1.3327	1.4329	0.5242
	k_{MU2}	1.8743	1.6067	1.6453	0.4081		k_{MU2}	1.7040	1.3491	1.4342	0.5061
	k_{MU3}	1.8743	1.6683	1.6496	0.3564		k_{MU3}	1.7040	1.4581	1.4424	0.4046
	k_{MU4}	1.8743	1.6684	1.6496	0.3563		k_{MU4}	1.7040	1.4588	1.4424	0.4093

Table 16: Estimated MSE when $n = 150$, $\phi = .95$, $p = 4$

σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}	σ	k	β_{RLS}	β_{RRR}	$\beta_{k,dCLASS}$	β_{RMUR}
5	k_{HK}	0.9112	1.0040	1.0025	0.7413	15	k_{HK}	0.9307	1.1384	1.1388	0.8129
	k_{HKB}	0.9112	1.0036	1.0024	0.7110		k_{HKB}	0.9307	1.1334	1.1385	0.8263
	k_{LW}	0.9112	1.0031	1.0024	0.6915		k_{LW}	0.9307	1.1300	1.1382	0.8346
	k_{HSL}	0.9112	1.0032	1.0024	0.6957		k_{HSL}	0.9307	1.1268	1.1380	0.8419
	k_{HMO}	0.9112	1.0034	1.0024	0.7028		k_{HMO}	0.9307	1.1298	1.1382	0.8352
	k_{AM}	0.9112	1.0033	1.0024	0.7003		k_{AM}	0.9307	1.1248	1.1379	0.8463
	k_{GM}	0.9112	1.0042	1.0025	0.7601		k_{GM}	0.9307	1.1409	1.1390	0.8110
	k_{MED}	0.9112	1.0043	1.0025	0.7748		k_{MED}	0.9307	1.1413	1.1390	0.8170
	k_{KS}	0.9112	1.0040	1.0025	0.7395		k_{KS}	0.9307	1.1408	1.1390	0.8103
	k_{sarith}	0.9112	1.0044	1.0025	0.7909		k_{sarith}	0.9307	1.1415	1.1390	0.8223
	k_{SMD}	0.9112	1.0043	1.0025	0.7876		k_{SMD}	0.9307	1.1415	1.1390	0.8214
	k_{MU1}	0.9112	0.9979	1.0021	0.6689		k_{MU1}	0.9307	1.1187	1.1375	0.8588
	k_{MU2}	0.9112	0.9974	1.0020	0.6699		k_{MU2}	0.9307	1.1270	1.1380	0.8415
	k_{MU3}	0.9112	1.0040	1.0025	0.7395		k_{MU3}	0.9307	1.1408	1.1390	0.8103

3.1 The discussion of simulation results

In this section, we present the results of simulation study of some restricted estimator to determine the good estimator, by using the mean square error MSE criterion. From Table 1 to Table 16, the performance of RLS, RRR, $(k - d)$ class and RMUR estimators for all cases of the sample size n , the coefficient correlation ϕ and the variance σ^2 . We can observe that the following results.

1. From Table 1 to Table 6, when $(n = 50, \phi = .85, .95, .99 \text{ and } \sigma = .1, 1, 15)$ the performance the RMUR estimator is better than of any estimator because the RMUR estimator has minimum mean square error MSE, while when $\sigma = 5$ and $\phi = .99$ the $(k - d)$ class estimator is better than of any estimator because the $(k - d)$ class has lowest MSE.
2. From Table 7 to Table 12, when $(n = 100, \phi = .85, .95, .99)$ and $(\sigma = .1, 1, 5, 15)$, the RMUR estimator has minimum mean square error, only in Table 12, when $\sigma = .5$ and $\phi = .99$ the RRR estimator has lowest mean square error.
3. From Table 13 to Table 16 for all cases of the coefficient correlation ϕ and the variance σ^2 the RMUR estimator is the best because the RMUR has lowest MEE comparing of any estimator.

4. Numerical example-

To illustrate the performance of RLS, RRR, $(k - d)$ class and RMUR estimators by using the real life data, numerical example is given. We consider the data about the total national product, which is cited by Akdeniz [1], Gruber [6] for comparison of the estimators that given in this study. This data shows the relationship between the dependent variable Y and the percentage that is the united states spent on four independent variables X_1, X_2, X_3 and X_4 representing the percentage spent by Farance, West Germany, Japan, and Soviet Union respectively. The goal is to compare the scalar mean square error for RLS, RRR, $(k - d)$ class and RMUR estimators. The scalar mean square of the RRR, $(k - d)$ class and RMUR are given in Eq (2.7), (2.10) and (2.15) respectively. Table 17 show the performance of the RLS, RRR, $(k - d)$ class and RMUR estimators by using scalar mean square error. we use the linear restrictions in (2) R and r as follows

$$R = [1 \ 1 \ 1 \ 1 ; 0 \ 1 \ 3 \ 1] \ , r = [1.2170 \ 1.0904]$$

Table 17 : The scalar mean square error of the RLS, RRR, $(k - d)$ class and RMUR estimators for different estimated ridge parameter k

k	β_{RLS}	β_{RRR}	$\beta_{k,d_{CLASS}}$	β_{RMUR}
0.0161	135.7475	92.9958	92.2758	83.1722
0.0243	135.7475	84.4906	81.6447	67.6833
0.050	135.7475	70.0587	64.7167	40.0855
0.020	135.7475	88.4942	86.678	75.1888
0.10	135.7475	58.2697	53.1129	20.2068
0.15	135.7475	52.8104	48.3190	13.0448
0.20	135.7475	49.5981	45.5077	9.6146

In Table 17, we can observe that, the RMUR estimator has lowest scalar mean square error for all different k comparing of other restricted estimators, that means the RMUR estimator is better than of any restricted estimators, and this is clear by looking to Figure 1,2 and 3.

Figure 1: Estimated mse of RLS, RRR, $(k - d)$ class and RMUR estimators for different estimated ridge parameter k

Figure 2: Estimated mse of RLS, RRR, $(k - d)$ class and RMUR estimators for different estimated ridge parameter k

Figure 3: Estimated mse of RLS, RRR, $(k - d)$ class and RMUR estimators for different estimated ridge parameter k

5 .Conclusion

When the multicollinearity problem and high variance there exists in linear regression model, the researchers tried to take advantage from prior information of the parameters through introduced the restricted biased estimators. In this paper, we make review of some restricted estimators through simulation study of this estimators. According to the simulation study, we observe that, the RMUR estimator is better than others in the science that it has minimum mean square error MSE comparing of other restricted estimators, that mean the RMUR estimator has good properties, also, we were able to illustrate that through a numerical example and some figures

References

- [1] Akdeniz, F., Erol, H. Mean squared error matrix comparisons of some biased estimators in linear regression. *Communications in Statistics Theory and Methods*, 32: (2003). 2389–2413
- [2] Alheety .M . I and S .D .Gore . A study of multiple linear regression model in presence of multicollinearity. Ph.D. Thesis published in Pune University. (2009).
- [3] Alkhamisi, M., Khalaf, G., & Shukur, G. Some modifications for choosing ridge parameters. *Communications in Statistics – Theory and Methods*, 35(11), (2006). 2005-2020.
- [4] Bader and Mustafa, I. N. New restricted biased estimator based on modified unbiased ridge regression estimator. Iop- Conference Series (in process). (2020)
- [5] Batah, F. S. M., & Gore, S. D.. Ridge regression estimator: Combining unbiased and ordinary ridge regression methods of estimation. *Surveys in Mathematics and its Applications*, 4, (2009). 99-109
- [6] Gruber, M. Improving Efficiency by Shrinkage: The James--Stein and Ridge Regression Estimators. Routledge. . (2017).
- [7] Hocking, R. R., Speed, F. M., & Lynn, M. J. A class of biased estimators in linear regression. *Technometrics*, 18(4), 55-67.
- [8] Hoerl, A. E., Kennard, R. W., & Baldwin, K. F.. Ridge regression: Some simulations. *Communications in Statistics*, 4(2), (1975). 105-123.
- [9] Khalaf, G. & Shukur, G.. Choosing ridge parameters for regression problems. *Communications in Statistics – Theory and Methods*, 34(5), (2005). 1177-1182.
- [10] Kibria, B. M. G . Performance of some new ridge regression estimators. *Communications in Statistics – Simulation and Computation*, 32(2), (2003). 419-435
- [11] Kibria, B. M., & Banik, S. Some ridge regression estimators and their performances. *Journal of Modern Applied Statistical Methods*, 15(1), (2016)..12.
- [12] Lawless, J. F. & Wang, P. A simulation study of ridge and other regression estimators. *Communications in Statistics – Theory and Methods*, 5(4), (1976).307-323.

- [13] Liu, K. J. Using Liu type estimator to combat collinearity. *Commun. Statist. Theoyr. Meth.* 32: . (2003). 1009–1020
- [14] M .N Lattef, M .I ALheety (2020) . Study of Some Kinds of Ridge Regression Estimators in Linear Regression Model. *Tikrit Journal of Pure Since*,[5.1] v.25,n.1 ,pp. 130-142. (2020).
- [15] Najarian, S., Arashi, M. and Kibria, B. M. G. A Simulation Study on Some Restricted Ridge Regression Estimators. *Comm. Statist. Sim. Comp.*, **42**, (2013). 871-879
- [16] Nomura, M. (1988). On the almost unbiased ridge regression estimation. *Communication in Statistics – Simulation and Computation*, 17(3), (1976). 729-743.
- [17] S. Kaçiranlar, S. Sakallıoğlu, F. Akdeniz, G.P.H. Styan, and H.J.Werner, *A new biased estimator in linear regression and a detailed analysis of the widely analyzed dataset on Portland Cement*. *Sankhya: Indian J. Stat. B* 61(3) . (1999) ,pp. 443–459
- [18] Sarkar. N . *A new estimator combining the ridge regression and the restricted least squares methods of estimation*, *Commun. Stat. Theory Methods* 21 , (1992). pp. 1987–2000.