A View On Symmetric Numerical Semigroups

Authors Name Sedat İLHAN	ABSTRACT
Article History Received on: 14/4 /2021 Revised on: 4/5/2021 Accepted on: 31/5/ 2021 Keywords: Symmetric numerical semigroups, Arf closure, Genus. DOI: https://doi.org/10.29350/jops..2021.26.3.1303	In this paper, we will give some results about the symmetric numerical semigroups such that $S_k = \langle 7, 7k + 4 \rangle$ where $k \geq 1, k \in \mathbb{N}$. Also, we will obtain Arf closure of these symmetric numerical semigroups.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}$ and ϕ be integer set. S is called a numerical semigroup if

$$(i) \ a_1 + a_2 \in S \text{ for } a_1, a_2 \in S$$

$$(ii) \ \gcd(S) = 1$$

$$(iii) \ 0 \in S$$

where $S \subseteq \mathbb{N}$ (Here, $\gcd(S)$ = greatest common divisor the elements of S).

A numerical semigroup S can be written that

$$S = \langle a_1, a_2, \dots, a_n \rangle = \left\{ \sum_{i=1}^n k_i a_i : k_i \in \mathbb{N} \right\}$$

(for details see [4]).

$U \nsubseteq S$ is minimal system of generators of S if $\langle U \rangle = S$ and there isn't any subset $V \subsetneq U$ such that $\langle V \rangle = S$. Also, $m(S) = \min \{x \in S : x > 0\}$ is called as multiplicity of S (see [3]). Let S be a numerical semigroup, then $F(S) = \max(\mathbb{N} \setminus S)$ is called as Frobenius number of S .

$$n(S) = \text{Card}(\{0, 1, 2, \dots, F(S)\} \cap S)$$

is called as the determine number of S (see [5]).

If S is a numerical semigroup such that $S = \langle a_1, a_2, \dots, a_n \rangle$, then we observe that

$$S = \langle a_1, a_2, \dots, a_n \rangle = \{s_0 = 0, s_1, s_2, \dots, s_{n-1}, s_n = F(S) + 1, \dots\},$$

where $s_i < s_{i+1}$, $n = n(S)$ and the arrow means that every integer greater than $F(S) + 1$ belongs to S for $i = 1, 2, \dots, n = n(S)$ (see [6]).

If $b \in \mathbb{N}$ and $b \notin S$, then b is called gap of S . We denote the set of gaps of S , by $H(S)$, i.e, $H(S) = \mathbb{N} \setminus S$. The $G(S) = \#(H(S))$ is called the genus of S . It known that

$$G(S) + n(S) = F(S) + 1$$

(see [4]).

S is called symmetric numerical semigroup if $F(S) - t$ belongs to S , for $t \in \mathbb{N} \setminus S$. It is known the numerical semigroup $S = \langle a_1, a_2 \rangle$ is symmetric and $F(S) = a_1 a_2 - a_1 - a_2$. In this case, we write

$$n(S) = \frac{F(S) + 1}{2}$$

(see [1]).

A numerical semigroup S is called Arf if $a_1 + a_2 - a_3 \in S$, for all $a_1, a_2, a_3 \in S$ such that $a_1 \leq a_2 \leq a_3$. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S , and it is denoted by $\text{Arf}(S)$ (for detail see [2, 3]). If S is a numerical semigroup such that $S = \langle a_1, a_2, \dots, a_n \rangle$, then $L(S) = \langle a_1, a_2 - a_1, a_3 - a_1, \dots, a_n - a_1 \rangle$ is called Lipman numerical semigroup of S and it is known that

$$L_0(S) = S \subseteq L_1(S) = L(L_0(S)) \subseteq L_2 = L(L_1(S)) \subseteq \dots \subseteq L_m = L(L_{m-1}(S)) \subseteq \dots \subseteq \square$$

(see [7]).

In this paper, we will give some results about the symmetric numerical semigroups such that $S_k = \langle 7, 7k + 4 \rangle$ where $k \geq 1, k \in \mathbb{N}$. Also, we will obtain Arf closure of these symmetric numerical semigroups.

2. Main Results

Theorem 1. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1, k \in \mathbb{N}$. Then, we have

$$(a) \quad F(S_k) = 42k + 17$$

$$(b) \quad n(S_k) = 21k + 9$$

$$(c) \quad G(S_k) = 21k + 9.$$

Proof. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1, k \in \mathbb{N}$. Then, S_k is symmetric and we find that

$$(a) \quad F(S_k) = 7(7k + 4) - 7 - 7k - 4 = 42k + 17.$$

$$(b) \quad n(S_k) = \frac{F(S_k) + 1}{2} = \frac{42k + 17 + 1}{2} = 21k + 9.$$

$$(c) \quad G(S_k) = 42k + 17 + 1 - 21k - 9 = 21k + 9 \text{ from } G(S_k) = F(S_k) + 1 - n(S_k).$$

Theorem 2. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1, k \in \mathbb{N}$. Then, $\text{Arf}(S_k) = \{0, 7, 14, 21, \dots, 7k, 7k + 4, 7k + 7, \dots\}$.

Proof. It is trivial $m_0 = 7$ since $L_0(S_k) = S_k = \langle 7, 7k + 4 \rangle$. Thus, we write $L_1(S_k) = \langle 7, 7k - 3 \rangle$. In this case,

$$(1) \text{ If } 7k - 3 < 7 \text{ (if } k = 1 \text{) then } S_1 = \langle 7, 11 \rangle \text{ and } L_1(S_1) = \langle 7, 4 \rangle = \langle 4, 7 \rangle, m_1 = 4.$$

$$L_2(S_1) = \langle 4, 3 \rangle = \langle 3, 4 \rangle, m_2 = 3 \text{ and } L_3(S_1) = \langle 3, 1 \rangle = \langle 1 \rangle = \mathbb{N}, m_3 = 1.$$

Thus, we obtain that $\text{Arf}(S_1) = \{0, 7, 11, 14, \dots\}$.

(2) If $7k - 3 > 7$ (if $k \geq 2$) then $L_1(S_k) = \langle 7, 7k - 3 \rangle$ and $m_1 = 7$. In this case, we write $L_2(S_k) = \langle 7, 7k - 10 \rangle$.

(a) If $k = 2$ then $L_2(S_2) = \langle 7, 4 \rangle = \langle 4, 7 \rangle$, $m_2 = 4$. $L_3(S_2) = \langle 4, 3 \rangle = \langle 3, 4 \rangle$, $m_3 = 3$ and $L_4(S_2) = \langle 3, 1 \rangle = \langle 1 \rangle = \mathbb{N}$, $m_4 = 1$. So, we have $\text{Arf}(S_2) = \{0, 7, 14, 18, 21, \mathbb{N} \dots\}$.

(b) If $k > 2$ then $L_2(S_k) = \langle 7, 7k - 10 \rangle$, $m_2 = 7$ and $L_3(S_k) = \langle 7, 7k - 17 \rangle$. In this case,

(i) if $k = 3$ then $L_3(S_3) = \langle 7, 4 \rangle = \langle 4, 7 \rangle$, $m_3 = 4$. $L_4(S_3) = \langle 4, 3 \rangle = \langle 3, 4 \rangle$, $m_4 = 3$ and $L_5(S_3) = \langle 1 \rangle = \mathbb{N}$, $m_5 = 1$. Thus we obtain that $\text{Arf}(S_3) = \{0, 7, 14, 21, 25, 28, \mathbb{N} \dots\}$.

(ii) If $k > 3$ then $L_3(S_k) = \langle 7, 7k - 17 \rangle$, $m_3 = 7$ and $L_4(S_k) = \langle 7, 7k - 24 \rangle$. In this case,

(1) if $k = 4$ then $L_4(S_4) = \langle 7, 4 \rangle = \langle 4, 7 \rangle$, $m_4 = 4$. $L_5(S_4) = \langle 4, 3 \rangle = \langle 3, 4 \rangle$, $m_5 = 3$

and

$$L_6(S_4) = \langle 1 \rangle = \mathbb{N}, m_6 = 1.$$

Thus we find $\text{Arf}(S_k) = \{0, 7, 14, 21, 28, 32, 35, \mathbb{N} \dots\}$.

(2) If $k > 4$ then $L_4(S_k) = \langle 7, 7k - 24 \rangle$, $m_4 = 7$ and we write $L_5(S_k) = \langle 7, 7k - 31 \rangle$. If we

continue the operations then we obtain Arf closure of $\text{Arf}(S_k)$ as follows

$$\text{Arf}(S_k) = \{0, 7, 14, 21, \dots, 7k, 7k + 4, 7k + 7, \mathbb{N} \dots\}.$$

Thus, the proof is completed.

Proposition 3. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1$, $k \in \mathbb{N}$. Then, we have

$$(a) F(\text{Arf}(S_k)) = 7k + 6$$

$$(b) n(\text{Arf}(S_k)) = k + 2$$

$$(c) G(\text{Arf}(S_k)) = 6k + 5.$$

Proof. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1$, $k \in \mathbb{N}$. Then,

we write that $F(\text{Arf}(S_k)) = 7k + 6$ from Theorem 2. On the other hand, we find that

$$n(\text{Arf}(S_k)) = \#(\{0, 1, 2, \dots, 7k + 6\} \cap \text{Arf}(S_k)) = \#(\{0, 7, 14, \dots, 7k, 7k + 4\}) = k + 2 \quad \text{and} \quad \text{we obtain} \\ G(\text{Arf}(S_k)) = 7k + 6 + 1 - k - 2 = 6k + 5 \text{ since } G(\text{Arf}(S_k)) = F(\text{Arf}(S_k)) + 1 - n(\text{Arf}(S_k)).$$

Corollary 4. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1, k \in \mathbb{N}$. Then, we have

$$(a) F(S_k) = F(\text{Arf}(S_k)) + 35k + 11$$

$$(b) n(S_k) = n(\text{Arf}(S_k)) + 20k + 7$$

$$(c) G(S_k) = G(\text{Arf}(S_k)) + 15k + 4.$$

Proof. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1, k \in \mathbb{N}$. We write that

$$(a) F(\text{Arf}(S_k)) + 35k + 11 = 7k + 6 + 35k + 11 = 42k + 17 = F(S_k). \text{ However, we find that}$$

$$(b) n(\text{Arf}(S_k)) + 20k + 7 = k + 2 + 20k + 7 = 21k + 9 = n(S_k),$$

$$(c) G(\text{Arf}(S_k)) + 15k + 4 = 6k + 5 + 15k + 4 = 21k + 9 = G(S_k).$$

Corollary 5. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1, k \in \mathbb{N}$. Then, it satisfies following conditions:

$$(a) F(S_{k+1}) = F(S_k) + 42$$

$$(b) n(S_{k+1}) = n(S_k) + 21$$

$$(c) G(S_{k+1}) = G(S_k) + 21.$$

Corollary 6. Let $S_k = \langle 7, 7k + 4 \rangle$ be numerical semigroups, where $k \geq 1, k \in \mathbb{N}$. Then, it satisfies following conditions:

$$(a) F(\text{Arf}(S_{k+1})) = F(\text{Arf}(S_k)) + 7$$

$$(b) n(\text{Arf}(S_{k+1})) = n(\text{Arf}(S_k)) + 1$$

$$(c) G(\text{Arf}(S_{k+1})) = G(\text{Arf}(S_k)) + 6.$$

Example 7. We put $k = 1$ in $S_k = \langle 7, 7k + 4 \rangle$ symmetric numerical semigroup. Then we have $S_1 = \langle 7, 11 \rangle = \{0, 7, 11, 14, 18, 21, 22, 25, 28, 29, 32, 33, 35, 36, 39, 40, 42, 43, 44, 46, 47, 49, 50, 51, 53, 54, 55, 56, 57, 58, 60, \dots\}$. In this case, we obtain $F(S_1) = 59, n(S_1) = 30,$

$H(S_1) = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 15, 16, 17, 19, 20, 23, 24, 26, 27, 30, 31, 34, 37, 38, 41, 45, 48, 52, 59\}$
 $G(S_1) = 30, \text{Arf}(S_1) = \{0, 7, 11, 14, \textcircled{R} \dots\}, F(\text{Arf}(S_1)) = 13, n(\text{Arf}(S_1)) = 3, G(\text{Arf}(S_1)) = 11.$ Thus, we find that

$$F(\text{Arf}(S_1)) + 46 = 13 + 46 = 59 = F(S_1), n(\text{Arf}(S_1)) + 27 = 3 + 27 = 30 = n(S_1)$$

$$\text{and } G(\text{Arf}(S_1)) + 19 = 11 + 19 = 30 = G(S_1).$$

If $k = 2$ then we write $S_2 = \langle 7, 18 \rangle = \{0, 7, 14, 18, 21, 25, 28, \dots, 100, 102, \textcircled{R} \dots\}.$

Thus, we have $F(S_2) = 101, n(S_2) = 51, G(S_2) = 51, \text{Arf}(S_2) = \{0, 7, 14, 18, 21, \textcircled{R} \dots\},$
 $F(\text{Arf}(S_2)) = 20, n(\text{Arf}(S_2)) = 4$ and $G(\text{Arf}(S_2)) = 17.$

So, we write that

$$F(S_1) + 42 = 59 + 42 = 101 = F(S_2),$$

$n(S_1) + 21 = 30 + 21 = 51 = n(S_2)$ and $G(S_1) + 21 = 30 + 21 = 51 = G(S_2).$ Also, we obtain that
 $F(\text{Arf}(S_1)) + 7 = 13 + 7 = 20 = F(\text{Arf}(S_2)), n(\text{Arf}(S_1)) + 1 = 3 + 1 = 4 = n(\text{Arf}(S_2))$ and
 $G(\text{Arf}(S_1)) + 6 = 11 + 6 = 17 = G(\text{Arf}(S_2)).$

References

- [1] J.C. Rosales, Fundamental gaps of numerical semigroups generated by two elements, Linear Algebra and its Applications, 405,(2005), 200-208.
- [2] J.C. Rosales, P.A.Garcia-Sanchez, J.I.Garcia-Garcia and M.B.Branco, Arf numerical semigroups, J.Algebra, 276,(2004),3-12.
- [3] S. İlhan and H.İ. Karakaş, Arf numerical semigroups, Turkish journal of Mathematics, 41, (2017),1448-1457.
- [4] J.C. Rosales and P.A. Garcia-Sanchez, Numerical semigroups. New York: Springer 181, 2009.
- [5] R.Froberg, C.Gotlieb and R. Haggkvist, On numerical semigroups. Semigroup Forum,35, (1987), 63 -68.
- [6] M.D'anna, Type Sequences of Numerical Semigroups, Semigroup Forum 56 (1998),1-31 .
- [7] J. Lipman, Stable ideals and Arf rings, Amer. J. Math.,93, (1971),649-685.