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Globally Null-Controllability and Complete stabilizability for Perturbed Linear Time-Varying System in Real Composition of Hilbert Spaces

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ABSTRACT

This paper deals with the problem of controllability and stabilizability of the perturbed linear time-varying systems defined in some suitable real Hilbert space. The aim of this paper is to show that any globally null-controllable system is completely stabilizability and conversely, under some additional condition the complete stabilizability implies global null-controllability.

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1. INTRODUCTION

Let $\mathbb{H}, U$ be a real Hilbert spaces, $\mathbb{H}^*, U^*$ are the dual of the spaces $\mathbb{H}, U$, respectively, and $\mathbb{H}^m, U^m$ are the Hilbert product spaces dotted.

Consider the perturbed linear time-varying nonlocal system of the form:

$$\left. \begin{aligned} \begin{bmatrix} \frac{dz_1(t)}{dt} \\ \vdots \\ \frac{dz_m(t)}{dt} \end{bmatrix} &= \left(\begin{bmatrix} \mathbb{A}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m(t) \end{bmatrix} \right) \begin{bmatrix} z_1(t) \\ \vdots \\ z_m(t) \end{bmatrix} + \\ &\quad \begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \forall t \geq 0 \\ \begin{bmatrix} z_1(0) \\ \vdots \\ z_m(0) \end{bmatrix} &= \begin{bmatrix} z_0 \\ \vdots \\ z_0 \end{bmatrix}, \quad z_i(0) \in Z^i, i \in \{1, \dots, m\} \end{aligned} \right\}$$

Where, $A(t): D(A(t)) \subset \mathbb{H} \rightarrow \mathbb{H}$ be linear operator and $A(t) = \text{diag}(\mathbb{A}_1(t), \dots, \mathbb{A}_m(t))$ with $\mathbb{A}_i(t)$ is generates a strongly continuous semigroup $\mathbb{T}_{t,i}$ of bounded linear operators on real Hilbert space $\mathbb{H}$ satisfies $\|\mathbb{T}_{t,i}\| \leq \mathbb{M}_i e^{\omega_i t}, t \geq 0$, for $\mathbb{M}_i \geq 1, \omega_i \in \mathbb{R}$, and all $i \in \{1, \dots, m\}$, and $B(t): \mathbb{H} \rightarrow \mathbb{H}$ with $B(t) = \text{diag}(\mathbb{B}_1(t), \dots, \mathbb{B}_m(t))$ are bounded linear operators i.e., there exists a real number c , such that $\|\mathbb{B}_i(t)\| \leq c$. Moreover, the operator $\mathbb{E}(t): U \rightarrow \mathbb{H}$ with $E(t) = \text{diag}(\mathbb{E}_1(t), \dots, \mathbb{E}_m(t))$ is bounded linear operator for all $i \in [1, m]$, $z_i(t) \in \mathbb{H}$ is the state, $u_i(t) \in U$ is the control, and $z_0 \in \mathbb{H}^m$, the inner product on the Hilbert product space dotted defined by $\langle z_i, z_i \rangle_{\mathbb{H}^m}^2 = \sum_{i=1}^m \langle z_i, z_i \rangle_{\mathbb{H}}^2$.

The problem of controllability and stabilizability of the perturbed linear time-varying system(1), with $B(t) = 0$, and $m = 1$, for all $i \in \{1, \dots, m\}$ has been studied extensively, see; e.g. [9,11,15,17], and the reference therein. This problem related as an extension of the classical Kalman result [6] on controllability and stabilizability of linear control system is to find an admissible control $u(t)$ such that the corresponding solution $z(t)$ of the linear control system has desirable properties. Depending on the properties interested, one defines different qualitative problem. For instance, the null-controllability problem concerns the question of finding an admissible control $u(t)$ which steers an arbitrary state z_0 of the linear control system into zero; stabilizability problem is to find a control $u(t) = \mathbb{K}(t)z(t)$ such that the zero solution of the closed-loop system

$$d/dt z(t) = [A(t) + B(t)\mathbb{K}(t)]z(t), t \geq 0$$

is asymptotically stable in the Lyapunov sense. Moreover, the linear control system is the stabilizable by the control $u(t) = \mathbb{K}(t)z(t)$, and it is the stabilizing feedback control of the linear control system. Different stabilizability concepts can be adopted to investigate the stability property of the linear control systems [4, 10, and 14]. Wonham [14] extended stability properties is the concept of the complete stabilizability, which relates to a strong exponentially stability of the linear control system. That is to say, the linear control system is completely stabilizability if for every number $\delta > 0$, there exists a feedback control $u(t) = \mathbb{K}(t)z(t)$ such that the solution $z(t, z_0)$ of the closed-loop system satisfies the following condition

$$\exists N > 0: \|z(t, z_0)\| \leq N e^{-\delta t} \|z_0\|, \forall t \geq 0$$

This means that for every positive number $\delta > 0$, the system zero-input restraint of the closed-loop system decays faster than $e^{-\delta t}$. In other words, for any given in advance the decay rate $\delta > 0$, the linear control system can be δ -exponentially stabilizable. It is well known that, If a finite-dimensional time-invariant linear control system is globally null-controllable in finite time then it is stabilizable, but the converse is not true, studied by Kalman [6], and Wonham [15]. Nevertheless, if the system is completely stabilizable, then it is globally null-controllability in finite time discuss by Wonham [14].

However, in the infinite-dimensional control theory, investigations of controllability and stabilizability are more complicated and require more techniques that are sophisticated. The difficulties increase to the same extent as passing from time-invariant to time-varying system. Some extensions have been developed by Slemrod [13], Zabczyk [16] for time-invariant control systems in Hilbert spaces. For time-varying control systems in finite-dimensional spaces, using Kalman's decomposition method, [5] proved that the system is completely stabilizable if it is uniformly globally null-controllable and [12] extends some of [5] to time-varying control systems. P. Niamsup and V. N. Path. [1], develop the result in [12] on the relationship between the globally null-controllability and complete stabilizability for the linear time-varying control system in the real Hilbert space H .

From the all above, one can find a reasonable justification to accomplish the study of this paper. In this paper, we develop the result of P. Niamsup and V. N. Path. [1] on the relationship between the exact controllability and complete stabilizability for the perturbed linear time-varying system (1) in Hilbert space $\mathbb{H}$. We show that the system is complete stabilizable if it is globally null-controllable in finite time, and conversely under some additional growth condition on the evolution operator of the system, the system is globally null-controllable in finite time if it is complete stabilizable in finite time.

2. CONCEPT AND MATHEMATICAL PRELIMINARIES

The following concepts will be used in this article.

1. All real numbers are represented by the letter $\mathbb{R}$, the set of all non-negative real numbers is denoted by $\mathbb{R}^+$
2. $\mathbb{H}, U$ denotes an infinite-dimensional real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$, $\mathbb{H}^*, U^*$ denotes the dual spaces of the spaces $\mathbb{H}, U$ respectively, $\mathbb{H}^m, U^m$ denotes the real Hilbert product spaces dotted with the inner product defined by $\langle z, z \rangle_{\mathbb{H}^m} = \sum_{i=1}^m \langle z_i, z_i \rangle_{\mathbb{H}}$.
3. $\mathcal{L}(\mathbb{H})$ denotes the Banach space of all linear bounded operators mapping $\mathbb{H}$ into $\mathbb{H}$, $L^2([t, s], \mathbb{H})$ denotes the set of all strongly measurable L^2 -integrable and $\mathbb{H}$ -valued functions on $[t, s]$.
4. $D(A), A^{-1}$ and A^* denote the domain, the inverse and the adjoint of the operator A respectively, $\bar{M}$ denotes the closure of a set M , I denotes the identity operator; the operator A is called self-adjoint if $A = A^*$, $\rho(A)$, and $R(\lambda, A)$ denotes the resolvent set and the resolvent of the operator A_i respectively.
5. An operator $\mathbb{Q}_i \in \mathcal{L}(\mathbb{H})$ is called non-negative definite ($\mathbb{Q}_i \geq 0$), if $\langle \mathbb{Q}_i z_i, z_i \rangle \geq 0$, for all $z_i \in \mathbb{H}$, and for all $i \in [1, m]$; $\mathcal{LO}(\mathbb{H})([t, +\infty], \mathbb{H}^+)$ denotes the set of all linear bounded self-adjoint non-negative definite operator-valued function on $[0, \infty)$, the operator $A(t): D(A(t)) \subset \mathbb{H} \rightarrow \mathbb{H}$ be linear operator and $A(t) = \text{diag}(A_1(t), \dots, A_m(t))$ with $A_i(t)$ is generates a strongly continuous semigroup $\mathbb{T}_{t,i}$ of bounded linear operators on real Hilbert space $\mathbb{H}$ satisfies $\|\mathbb{T}_{t,i}\| \leq M_i e^{\omega_i t}$, $t \geq 0$, for $M_i \geq 1, \omega_i \in \mathbb{R}$ for all $i \in \{1, \dots, m\}$ if:

$$A_i(t)z_i = \lim_{t \rightarrow 0^+} \frac{1}{t} [\mathbb{T}_{t,i} - I_i]z_i, \forall z_i \in D(A_i(t)), i \in \{1, \dots, m\}$$

Where, $D(A_i(t)) = \{z_i \in \mathbb{H}: \lim_{t \rightarrow 0^+} \frac{1}{t} [\mathbb{T}_{t,i} - I_i]z_i\}$ exists.

Consider the perturbed linear time-varying system (1), where $\mathbb{H}, U$ infinite-dimensional real Hilbert spaces and $\mathbb{H}^m, U^m$ infinite-dimensional real Hilbert product spaces dotted.

Assume the following assumptions satisfying.

1. $A(t): D(A(t)) \subset \mathbb{H} \rightarrow \mathbb{H}$ be linear operator and $A(t) = \text{diag}(\mathbb{A}_1(t), \dots, \mathbb{A}_m(t))$ with $\mathbb{A}_i(t)$ is generates a strongly continuous semigroup $\mathbb{T}_{t,i}$ of bounded linear operators on real Hilbert space $\mathbb{H}$ satisfies $\|\mathbb{T}_{t,i}\| \leq \mathbb{M}_i e^{\omega_i t}, t \geq 0$, for $\mathbb{M}_i \geq 1, \omega_i \in \mathbb{R}$ for all $i \in \{1, \dots, m\}$.
2. $B(t): \mathbb{H} \rightarrow \mathbb{H}$ with $B(t) = \text{diag}(\mathbb{B}_1(t), \dots, \mathbb{B}_m(t))$ are bounded linear operators i.e., there exists a real number c_i , such that $\|\mathbb{B}_i(t)\| \leq c_i$, for all $i \in \{1, \dots, m\}$.
3. $E(t) = \text{diag}(\mathbb{E}_1(t), \dots, \mathbb{E}_m(t))$ is bounded linear operator for all $i \in [1, m]$, $z_i(t) \in \mathbb{H}$ is the state, $u_i(t) \in U$ is the control.

Therefore, by "bounded perturbation theorem" [18], the operator $A(t) + B(t): D(A(t)) \subset \mathbb{H} \rightarrow \mathbb{H}$ with

$A(t) + B(t) = \text{diag}(\mathbb{A}_1(t) + \mathbb{B}_1(t), \dots, \mathbb{A}_m(t) + \mathbb{B}_m(t))$ is infinitesimal generator a strongly continuous semigroup $\mathbb{S}_{t,i}$ satisfies

$$\|\mathbb{S}_{t,i}\| \leq \mathbb{M}_i e^{(\omega_i + \mathbb{M}_i c_i)t}, \text{ for all } t \geq 0, \text{ omits } B(t) \neq -A(t), \text{ and } B(t) \neq \alpha A(t), \text{ for all } \alpha > 1, i \in \{1, \dots, m\}.$$

Moreover, the following conditions that guarantee the existence and uniqueness of the solution for the perturbed linear time-varying system (1).

4. The linear operators $[A(\cdot) + B(\cdot)]z(\cdot)$, and $E(\cdot)u(\cdot)$ are continuous and bounded in $t \in \mathbb{R}^+$ for all $z(\cdot) \in \mathbb{H}, u(\cdot) \in U$, and for all $i \in \{1, \dots, m\}$.
5. $\overline{D(A(t) + B(t))} = \overline{D(A(t))} = \mathbb{H}, t \in \mathbb{R}^+, A(\cdot) + B(\cdot)$ generates a strongly continuous semigroup $\mathbb{S}_{t,i}$ on $\mathbb{H}$, and there is an evaluation operator $\mathbb{W}_{i,A+B}(t, s): \{(t, s): t \geq s \geq 0\} \rightarrow \mathcal{L}(\mathbb{H})$, such that $\mathbb{W}_{i,A+B}^*(t, s)$ is continuous in t, s and for each $z_i \in D(A_i(t))$, $\mathbb{W}_{i,A+B}(t, s)z_i \in D(A_i(t))$ for all $i \in \{1, \dots, m\}$.

$$(i) \frac{d\mathbb{W}_{i,A+B}(t, s)z_i}{dt} = [A(t) + B(t)]\mathbb{W}_{i,A+B}(t, s)z_i, \mathbb{W}_{i,A+B}(s, s) = I.$$

$$(ii) \mathbb{W}_{i,A+B}(t, s) = \mathbb{W}_{i,A+B}(t, r)\mathbb{W}_{i,A+B}(r, s), \text{ for all } t \geq s \geq 0, i \in \{1, \dots, m\}.$$

In this case, we say that $[A(t) + B(t)]$ generates a strongly continuous evolution perturbation operator $\mathbb{W}_{i,A+B}(t, s)$, then for every initial state $z_0 \in \mathbb{H}$, and for every admissible control $u(t)$, the perturbed linear time-varying system (1) has a mild solution given by:

Let $z(\cdot) \in \mathbb{H}$ be a solution of the perturbed linear time-varying system (1), then by definition of evolution family we have $\mathbb{W}_{i,A+B}(t, s), z_i(s)$ is differentiable, that implies the $\mathbb{H}$ -value function $g(s) = \mathbb{W}_{i,A+B}(t, s)z(s)$ is differentiable for $t \geq s \geq 0$; and :

$$\begin{aligned} \frac{\partial g(s)}{\partial s} &= \mathbb{W}_{i,A+B}(t, s) \frac{\partial}{\partial s} z(s) + z(s) \frac{\partial}{\partial s} \mathbb{W}_{i,A+B}(t, s) \\ &= \mathbb{W}_{i,A+B}(t, s) z(s) ([A(t) + B(t)]z_i(s) + E(s)u(s)) z(s) (-[A(t) + B(t)]\mathbb{W}_{i,A+B}(t, s)Z(s)) \end{aligned}$$

=

$$\mathbb{W}_{i,A+B}(t, s)E(s)u(s),$$

Integration (2) from 0 to t, yields:

$$g(t) - g(0) = \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s) ds$$

Since:

$$g(s) = \mathbb{W}_{i,A+B}(t,s)z(s), t \geq s \geq 0$$

Then

$$\mathbb{W}_{i,A+B}(t,t)z(s) - \mathbb{W}_{i,A+B}(t,0)z(s) = \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s) ds$$

Hence,

$$\mathbb{W}_{i,A+B}(t,t)z(s) = \mathbb{W}_{i,A+B}(t,0)z(s) + \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s) ds$$

By definition of semigroup, we obtain the solution

$$\begin{aligned} \mathbb{W}_{i,A+B}(t,s)z(s) &= z(t) \\ &= \mathbb{W}_{i,A+B}(t,0)z_0 \\ &\quad + \int_0^t \mathbb{W}_{i,A+B}(t,s)E(s)u(s) ds, \end{aligned} \tag{3}$$

The evaluation-perturbed operator $\mathbb{W}_i(t,s)$ is a natural extension to the strongly continuous semigroup of time-invariant perturbed linear systems. For example, if $A(t) + B(t) = A + B$ is independent of t then $\mathbb{W}_{i,A+B}(t,s)\mathbb{S}_{t,i}\mathbb{S}_{t,i}$, and the two parameter family of semigroup operators reduces to one parameter family $\mathbb{S}_{t,i}$, which is standard a strongly continuous semigroup generated by $A + B \in \mathcal{L}(\mathbb{H})$, $t \in \mathbb{R}^+$, then the semigroup evolution operators $\mathbb{W}_{i,A+B}(t,s)$ satisfying the above conditions always exists. However, if $A(t) + B(t)$, $t \in \mathbb{R}^+$ is unbounded perturbed operator, then the evolution-perturbed operator $\mathbb{W}_{i,A+B}(t,s)$ exists provided additional assumptions [2,8].

1. Definition

The perturbed linear time-varying system (1), is called globally null-controllability (GNC) in finite time if, for every $z_0 \in \mathbb{H}$, there exist a number $T > 0$ and an admissible control $u(t)$ such that

$$\mathbb{W}_{i,A+B}(t,0)z_0 + \int_0^T \mathbb{W}_{i,A+B}(T,s)E(s)u(s) ds = 0$$

2. Remark [2,3]

The perturbed linear time-varying system (1) is called globally null-controllability if and only if $\exists T > 0, \mathbb{b}_i > 0$ such that

$$\int_0^T \|E^*(s)W_{i,A+B}^*(T,s)z^*\|^2 ds \geq \mathbb{b}_i \|W_{i,A+B}^*(T,0)z^*\|^2, \forall z^* \in \mathbb{H}^*$$

3. Definition

The perturbed linear time-varying system (1) is called completely stabilizability (CZs) if every number $(\omega_i + \mathbb{M}_i c_i) > 0$ there exists a feedback control $u(t) = \mathbb{K}(t)z(t)$, where $\mathbb{K}(t) \in \mathcal{L}(\mathbb{H}, U)$ is bounded on $\mathbb{R}^+$, such that the solution $z(t, z_0)$ of the closed-loop nonlocal system

$$\begin{aligned} \begin{bmatrix} \frac{dz_1(t)}{dt} \\ \vdots \\ \frac{dz_m(t)}{dt} \end{bmatrix} &= \left(\left(\begin{bmatrix} \mathbb{A}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m(t) \end{bmatrix} \right) \right. \\ &\quad \left. + \begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix} \begin{bmatrix} \mathbb{K}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{K}_m(t) \end{bmatrix} \right) \begin{bmatrix} z_1(t) \\ \vdots \\ z_1(t) \end{bmatrix} \\ \begin{bmatrix} z_1(0) \\ \vdots \\ z_1(0) \end{bmatrix} &= \begin{bmatrix} z_0 \\ \vdots \\ z_0 \end{bmatrix} \end{aligned}$$

Satisfies

$\exists N > 0: \|z_i(t, z_0)\| \leq N e^{-(\omega_i + \mathbb{M}_i c_i)t} \|z_0\|, \forall t \geq 0, i \in [1, m]$. The solution for the stabilizability nonlocal problem involves a perturbed Riccati operator equation (PROE) of the form:

$$\begin{aligned}
& \begin{bmatrix} \frac{dP_1(t)}{dt} \\ \vdots \\ \frac{dP_m(t)}{dt} \end{bmatrix} + \left(\begin{bmatrix} \mathbb{A}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m^*(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \\
& + \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \mathbb{A}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m(t) \end{bmatrix} \right) \\
& - \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix} \begin{bmatrix} \mathbb{E}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{Q}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_m(t) \end{bmatrix} \\
& = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4}
\end{aligned}$$

Where, $\mathbb{Q}_i(t) \geq 0$, is given self-adjoint operators for all $i \in \{1, \dots, m\}$, it is not clear a priori what a solution of (PROE) means. We will defined the solution of (PROE)(4) as follows.

4. Definition

The (mild) solution of (PROE)(4) is a linear operator $P_i(t) \in \mathcal{L}(\mathbb{H})$ satisfying the following conditions:

- 1) The scalar function $\langle P_i(t)z_i, y_i \rangle$ is differentiable on $[0, \infty)$ for every $z_i, y_i \in D(A(t) + B(t)) = D(A(t))$, $i \in \{1, \dots, m\}$.
- 2) For all $z_i, y_i \in D(A(t) + B(t)) = D(A(t))$, $t \in \mathbb{R}^+$, $i \in \{1, \dots, m\}$:

$$\begin{aligned}
& \frac{d}{dt} \langle P_i(t)z_i, y_i \rangle + \langle P_i(t)z_i, (A(t) + B(t))y_i \rangle + \langle P_i(t)(A(t) + B(t))z_i, y_i \rangle - \langle P_i(t)E(t)E^*(t)P_i(t)z_i, y_i \rangle + \\
& \langle \mathbb{Q}_i(t)z_i, y_i \rangle = 0.
\end{aligned}$$

We state the following sufficient condition, which guarantees the existence of a bounded solution $P_i(t)$ of (PROE)(4).

5. Definition .

Let $\mathbb{Q}_i(t) \in \mathcal{LO}([0, \infty), \mathbb{H}^+)$. The perturbed linear time-varying system (1) is called $\mathbb{Q}_i(t)$ -stabilizable if for every initial state z_0 , there is a control $u(t) \in L^2([0, \infty), U)$ such that the cost function:

$$\begin{bmatrix} J_1(u_1) \\ \vdots \\ J_m(u_i) \end{bmatrix} = \begin{bmatrix} \int_0^\infty [\|u_1(t)\|^2 + \langle Q_1(t)z_1(t, z_0), z_1(t, z_0) \rangle] dt \\ \vdots \\ \int_0^\infty [\|u_1(t)\|^2 + \langle Q_1(t)z_1(t, z_0), z_1(t, z_0) \rangle] dt \end{bmatrix} \quad (5)$$

Exists and is finite for all $i \in \{1, \dots, m\}$.

6. Remark [2]

If the perturbed linear time-varying system (1) is $Q_i(t)$ -stabilizable, then the (PROE)(4) has the solution $P_i(t) \in \mathcal{LO}([0, \infty), \mathbb{H}^+)$ bounded on $\mathbb{R}^+$.

3. MAIN RESULT

Consider the perturbed linear time-varying system (1) in real Hilbert space. As we have already hinted at introduction, for time-invariant control system in finite-dimensional spaces Wonham [13] studied the equivalence of complete stabilizability and global null-controllability and for the case of infinite-dimensional systems, assuming a compactness property on the semigroup Slemrod [11] showed that the time-invariant control system in Hilbert spaces is completely stabilizable if and only if it is globally null-controllability in finite time. P. Niamsup and V. N. Path. [1], develop the result in [13] on the relationship between the globally null-controllability and complete stabilizability for the linear time-varying control system in the real Hilbert space $\mathbb{H}$. In this paper, we develop the result in [1] on the relationship between the exact controllability and complete stabilizability for the perturbed linear time-varying nonlocal system (1) in Hilbert space $\mathbb{H}$. We show that the system is complete stabilizable if it is globally null-controllable in finite time, and conversely under some additional growth condition on the evolution operator of the system, the system is globally null-controllable in finite time if it is complete stabilizable in finite time. In this section we use the following growth condition on the perturbed evolution $\mathbb{W}_{i, A+B_i}(t, s)$:

$$(H) \exists M_i > 0, \gamma_i + M_i c_i > 0: \|\mathbb{W}_{i, A+B}(t, s)\| \leq M_i e^{(\gamma_i + M_i c_i)|t-s|}, \forall t, s \geq 0, i \in \{1, \dots, m\}.$$

From [8] that the growth condition $\mathbb{H}$ hold for time-invariant system when $m = 1, B_1(t) = 0$, for, and $A(t) \in \mathcal{L}(\mathbb{H})$ is a linear continuous constant operator as well as for time-varying system when $A(t)$ is a matrix function uniformly bounded in $t \in \mathbb{R}^+$.

Proposition 3.1

The perturbed linear time-varying system (1) is completely stabilizable (CS_2) if it is globally null-controllability (GNC) in finite time.

Proof:

Assume that the perturbed linear time-varying system (1) is globally null-controllability (GNC) in finite time. Let $(\omega_i + M_i c_i) > 0$, for all $i \in \{1, \dots, m\}$ be any given number. We take a change of the state variable $y_i(t) = e^{(\omega_i + M_i c_i)t} z_i(t)$, then the perturbed linear time-varying system (1) is transformed to the system given by:

$$\begin{bmatrix} \frac{d}{dt}y_1(t) \\ \vdots \\ \frac{d}{dt}y_m(t) \end{bmatrix} = \left(\begin{bmatrix} \tilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m(t) \end{bmatrix} \right) \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{E}}_m(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ \vdots \\ y_m(0) \end{bmatrix} = \begin{bmatrix} y_0 z_0 \\ \vdots \\ y_0 z_0 \end{bmatrix}, \forall i \in [1, m] = \{1, \dots, m\}, \quad (6)$$

Where,

$$\begin{aligned} & \left(\begin{bmatrix} \tilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m(t) \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} \mathbb{A}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{A}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{B}_m(t) \end{bmatrix} \right) \times \begin{bmatrix} e^{(\omega_1 + \mathbb{M}_1 c_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{(\omega_m + \mathbb{M}_m c_m)} \end{bmatrix} \times I \end{aligned}$$

and,

$$\begin{bmatrix} \tilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{E}}_m(t) \end{bmatrix} = \begin{bmatrix} e^{(\omega_1 + \mathbb{M}_1 c_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{(\omega_m + \mathbb{M}_m c_m)} \end{bmatrix} \begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix}$$

We choose an operator function $\mathbb{Q}_i(t) \in \mathcal{LO}([0, \infty), \mathbb{H}^+)$ bounded on $\mathbb{R}^+$, such that

$$\begin{aligned} \begin{bmatrix} \mathbb{Q}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_m(t) \end{bmatrix} &\geq 2 \left(\begin{bmatrix} \tilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m(t) \end{bmatrix} \right) \\ &+ \left(\begin{bmatrix} \mathbb{E}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m(t) \end{bmatrix} \begin{bmatrix} \mathbb{E}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{E}_m^*(t) \end{bmatrix} \right), \quad (7) \end{aligned}$$

We now show that the perturbed linear time-varying system

$$[\tilde{\mathbb{A}}_i(t) + \tilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)]:$$

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \vdots \\ \frac{d}{dt}x_m(t) \end{bmatrix} = \left(\begin{bmatrix} \tilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m(t) \end{bmatrix} \right) \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{E}}_m(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ \vdots \\ x_m(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \vdots \\ x_0 \end{bmatrix}, \forall i \in \{1, \dots, m\},$$

is globally null-controllability in finite time.

Moreover, by the globally null-controllability of previous system $[(\mathbb{A}_i(t) + \mathbb{B}_i(t)), \mathbb{E}_i(t)]$, for every $x_0 \in \mathbb{H}$ there exists a time $T > 0$, and admissible control $u(t) \in L^2([0, T], U)$ such that:

$$\mathbb{W}_{i,A+B}(T, 0)x_0 + \int_0^T \mathbb{W}_{i,A+B}(T, s)\mathbb{E}_i(s)u_i(s) ds = 0, \forall i \in \{1, \dots, m\}, \quad (8)$$

Multiplying both sides of equation (8) with $e^{(\omega_i + \mathbb{M}_i c_i)T}$ and observing that

$$\mathbb{W}_{i,\tilde{A}+\tilde{B}}(t, s) = e^{(\omega_i + \mathbb{M}_i c_i)|t-s|} \mathbb{W}_{i,A+B}(t, s)$$

We find

$$\mathbb{W}_{i,\tilde{A}+\tilde{B}}(T, 0)x_0 + \int_0^T \mathbb{W}_{i,\tilde{A}+\tilde{B}}(T, s)\mathbb{E}_i(s)\tilde{u}_i(s) ds = 0, \forall i \in \{1, \dots, m\}$$

Where $\tilde{u}_i(s) = e^{(\omega_i + \mathbb{M}_i c_i)s}u_i(s)$,

This implies that the perturbed linear time-varying system $[\tilde{\mathbb{A}}_i(t) + \tilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)]$ is globally null-controllability in finite time. Let $u_{i,x_i}(t)$ be an admissible control as stated by to the solution $x_i(t)$ for all $i \in \{1, \dots, m\}$ of the perturbed linear time-varying nonlocal system $[\tilde{\mathbb{A}}_i(t) + \tilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)]$ steers $x_0 \in \mathbb{H}$ into zero in time T .

For every initial state $x_0 \in \mathbb{H}$ there is an admissible control $u_{i,x_i}(t) \in L^2([0, T], U)$ such that the solution $x_i(t)$ of the system as maintained by the control $u_{i,x_i}(t)$ satisfies $x_i(0) = x_0, x_i(T) = 0$, for all $i \in \{1, \dots, m\}$.

Defined the admissible control $\tilde{u}_{i,x_i}(t) \in L^2([0, T], U), t \geq 0$ by

$$\tilde{u}_{i,x_i}(t) = \begin{cases} u_{i,x_i}(t), & \text{if } t \in [0, T] \\ 0, & \text{if } t > T \end{cases}$$

Therefore, we have

$$\begin{aligned}
J_i(\tilde{u}_{i,x_i}(t)) &= \int_0^\infty \left[\|\tilde{u}_{i,x_i}(t)\|^2 + \langle \mathbb{Q}_i(t)x_i(t), x_i(t) \rangle \right] dt \\
&= \int_0^T \left[\|\tilde{u}_{i,x_i}(t)\|^2 + \langle \mathbb{Q}_i(t)x_i(t), x_i(t) \rangle \right] dt < +\infty
\end{aligned}$$

Therefore, the perturbed linear time-varying nonlocal system $[\tilde{\mathbb{A}}_i(t) + \tilde{\mathbb{B}}_i(t), \mathbb{E}_i(t)]$ is $\mathbb{Q}_i(t)$ -stabilizable. By using remark 2 to the cost function (5), we can find an operator function $P_i \in \mathcal{LO}([0, \infty), \mathbb{H}^+)$ which a solution of the following PROE

$$\begin{aligned}
&\begin{bmatrix} \frac{dP_1(t)}{dt} \\ \vdots \\ \frac{dP_m(t)}{dt} \end{bmatrix} + \left(\begin{bmatrix} \tilde{\mathbb{A}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m^*(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{B}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \tilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m(t) \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} \tilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m(t) \end{bmatrix} \right) - \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \tilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{E}}_m(t) \end{bmatrix} \begin{bmatrix} \tilde{\mathbb{E}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{E}}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} \mathbb{Q}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_m(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}
\end{aligned}$$

or equivalently

$$\begin{aligned}
& \begin{bmatrix} \frac{dP_1(t)}{dt} \\ \vdots \\ \frac{dP_m(t)}{dt} \end{bmatrix} + \left(\begin{bmatrix} \tilde{\mathbb{A}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m^*(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{B}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \\
& + \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \tilde{\mathbb{A}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{A}}_m(t) \end{bmatrix} + \begin{bmatrix} \tilde{\mathbb{B}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{B}}_m(t) \end{bmatrix} \right) \\
& - 2e^{(\omega_i + \mathbb{M}_i c_i)t} \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \left(\begin{bmatrix} \tilde{\mathbb{E}}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{E}}_m(t) \end{bmatrix} \begin{bmatrix} \tilde{\mathbb{E}}_1^*(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{\mathbb{E}}_m^*(t) \end{bmatrix} \right) \begin{bmatrix} P_1(t) \\ \vdots \\ P_m(t) \end{bmatrix} \\
& + \begin{bmatrix} \mathbb{Q}_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{Q}_m(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (9)
\end{aligned}$$

We consider a Lyapunov-like function

$$\mathbb{W}_{i,A+B}(t, y_i) = \langle P_i(t)y_i, y_i \rangle + \|y_i\|^2$$

and construct a feedback control of the form

$$u_i(t) = -\frac{1}{2}e^{-2(\omega_i + \mathbb{M}_i c_i)t} \tilde{\mathbb{E}}_i(t)[P_i(t) - I]y_i(t), \quad (10)$$

By taking the derivative of $W_{i,A+B}(\cdot)$ in t along the solution $y_i(t)$ of the system (6), and using the chosen feedback control and the PROE (9), we have

$$\begin{aligned}
\frac{d}{dt} W_{i,A+B}(t, y_i) &= \left\langle \frac{d}{dt} P_i(t) y_i(t), y_i(t) \right\rangle + 2 \left\langle P(t) \frac{d}{dt} y_i(t), y_i(t) \right\rangle + 2 \left\langle \frac{d}{dt} y_i(t), y_i(t) \right\rangle, \\
&= \langle (-\tilde{\mathbb{A}}_i^* + \tilde{\mathbb{B}}_i^*)P_i - P_i(\tilde{\mathbb{A}}_i + \tilde{\mathbb{B}}_i) + 2e^{-2(\omega_i + \mathbb{M}_i c_i)t} P_i \tilde{\mathbb{E}}_i \tilde{\mathbb{E}}_i^* \mathbb{Q}_i \rangle y_i, y_i \rangle \\
&\quad + 2 \langle P_i[(\tilde{\mathbb{A}}_i^* + \tilde{\mathbb{B}}_i^*)y_i + \tilde{\mathbb{E}}_i u_i], y_i \rangle + 2 \langle (\tilde{\mathbb{A}}_i^* + \tilde{\mathbb{B}}_i^*)y_i + \tilde{\mathbb{E}}_i u_i, y_i \rangle, \\
&= e^{-2(\omega_i + \mathbb{M}_i c_i)t} \langle P_i \tilde{\mathbb{E}}_i \tilde{\mathbb{E}}_i^* P_i y_i, y_i \rangle + 2 \langle P_i \tilde{\mathbb{E}}_i u_i, y_i \rangle + 2 \langle (\tilde{\mathbb{A}}_i + \tilde{\mathbb{B}}_i)y_i, y_i \rangle + 2 \langle \tilde{\mathbb{E}}_i u_i, y_i \rangle - \langle \mathbb{Q}_i y_i, y_i \rangle \\
&= -\langle [\mathbb{Q}_i(t) - 2(\tilde{\mathbb{A}}_i + \tilde{\mathbb{B}}_i)(t) - e^{-2(\omega_i + \mathbb{M}_i c_i)t} \tilde{\mathbb{E}}_i(t) \tilde{\mathbb{E}}_i^*(t)] y_i(t), y_i(t) \rangle \\
&= -\langle [\mathbb{Q}_i(t) - 2(\tilde{\mathbb{A}}_i + \tilde{\mathbb{B}}_i)(t) - \mathbb{E}_i(t) \mathbb{E}_i^*(t)] y_i(t), y_i(t) \rangle
\end{aligned}$$

By choosing $\mathbb{Q}_i(t)$ from (7), we find $[\mathbb{Q}_i(t) - 2(\tilde{\mathbb{A}}_i + \tilde{\mathbb{B}}_i)(t) - \mathbb{E}_i(t)\mathbb{E}_i^*(t)] \geq 0$, and hence:

$$\frac{d}{dt} \mathbb{W}_{i,A+B}(t, y_i) \leq 0, \forall t \geq 0, i \in \{1, \dots, M\}, \quad (11)$$

This inequality shows the property boundedness of the solution $y_i(t)$ for all $i \in \{1, \dots, m\}$ for the nonlocal system (6).

Actually, by integrating the inequality (11) from zero to t , we obtain that:

$$\mathbb{W}_{i,A+B}(t, y_i) - \mathbb{W}_{i,A+B}(0, y_0) \leq 0,$$

and hence

$$\langle P_i(t)y_i(t), y_i(t) \rangle - \|y_i(t)\|^2 \leq \langle P_i(0)y_0, y_0 \rangle - \|y_0\|^2.$$

Since $P_i(t) \geq 0$, for all $t \in \mathbb{R}^+, i \in \{1, \dots, m\}$,

$$\|y_i(t)\| \leq \langle P_i(0)y_0, y_0 \rangle - \|y_0\|^2, \forall t \geq 0, i \in \{1, \dots, m\},$$

We can find a number $N > 0$ such that:

$$\|y_i(t)\| \leq N\|y_0\|, \forall t \geq 0, i \in \{1, \dots, m\}.$$

Therefore, by returning to the solution $z_i(t, z_0)$ of the perturbed linear time-varying system(1), we finally obtain that:

$$\|z_i(t, z_0)\| \leq Ne^{-(\omega_i + \mathbb{M}_i c_i)t} \|z_0\|, \forall t \geq 0, i \in \{1, \dots, m\}$$

Then, the perturbed linear time-varying nonlocal system(1) is completely stabilizable by the feedback control(10) trasfomed in the state $z_i(t)$ as

$$u_i(t) = -\frac{1}{2}e^{-2(\omega_i + \mathbb{M}_i c_i)t} \tilde{\mathbb{E}}_i(t)[P_i(t) - I]y_i(t) = \mathbb{K}_i(t)z_i(t),$$

Where $\mathbb{K}_i(t) = -\frac{1}{2}\tilde{\mathbb{E}}_i(t)[P_i(t) - I]$, which is bounded on $\mathbb{R}^+$.

Then, the globally null-controllability implies completely stabilizable. ■

Proposition 3.2

If the perturbed linear time-varying nonlocal system (1) is globally null-controllability (GNC) with condition $\mathbb{H}$, then it is completely stabilizable (CS_2) in finite time.

Proof:

Suppose that perturbed linear time-varying system (1) completely stabilizable. By property of the perturbed evolution operator we have:

$$\exists M_i > 0, \gamma_i + M_i c_i > 0: \|\mathbb{W}_i^*(t, s)\| \leq M_i e^{(\gamma_i + M_i c_i)|t-s|}, \forall t, s \geq 0, i \in \{1, \dots, m\}$$

For chosen number $\omega_i + \mathbb{M}_i c_i > \gamma_i + M_i c_i > 0$, there is an operator $\mathbb{K}_i \in \mathcal{L}(\mathbb{H}, U)$, which is bounded on $\mathbb{R}^+$ such that the solution $z_i(t, z_0) = \mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}(t, 0)z_0$, where $\mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}(t, 0)z_0$ is the evolution operator generated by the operator $[(\mathbb{A}_i(t) + \mathbb{B}_i(t)) + \mathbb{K}_i(t)\mathbb{E}_i(t)]$, satisfies

$\exists N > 0$ such that

$$\|z_i(t, z_0)\| = \|\mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}(t, 0)z_0\| \leq Ne^{-(\omega_i + \mathbb{M}_i c_i)t} \|z_0\|, \forall t \geq 0, i \in \{1, \dots, m\} \quad (12)$$

Furthermore, for every $z_0 \in \mathbb{H}$ feedback control $u_i(t) = \mathbb{K}_i(t)z_i(t)$, the solution $z(t, z_0)$ of the perturbed linear time-varying system (1) is defined as

$$z_i(t, z_0) = \mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}(t, 0)z_0 + \int_0^t \mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}(t, s)\mathbb{B}_i(s)u_i(s) ds.$$

Therefore,

$$\mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}(t, 0)z_0 = \mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}(t, 0)z_0 - \int_0^t \mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}(t, s)\mathbb{E}_i(s)\mathbb{K}_i(s)\mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}(t, 0)z_0,$$

For all $t \in \mathbb{R}^+$, and $i \in \{1, \dots, m\}$.

Since the above relation holds for every $z_0 \in \mathbb{H}$, the following estimate holds for every $z_i^* \in \mathbb{H}^*$, we have

$$\|\mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}^*(t, 0)z_i^*\| \leq \|\mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}^*(t, 0)z_i^*\| + \int_0^t \|\mathbb{U}_{i, \mathbb{A}_i + \mathbb{B}_i + \mathbb{K}_i}^*(s, 0)\mathbb{K}_i^*(s)\mathbb{E}_i^*(s)\mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}^*(t, s)z_i^*\| ds$$

By using condition (12), we obtain that:

$$\begin{aligned} \|\mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}^*(t, 0)z_i^*\| &\leq Ne^{-(\omega_i + \mathbb{M}_i c_i)t} \|z_0\| + NK \int_0^t e^{-(\omega_i + \mathbb{M}_i c_i)t} \|\mathbb{E}_i^*(s)\mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}^*(t, s)\| ds \\ &\leq Ne^{-(\omega_i + \mathbb{M}_i c_i)t} \|z_0\| + NK \left(\int_0^t e^{-(\omega_i + \mathbb{M}_i c_i)t} \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t \|\mathbb{E}_i^*(s)\mathbb{W}_{i, \mathbb{A}_i + \mathbb{B}_i}^*(t, s)\|^2 ds \right)^{\frac{1}{2}}, \end{aligned} \quad (13)$$

Where $K = \sup\{\mathbb{K}_i(s): s \in [0, \infty)\}$. Setting $\mu_i(t) = \left(\int_0^t e^{-(\omega_i + \mathbb{M}_i c_i)t} \right)^{\frac{1}{2}}$, we obtain that:

$$\mu_i(t) = \left(\frac{1}{2(\omega_i + \mathbb{M}_i c_i)} - \frac{1}{2(\omega_i + \mathbb{M}_i c_i)} e^{-2(\omega_i + \mathbb{M}_i c_i)t} \right)^{\frac{1}{2}}, \quad (14)$$

We prove the globally null-controllable of the perturbed linear time-varying system(1), by contradiction. Suppose that the system(1) is not globally null-controllable in any finite time $t > 0$. Then, by using remark 2, for every $t > 0$, and for some chosen number $\mathbb{b}_i > 0, \kappa \in (0,1)$, satisfying

$$\mathbb{b}_i < \left[\frac{(1 - \kappa)\sqrt{2(\omega_i + \mathbb{M}_i c_i)}}{NK} \right]^2, \quad (15)$$

there is $z_0^* \in \mathbb{H}^*$, such that:

$$\int_0^t \|\mathbb{B}_i^*(s)\mathbb{W}_{i,A_i+B_i}^*(t,s)z_0^*\|ds \leq \mathbb{b}_i \|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\|^2, \quad (16)$$

This means that, the inequality (16) shows that $z_0^* \neq 0$, we may assume that the inequality (16) satisfying for all $z_0: \|z_0^*\| = 1$; otherwise we can take $z_i^* = \frac{z_0^*}{\|z_0^*\|}$ for all $i \in [1, m] = \{1, \dots, m\}$. Therefore, by using inequality(13), (16), we obtain that:

$$\begin{aligned} & \|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\| \\ & < Ne^{-(\omega_i + \mathbb{M}_i c_i)t} \\ & + \sqrt{\kappa}NK\mu_i(t)\|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\|, \end{aligned} \quad (17)$$

On other hand, we observe that:

$$1 = \|z_0^*\| = \|\mathbb{W}_{i,A_i+B_i}^*(t,0)\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\| \leq \|\mathbb{W}_{i,A_i+B_i}^*(t,0)\| \|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\|,$$

Which, due to (12), and $\|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\| \neq 0$, we have:

$$\frac{1}{\|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\|} \leq \|\mathbb{W}_{i,A_i+B_i}^*(t,0)\| < M_i e^{(\gamma_i + M_i c_i)t}, \quad (18)$$

Thus, by using (17) and (18), we obtain that:

$$1 < \frac{N_i e^{-(\omega_i + \mathbb{M}_i c_i)t}}{\|\mathbb{W}_{i,A_i+B_i}^*(t,0)z_0^*\|} + \sqrt{\kappa}NK\mu_i(t) < NM_i e^{-[(\omega_i + \mathbb{M}_i c_i) - (\gamma_i + M_i c_i)]t} + \sqrt{\kappa}NK\mu_i(t)$$

Hence

$$1 - \sqrt{\kappa}NK\mu_i(t) < NM_i e^{-[(\omega_i + \mathbb{M}_i c_i) - (\gamma_i + M_i c_i)]t}, \forall t > 0$$

This means that, the above relation does not depend on z_0^* , we can let t go to infinity and observation from (14) that

$\mu_i(t) \rightarrow \left(\frac{1}{\sqrt{2(\omega_i + \mathbb{M}_i c_i)}} \right)$, the right hand-side goes to zero because of $(\omega_i + \mathbb{M}_i c_i) > (\gamma_i + M_i c_i)$, we obtain that

$$1 - \sqrt{\kappa}NK \frac{1}{\sqrt{2(\omega_i + \mathbb{M}_i c_i)}} \leq 0.$$

Then, by using (15) it follows the condition

$$\kappa < 1 - \sqrt{\kappa NK} \frac{1}{\sqrt{2(\omega_i + \mathbb{M}_i c_i)}} \leq 0$$

Which is contradiction. This means that the perturbed linear time-varying nonlocal system(1) is globally null-controllability in finite time.

Theorem 3.1

The perturbed linear time-varying system (1) is completely stabilizable (CS_z) iff it is globally null-controllability (GNC) in finite time. Conversely, assume the condition III, the perturbed linear time-varying nonlocal system (1) is globally null-controllability (GNC) if it is completely stabilizable (CS_z) in finite time.

Proof:

The proof is direct way on proposition 3.1, and proposition 3.2.

■

4. CONCLUSIONS

In this paper, we have established the equivalence of globally null controllability and complete stabilizability for perturbed linear time-varying control systems in real Hilbert space. The result obtained by extends existing results in the literature to infinite-dimensional and time-varying perturbed control systems.

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