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A Quasi-Hilbert Space and Its Properties

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ABSTRACT

This paper studies the concept of a quasi-inner product space and its completeness to get and prove some properties of quasi-Hilbert spaces. The best examples of this notion are spaces L_p , where $0 < p < \infty$.

1. Introduction

The most important in functional analysis is the concept of normed space and its completeness which is related to other concepts such as a metric space, inner product space, and a quasi-normed space. The set of all measurable functions $L_p[a, b]$, $0 < p < \infty$, $[a, b] \subset \mathbb{R}$ is a good example on these concepts ([3,4,8]).

In the normed spaces, mathematicians have used Gâteaux derivatives to introduce a notion of quasi-inner product space ([5,7]). This paper has used a quasi-normed space to give a quasi-inner product space and quasi-Hilbert space. It is studied the relationship between a notion of quasi-Hilbert space and others, with study some properties of it.

Section one of the paper includes definitions of a quasi-normed space, a quasi-Banach space and others with some valuable results. The second section presents a quasi-inner product space and quasi-Hilbert space with some essential properties.

2. Quasi-Banach Spaces

Definition (2.1). Quasi-metric space (U, d_q) , where U is a nonempty set X with a quasi-metric d_q which differs from a metric function by the inequality :

$$d_q(u, v) \leq C \left(d_q(u, w) + d_q(w, v) \right) \text{ for all } u, v, w \in U, \text{ where } 1 \leq C < \infty.$$

A function d_q be a metric if $C = 1$, thus it is generalization of a metric. Every a metric function is quasi-metric, but not the converse in generality [6].

Definition (2.2). A quasi-norm ${}_q\|\cdot\|$ on a vector space U over the field of real numbers $\mathbb{R}$ is a function ${}_q\|\cdot\|: U \rightarrow [0, +\infty)$ with the properties:

$$(1) \quad {}_q\|v\| \geq 0, \forall v \in U, \quad {}_q\|v\| = 0 \Leftrightarrow v = 0.$$

$$(2) \quad {}_q\|\alpha v\| = |\alpha| {}_q\|v\|, \quad \forall v \in U, \quad \forall \alpha \in \mathbb{R}.$$

$$(3) \quad {}_q\|v + w\| \leq C \left({}_q\|v\| + {}_q\|w\| \right) \quad \forall v, w \in U, \text{ where a constant } C \geq 1.$$

If $C = 1$, then the quasi-norm is a norm function. A quasi-normed space is $(U, {}_q\|\cdot\|)$ or simply U . Since every quasi-normed space U is a quasi-metric space by $d_q(v, w) = {}_q\|v - w\|$, the concept of completeness is given. A quasi-Banach space is a complete quasi-normed space ([3,6]).

Remark (2.3). It is clear, every quasi-normed space is a quasi-metric space, conversely may be not true, indeed,

Take (U, d_q) , where $d_q(u, v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|u_k - v_k|}{1 + |u_k - v_k|}$ is a quasi-metric space, but it is not a quasi-normed space (see[8]).

Remark (2.4). It is obvious, any inner product function satisfies Definition (2.2) and generates a quasi-norm which is ${}_q\|v\| = (\langle v, v \rangle)^{1/2} \quad \forall v \in U$.

Theorem (2.5) [1]. A quasi-normed space U is an inner product space iff equality:

$${}_q\|v + w\|^2 + {}_q\|v - w\|^2 = 2 {}_q\|v\|^2 + 2 {}_q\|w\|^2 \quad \forall u, v \in U, \quad (2.1)$$

is satisfied by the quasi-norm of U .

Definition (2.6) [4]. The vector space $L_p[a, b]$, $0 < p < \infty$, $[a, b] \subset \mathbb{R}$ is a set of all measurable

functions f on $[a, b]$ for which $\int_a^b |f(x)|^p dx < \infty$.

Theorem (2.7) [3,4]. A space $L_p [a, b]$, with the function $\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$ is a quasi-Banach space when $0 < p < \infty$, a Banach space when $1 < p < \infty$, and $L_2 [a, b]$ is only a Hilbert space.

3. Some Properties of Quasi-inner Product Spaces.

Definition (3.1) [1]. A quasi-normed space U is called a quasi-inner product space, if:

$$\|v + w\|^4 - \|v - w\|^4 = 8 (\|v\|^2 \tau(v, w) + \|w\|^2 \tau(w, v)), \quad \forall v, w \in U, \quad (3.1)$$

is satisfied, where $\tau(v, w)$ and $\tau(w, v)$ are Gateaux derivatives, A Gateaux derivative of $\|v\|$, $\tau(v, w)$ at $v \in U$ in the direction $w \in U$ is defined as:

$$\tau(v, w) = \frac{\|v\|}{2} \left(\lim_{h \rightarrow +0} \frac{\|v + hw\| - \|v\|}{h} + \lim_{h \rightarrow -0} \frac{\|v + hw\| - \|v\|}{h} \right), \text{ where } h \in \mathbb{R}.$$

Similarly, $\tau(w, v)$ at $w \in U$ in the direction v is defined. If U is a quasi-Banach space then it is called a quasi-Hilbert space.

Proposition (3.2). If U is a quasi-normed space then $\forall v, w \in U$,

$$(1) \tau(v, w) \geq 0, \tau(0, w) = \tau(v, 0) = 0, \text{ and } \tau(v, v) = \|v\|^2.$$

$$(2) \text{ If the right limit = the left limit then: } \tau(v, w) = \|v\| \lim_{h \rightarrow 0} h^{-1} (\|v + hw\| - \|v\|).$$

Proof:

The proof is obvious from Definition (3.1).

Proposition (3.3). If U is an inner-product space then $\tau(v, w) = \langle v, w \rangle$ and $\tau(w, v) = \langle w, v \rangle$

$$\forall v, w \in U.$$

Proof:

From Definition (3.1), $\forall v, w \in U$.

$$\tau(v, w) = \frac{\|v\|}{2} \left(\left(\lim_{h \rightarrow +0} \frac{\|v + hw\|^2 - \|v\|^2}{h \|v + hw\| + \|v\|} \right) + \left(\lim_{h \rightarrow -0} \frac{\|v + hw\|^2 - \|v\|^2}{h \|v + hw\| + \|v\|} \right) \right).$$

Using Remark (2.4), we get $\tau(v, w) = \langle v, w \rangle$. Similarly, $\tau(w, v) = \langle w, v \rangle$.

Remark (3.4). Using the proof of Proposition (3.3) with the binomial theorem [2] in Definition (3.1) to get the functional $\tau(f, g)$ of the following:

(1) $\forall f, g \in L_1[a, b]$, we have $\tau(f, g) = {}_q\|f\| \int_a^b (\text{sng } f(x))g(x) dx$. where,

$$\text{sng } f(x) = \begin{cases} 1, & f(x) > 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0 \end{cases}.$$

(2) $\forall f, g \in L_4[a, b]$, we have $\tau(f, g) = {}_q\|f\|^{-2} \int_a^b |f(x)|^3 (\text{sng } f(x))g(x) dx$.

Similarly, $\tau(g, f)$ is defined for (1) and (2).

Example (3.5). If $f, g \in L_1[0, 1]$, such that $f(x) = x \quad \forall x \in [0, \frac{1}{2}]$ and $g(x) = x \quad \forall x \in [\frac{1}{2}, 1]$.

From Remark (3.4), $\tau(f, g) = \tau(w, v) = \frac{3}{64}$ and the right hand of equation (3.1) = $\frac{15}{32}$, but the left hand = $\frac{3}{64}$, thus $L_1[0, 1]$ is not a quasi-inner product space. Also, it is not an inner product space, since $1 =$ the right hand of equation (2.1) $\neq$ the left hand = $\frac{5}{16}$.

Theorem (3.6). A quasi-inner product space U is an inner-product space if and only if the equality (2.1) is satisfied.

Proof:

If U is a quasi-inner product space such that the equality (2.1) is satisfied, then by Theorem (2.5), it is an inner product space.

If U is an inner-product space, using Remark (2.4) we get :

$${}_q\|v + w\|^2 = \langle v + w, v + w \rangle = {}_q\|v\|^2 + 2\langle v, w \rangle + {}_q\|w\|^2 \Rightarrow \left({}_q\|v + w\|^2 \right)^2 = \left({}_q\|v\|^2 + {}_q\|w\|^2 \right)^2 + 4\langle v, w \rangle \left({}_q\|v\|^2 + {}_q\|w\|^2 \right) + 4(\langle v, w \rangle)^2, \quad \forall v, w \in U.$$

$$\text{Also, } {}_q\|v - w\|^2 = {}_q\|v\|^2 - 2\langle v, w \rangle + {}_q\|w\|^2 \Rightarrow$$

$${}_q\|v - w\|^4 = \left({}_q\|v\|^2 + {}_q\|w\|^2 \right)^2 - 4\langle v, w \rangle \left({}_q\|v\|^2 + {}_q\|w\|^2 \right) + 4(\langle v, w \rangle)^2.$$

This implies that, ${}_q\|v + w\|^4 - {}_q\|v - w\|^4 = 8({}_q\|v\|^2 + {}_q\|w\|^2)\langle v, w \rangle$.

Thus, from Proposition (3.3), an equation (3.1) is satisfied, and the proof is finished.

Example (3.7). Since the right hand of an equation (3.1) = the left hand =

$8 \left(\int_a^b |f(x)|^3 (\text{sng } f(x))g(x) dx + \int_a^b |g(x)|^3 (\text{sng } g(x))f(x) dx \right)$ with $L_4[a, b]$, then it is a quasi-inner product spaces, but an equation (2.1) fails, indeed,

If we take $L_4 [-1,1]$ such that $f(x) = -x \quad \forall x \in [-1,0]$ and $g(x) = x \quad \forall x \in [0,1]$, then it is not an inner product space, since $1 =$ the right hand of equation (2.1) $\neq$ the left hand $\cong 1$. 265.

Proposition (3.8). A quasi-inner product space U is an inner-product space if and only if the following equivalence holds:

$$_q\|v + w\| = _q\|v - w\| \leftrightarrow \tau(v, w) = 0, \quad \forall v, w \in U \quad (3.2)$$

Proof:

The proof of this proposition proceeds in a same way into version in a normed space [5].

Remark (3.9). (1) Since a space $L_p [a, b]$ is a quasi-Banach space, then it is a quasi-Hilbert space if it is a quasi-inner product space for example a space $L_4 [a, b]$ is a quasi-Hilbert space, while $L_1 [a, b]$ is not.

(2) A space $L_2 [a, b]$ is one only which be a quasi-Hilbert space and Hilbert space together, as shown in the following:

Theorem (3.10). $L_2 [a, b], [a, b] \subset \mathbb{R}$, is a quasi-Hilbert space.

Proof:

Since a space $L_2 [a, b]$ is Hilbert space from Theorem (2.6), then $\forall f, g \in L_2 [a, b]$,

$\langle f, g \rangle = \tau(f, g) = \int_a^b |f(x)| (\text{sng } f(x)) g(x) dx$, and $\langle g, f \rangle = \tau(g, f) = \int_a^b |g(x)| (\text{sng } g(x)) f(x) dx$, according to Proposition (3.3) and application the binomial theorem in Definition (3.1). Thus, the right hand of equation (3.1) is $8(\int_a^b |f(x)|^3 (\text{sng } f(x)) g(x) dx \int_a^b |g(x)|^3 (\text{sng } g(x)) f(x) dx)$ which is the same value of its left hand. Therefore, $L_2 [a, b]$ is a quasi-Hilbert space.

References

- [1] J. K. Al-Delfi, Continuous Linear Operators On Infinite Quasi-Sobolev Spaces ℓ_∞^m . Journal of Physics: Conference Series (IOP Publishing), Sixth International Scientific Conference for Iraqi Al Khwarizmi Society (FISCAS), Cairo, Egypt. 1897 (2021) 012044. doi:10.1088/1742-6596/1897/1/012044. 1-6.
- [2] P. R. Halmose, Polynomials. Springer-Verlag, Inc. New York, (1989).
- [3] N. Kalton, Linear Operators on L_p for $0 < p < 1$. Transaction of The American Mathematical Society, Missouri. Vol. 259, No. 2 (1980). 319-355.
- [4] E. Kreyszig, Introductory Functional Analysis with Applications. John Wiley and Sons, Inc (1978).

- [5] P.M. Milicic , On The Quasi-Inner Product Spaces. Mat Bulletin. Skopje, Macedonia, 22 (XLVIII) (1998). 19-30.
- [6] G. Rano and T. Bag , Quasi-Metric Space and Fixed Point Theorems. International Journal of Mathematics and Scientific Computing (ISSN: 2231-5330), VOL. 3, NO. 2 (2013). 27-31.
- [7] R . A . Tapia, A characterization of inner product spaces. Proc. Amer. Math. Soc., 41(1973). 569-574.
- [8] A. H. Siddiqi, Functional Analysis with Applications. Tata McGraw-Hill Publishing Company, Ltd. New Delhi, India, (1986).