On Commutativity of Prime Rings with Symmetric Left θ -3- Centralizers

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Article History Received on:29/6/2021 Revised on:30/8/2021 Accepted on:28/10/2021 Keywords: <i>Prime rings , Left θ-3-centralizer , symmetric left θ-3-centralizer .</i>	Let R be an associative ring with center $Z(R)$, I be a nonzero ideal of R and θ be an automorphism of R . An 3-additive mapping $M:R \times R \times R \rightarrow R$ is called a symmetric left θ -3-centralizer if $M(u_1y, u_2, u_3) = M(u_1, u_2, u_3)\theta(y)$ holds for all $y, u_1, u_2, u_3 \in R$. In this paper , we shall investigate the commutativity of prime rings admitting symmetric left θ -3-centralizer satisfying any one of the following conditions : (i) $M([u, y], u_2, u_3) \pm [\theta(u), \theta(y)] = 0$, (ii) $M((u \circ y), u_2, u_3) \pm (\theta(u) \circ \theta(y)) = 0$ (iii) $M(u^2, u_2, u_3) \pm \theta(u^2) = 0$, (iv) $M(u y, u_2, u_3) \pm \theta(uy) = 0$ (v) $M(u y, u_2, u_3) \pm \theta(uy) \in Z(R)$ For all $u_2, u_3 \in R$ and $u, y \in I$
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1.Introduction

Let R be an associative ring with center $Z(R)$. In [3], Ashraf , and Ali, investigate the commutativity of prime rings satisfying certain identities involving left multiplier on a nonzero ideal . In [1] , Ali and Huang , get many sufficient conditions of commutativity on left α -multipliers when R is semiprime ring on a nonzero ideal . In [2] , Abduljaleel , and Majeed, discussed the commutativity of prime ring admitting a right α - centralizer where α is an endomorphism of R on a lie ideal . For more information see [5-9] . In [10] , Saed , study the commutativity of prime Γ -ring by using the notion of symmetric left Γ -n- centralizers .

This paper is organized as follows . In section two , we recall some well-known definitions and examples that will be used in this paper . In section three , we present the notion of symmetric left θ -3-centralizer of R , where θ is a mapping on R , and study the commutativity of prime rings

admitting a symmetric left θ -3-centralizer , where θ is an automorphism of R satisfying many conditions on a nonzero ideal .

2. Basic Concept

Definition 2.1:[4] A nonempty set R is said to be associative ring if in R there are defined two operations , denoted “ + ” and “ . ” respectively, such that for all a, b, c in R :

1. $a+b$ is in R
2. $a + b = b + a$
3. $(a + b) + c = a + (b + c)$
4. There is an element 0 in R such $a + 0 = a$ (for every a in R)
5. There exists an element $-a$ in R such that $a + (-a) = 0$
6. $a . b$ is in R
7. $a.(b.c) = (a.b).c$
8. $a . (b + c) = a.b + a.c$ and $(b + c).a = b.c + c.a$

Definition 2.2:[4] A subring I of a ring R is called (two-sided) ideal of R if for every $r \in R$ and every $a \in I$, $ra \in I$ and $ar \in I$.

Examples 2.3:[4] (1) In a ring R the trivial subrings : R and $\{0\}$ are both ideals .

(2) The set of even integers is an ideal in the ring of integers .

Definition 2.4:[4] Let R be a ring , the center of R denoted by $Z(R)$ and is defined by :
 $Z(R) = \{ x \in R : xr = rx, \text{ for all } r \in R \}$.

Definition 2.5:[3] A ring R is called a prime ring if for any $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$.

Example 2.6:[3] The ring of real numbers with the usual operation of addition and multiplication is prime ring .

Definition 2.7:[3] Let R be a ring . An additive mapping $H : R \rightarrow R$ is called a left (resp. right) centralizer of R if $H(xy) = H(x)y$ (resp. $H(xy) = xH(y)$), holds for all $x, y \in R$. A centralizer of a ring R is both left and right centralizer .

Definition 2.8:[1] Let R be a ring . An additive mapping $H : R \rightarrow R$ is called a left (resp. right) α -centralizer of R if $H(xy) = H(x) \alpha(y)$ (resp. $H(xy) = \alpha(x) H(y)$) holds for all $x, y \in R$, where α is an endomorphism of R . A α - centralizer of a ring R is both left and right α - centralizer .

Definition 2.9:[2] Let R be a ring . For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anticommutator $xy + yx$.

Remark 2.10 :[2]

Let R be a ring and $x, y, z \in R$, then

$$(i) [xy, z] = x[y, z] + [x, z]y$$

$$(ii) [x, yz] = y[x, z] + [x, y]z$$

$$(iii) (xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$

$$(iv) x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(v) (xy) \circ z = x(y \circ z)$$

Definition 2.11:[10] Let M be a Γ -ring and n be a fixed positive integer. An n -additive mapping $T: M \times M \times M \times \dots \times M \rightarrow M$ is said to be left Γ - n -centralizer if the following equations hold for all $y, r_1, r_2, \dots, r_n \in M$ and $\gamma \in \Gamma$.

$$T_1(r_1 \gamma y, r_2, \dots, r_n) = T_1(r_1, r_2, \dots, r_n) \gamma y$$

$$T_2(r_1, r_2 \gamma y, \dots, r_n) = T_2(r_1, r_2, \dots, r_n) \gamma y$$

$$\vdots$$

$$T_n(r_1, r_2, \dots, r_n \gamma y) = T_n(r_1, r_2, \dots, r_n) \gamma y$$

T is said to be a symmetric left Γ - n -centralizers if all the above equations are equivalent to all other. That is,

$$T(r_1 \gamma y, r_2, \dots, r_n) = T(r_1, r_2, \dots, r_n) \gamma y$$

For all $y, r_1, r_2, \dots, r_n \in M$ and $\gamma \in \Gamma$.

3. On Symmetric Left θ -3-Centralizers and Commutativity of Prime Rings

Now, we introduce the concept of symmetric left θ -3-centralizer

Definition 3.1 :

Let R be a ring and θ is a mapping on R . An 3-additive mapping $M: R \times R \times R \rightarrow R$ is said to be left θ -3-centralizer if the following equations hold for all $y, u_1, u_2, u_3 \in R$

$$M_1(u_1 y, u_2, u_3) = M_1(u_1, u_2, u_3) \theta(y)$$

$$M_2(u_1, u_2 y, u_3) = M_2(u_1, u_2, u_3) \theta(y)$$

$$M_3(u_1, u_2, u_3 y) = M_3(u_1, u_2, u_3) \theta(y)$$

M is said to be a symmetric left θ -3-centralizer if all the above equations are equivalent to all other. That is ,

$$M(u_1y, u_2, u_3) = M(u_1, u_2, u_3)\theta(y)$$

For all $y, u_1, u_2, u_3 \in R$.

Example 3.2: Consider $R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z} \right\}$ be a ring and $\mathbb{Z}$ is a ring of integer numbers .

Let $M : R \times R \times R \rightarrow R$ be a mapping defined by

$$M\left(\begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_3 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1a_2a_3 \\ 0 & 0 \end{pmatrix}, \text{ for all}$$

$$\begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_3 \\ 0 & 0 \end{pmatrix} \in R$$

And $\theta : R \rightarrow R$ is defined by

$$\theta\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \text{ for } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in R$$

Then M is a symmetric left θ -3-centralizer .

Theorem 3.3 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ -3-centralizer M such that $M(u, u_2, u_3) \neq \theta(u)$, for all $u \in I$. Further , if $M([u, y], u_2, u_3) - [\theta(u), \theta(y)] = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative .

Proof :

By the given hypothesis we have

$$M([u, y], u_2, u_3) - [\theta(u), \theta(y)] = 0, \text{ for all } u_2, u_3 \in R \text{ and } u, y \in I \quad (3.1)$$

This can be rewritten as

$$(M(u, u_2, u_3) - \theta(u))\theta(y) - (M(y, u_2, u_3) - \theta(y))\theta(u) = 0 \quad (3.2)$$

Replacing u by ur in (3.2), where $r \in R$, we find that

$$(M(u, u_2, u_3) - \theta(u))\theta(r)\theta(y) - (M(y, u_2, u_3) - \theta(y))\theta(u)\theta(r) = 0 \quad (3.3)$$

Multiply equation (3.2) from the right side by $\theta(r)$

$$(M(u, u_2, u_3) - \theta(u))\theta(y)\theta(r) - (M(y, u_2, u_3) - \theta(y))\theta(u)\theta(r) = 0 \quad (3.4)$$

Comparing equation (3.3) and (3.4), we get

$$(M(u, u_2, u_3) - \theta(u))(\theta(r)\theta(y) - \theta(y)\theta(r)) =$$

$$(M(u, u_2, u_3) - \theta(u))[\theta(r), \theta(y)] = 0 \quad (3.5)$$

Again , replace r by rs in (3.5) , where $s \in R$, and using it , to get

$$(M(u, u_2, u_3) - \theta(u)) \theta(r) [\theta(s), \theta(y)] = 0$$

$$\text{i.e., } (M(u, u_2, u_3) - \theta(u)) R[\theta(s), \theta(y)] = \{0\}$$

Since R is prime ring , then either $(M(u, u_2, u_3) - \theta(u)) = 0$ or $[\theta(s), \theta(y)] = 0$, for all $s, u_2, u_3 \in R$ and $u, y \in I$.

Thus application of our hypotheses implies that $[\theta(s), \theta(y)] = 0$, for all $y \in I$ and $s \in R$ and therefore $I \subseteq Z(R)$.

Hence , R is commutative .

Using similar arguments as used in proof of the Theorem 3.3 , we can prove the following :

Theorem 3.4 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ -3-centralizer M such that $M(u, u_2, u_3) \neq -\theta(u)$, for all $u \in I$. Further , if $M([u, y], u_2, u_3) + [\theta(u), \theta(y)] = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative .

Theorem 3.5 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ -3-centralizer M such that $M(u, u_2, u_3) \neq \theta(u)$, for all $u \in I$. Further , if $M((u \circ y), u_2, u_3) - (\theta(u) \circ \theta(y)) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative

Proof :

By the hypotheses , we have

$$M((u \circ y), u_2, u_3) - (\theta(u) \circ \theta(y)) = 0 , \text{ for all } u_2, u_3 \in R \text{ and } u, y \in I \quad (3.6)$$

This implies that

$$(M(u, u_2, u_3) - \theta(u)) \theta(y) + (M(y, u_2, u_3) - \theta(y)) \theta(u) = 0 \quad (3.7)$$

Replacing u by ur in (3.7) , where $r \in R$, we obtain

$$(M(ur, u_2, u_3) - \theta(ur)) \theta(y) + (M(y, u_2, u_3) - \theta(y)) \theta(ur) = 0 \quad (3.8)$$

Multiply equation (3.7) from the right side by $\theta(r)$

$$(M(u, u_2, u_3) - \theta(u)) \theta(y) \theta(r) + (M(y, u_2, u_3) - \theta(y)) \theta(u) \theta(r) = 0 \quad (3.9)$$

Comparing equation (3.8) and (3.9) , we get

$$(M(u, u_2, u_3) - \theta(u)) (\theta(r) \theta(y) - \theta(y) \theta(r)) =$$

$$(M(u, u_2, u_3) - \theta(u)) [\theta(r), \theta(y)] = 0 \quad (3.10)$$

Again , replace r by rs in (3.5) , where $s \in R$, and using it , to get

$$(M(u, u_2, u_3) - \theta(u)) \theta(r) [\theta(s), \theta(y)] = 0$$

i.e., $(M(u, u_2, u_3) - \theta(u)) R[\theta(s), \theta(y)] = \{0\}$

Since R is prime ring, then either $(M(u, u_2, u_3) - \theta(u)) = 0$ or $[\theta(s), \theta(y)] = 0$, for all $s, u_2, u_3 \in R$ and $u, y \in I$.

Thus, application of our hypotheses implies that $[\theta(s), \theta(y)] = 0$, for all $y \in I$ and $s \in R$ and therefore $I \subseteq Z(R)$.

Hence, R is commutative.

Using similar arguments as used in proof of the Theorem 3.5, we can prove the following :

Theorem 3.6 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and R admits a nonzero symmetric left θ -3-centralizer M such that $M(u, u_2, u_3) \neq -\theta(u)$, for all $u \in I$. Further, if $M((u \circ y), u_2, u_3) + (\theta(u) \circ \theta(y)) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative.

Theorem 3.7 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ -3-centralizer. If $M(u^2, u_2, u_3) = \theta(u^2)$, for all $u_2, u_3 \in R$ and $u \in I$, then R is commutative.

Proof :

We are given that

$$M(u^2, u_2, u_3) = \theta(u^2), \text{ for all } u_2, u_3 \in R \text{ and } u \in I \quad (3.11)$$

For all $u, y \in I$, we have

$$M((u+y)^2, u_2, u_3) = \theta((u+y)^2), \text{ which implies that}$$

$$M((u \circ y), u_2, u_3) - (\theta(u) \circ \theta(y)) = 0, \text{ for all } u_2, u_3 \in R \text{ and } u, y \in I$$

By using Theorem 3.5, we get R is commutative.

Similarly, we can prove the following :

Theorem 3.8 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ -3-centralizer. If $M(u^2, u_2, u_3) = -\theta(u^2)$, for all $u_2, u_3 \in R$ and $u \in I$, then R is commutative.

Theorem 3.9 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ -3-centralizer such that $M(u, u_2, u_3) \neq \pm \theta(u)$, for all $u \in I$. Further, if $M(u y, u_2, u_3) \pm \theta(u y) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative.

Proof :

For any $u_2, u_3 \in R$ and $u, y \in I$

$$\text{We have } M(u y, u_2, u_3) = \theta(u y)$$

Then we have $M(u y - y u, u_2, u_3) = \theta(u y - y u)$

This implies that $M([u, y], u_2, u_3) - [\theta(u), \theta(y)] = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$, and hence by Theorem 3.3, R is commutative.

On the other hand if R satisfy the condition $M(u y, u_2, u_3) + \theta(u y) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$

Then we have $M(u y + y u, u_2, u_3) = -\theta(u y + y u)$

This implies that $M((u \circ y), u_2, u_3) + (\theta(u) \circ \theta(y)) = 0$, for all $u_2, u_3 \in R$ and $u, y \in I$. Thus, by Theorem 3.6, R is commutative.

Theorem 3.10 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ -3-centralizer such that $M(u, u_2, u_3) \neq \theta(u)$, for all $u \in I$. Further, if $M(u y, u_2, u_3) - \theta(u y) \in Z(R)$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative.

Proof :

For all $u_2, u_3 \in R$ and $u, y \in I$, we have

$$M(u y, u_2, u_3) - \theta(u y) \in Z(R)$$

This can be rewritten as

$$(M(u, u_2, u_3) - \theta(u))\theta(y) \in Z(R) \quad (3.12)$$

Replacing u by ur in (3.12), where $r \in R$, we obtain

$$(M(ur, u_2, u_3) - \theta(ur))\theta(y) \in Z(R) \quad (3.13)$$

Multiply equation (3.12) from the right side by $\theta(r)$

$$(M(u, u_2, u_3) - \theta(u))\theta(y)\theta(r) \in Z(R) \quad (3.14)$$

$$[M(u, u_2, u_3) - \theta(u))\theta(y), \theta(r)] = 0,$$

$$\text{For all } u_2, u_3 \in R \text{ and } u, y \in I \quad (3.15)$$

This implies that,

$$(M(u, u_2, u_3) - \theta(u))[\theta(y), \theta(r)] + [M(u, u_2, u_3) - \theta(u), \theta(r)]\theta(y) = 0$$

$$\text{For all } u_2, u_3 \in R \text{ and } u, y \in I \quad (3.16)$$

Again, replace u by us in (3.16), where $s \in R$, and using it, to get

$$(M(us, u_2, u_3) - \theta(us))\theta(s)[\theta(y), \theta(r)] + [(M(us, u_2, u_3) - \theta(us))\theta(s), \theta(r)]\theta(y) = 0 \quad (3.17)$$

Comparing equation (3.16) and (3.17), we get

$$(M(u, u_2, u_3) - \theta(u))\theta(s)[\theta(y), \theta(r)] = 0$$

For all $r, s, u_2, u_3 \in R$ and $u, y \in I$

This yields that

$(M(u, u_2, u_3) - \theta(u)) R [\theta(y), \theta(r)] = \{0\}$. For all $r, u_2, u_3 \in R$ and $u, y \in I$

The primeness of R implies that either $[\theta(y), \theta(r)] = 0$ or

$M(u, u_2, u_3) - \theta(u) = 0$, for all $r, u_2, u_3 \in R$ and $u, y \in I$.

Since $M(u, u_2, u_3) \neq \theta(u)$, implies that $[\theta(y), \theta(r)] = 0$, for all $y \in I$ and $r \in R$ and therefore $I \subseteq Z(R)$.

Hence , R is commutative .

Using similar arguments as used in proof of the Theorem 3.10 , we can prove the following :

Theorem 3.11 : Let R be a prime ring and I be a nonzero ideal of R . Suppose that θ is an automorphism of R and M is a nonzero symmetric left θ -3- centralizer such that $M(u, u_2, u_3) \neq -\theta(u)$, for all $u \in I$. Further , if $M(u, y, u_2, u_3) + \theta(u, y) \in Z(R)$, for all $u_2, u_3 \in R$ and $u, y \in I$, then R is commutative .

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