Some results on Cayley graph

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<p>a. Nihad ABDEL - JALIL</p> <p>Article History</p> <p>Received on: 1/9/2021 Revised on: 1 /11/2021 Accepted on: 8/11/2021</p> <p>DOI: https://doi.org/10.29350/jops.2021.26.5.1446</p>	<p>Cayley graph has been introduced by A . Cayley , which is point – symmetric . However in this paper , I have found another type of symmetric graph , which is not Cayley graph . This graph is called peterson graph with 10 vertices is proposed</p>

1 - Introduction:

In recent years ,graph theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from operational research and chemistry to genetics and linguistics, and from electrical engineering and geography to sociology and architecture .At the same time it has also emerged as a worthwhile mathematical discipline in its own right.

A . Cayley gave a process to construct the graphes . For that we consider here the finite graph Let G be any finite group . $G = \{ X_1, X_2 \dots X_p \}$ G is generated by the element $S_1, S_2, \dots, S_k$ (i.e) each element of G can be written as a product of finite number of generators and their inverses the identity is not include in the set of generators we associate to G and S one graph , noted by $\text{Cay}(G, S)$

Then the set of vertices correspond to the set of G elements and defined in the following form let V_i be a vertex of $\text{cay}(G, S)$ corresponding to X_i for $i=1, 2, \dots, p$ the pair (V_1, V_2) is an arc of $\text{cay}(G, S)$ if

and only if there exists $s_i \in S$ such that $X_2 = X_1 s_i$ or this gives a graph of Cayley of G related to S and take different values obtain different graphs of Cayley for the same group

The symbol S is included in the notation $\text{Cay}(G, S)$ to indicate that the ordinary graph of Cayley depends on the set S and this set S is a generator of G .

Generally let G be a finite group and A is any part of $G - \{1\}$, A is not necessary a set of generators we replace S by A when A is the set of generators the graph of Cayley is defined by A is then given by $\text{Cay}(G, A) = (G, E)$ where $E = \{(X, Y) / Y \in A\}$ let T_a be a left translation defined by $T_a = x \rightarrow ax$

Where $a \in G$ Then we have

$$(X, Y) \in E \iff Y \in A$$

$$\iff (ay) \in A$$

$$\iff (T_{ax}, T_{ay}) \in E$$

C. Delorme gave a process to construct of graphs which is point symmetric graph, let G be any finite group H subgroup of G and A is subset of G the graph $[G, H, A]$ which G acts transitively by automorphisms in the following method. A graph automorphism is a type of symmetry in which the graph is mapped onto itself

The vertices of graph are the parts of G in the form xH number is the index of H in G neighbor of xH are xaH , where $a \in A$ we say that y in neighbor of x if $\{x, y\}$ is an arc when H is reduced to a natural element of the group we find Cayley graph associate to group G and a part A, G give the automorphism of $[G, H, A]$ if $g \in G$, then

$$xH \rightarrow gxH \text{ in an automorphism}$$

2 - Definitions :

A graph G is the pair (V, E) where $V(G)$ is the set of vertices and $E(G)$ is the set of arcs such that the pair (x, y) with $G(x, y) = +$ is an arc and the $G(y, x) = +$, the pair $\{x, y\}$ is an arc, the graph is finite if its number of vertices is finite, the graph in which there does not have the ~~arêtes~~ is called symmetric.

The graph is point – symmetric when - for all $x, y \in E$ (the base of G)

There exists an automorphism of G such that $f(x)=y$.

Generally, let G be a group and A is any part from $G - \{1\}$. A is not a necessary set of generators by relocated S in state of A when A is the set of generators Cayley graph, defined by $A : \text{cay}(G, A) = (G, E)$ where

$$E = \{(X, Y) / Y \in A\}$$

Let T_a the left translation defined by $T_a = x \rightarrow ax$

Where $a \in G$ Then we have

$$(X, Y) \in E \Leftrightarrow Y \in A$$

$$\Leftrightarrow (ay) \in A$$

$$\Leftrightarrow (T_{ax}, T_{ay}) \in A$$

The oriented path of G T_{ax} is a sequence of a T_{ay} vertices $x_0, x_1, \dots, x_n$ such that $G(x_i, x_{i+1}) = +$ for all i

The graph is called strong when for all x, y : there exists a path from x to y . We have the following lemma

Lemma 1

The left translations of G are the automorphisms of $\text{cay}(G, S)$

They $\text{cay}(G, S)$ is point symmetric let $X \in G$ the factorization of x with respect to S is a sequence

$$\{s_i ; 1 \leq i \leq K\} \text{ of elements of } S \text{ such that } X = \prod_{1 \leq i \leq K} s_i$$

Where K is length of the factorization, we have the following lemma

Lemma 2

Let G is finite group . $S \subset G - \{1\}$ and $x \in G$

Then the factorization $x = \prod_{1 \leq i \leq K} s_i$

Is defined the path of length K from 1 to x

And inversely each path from 1 to x defines a factorization of x with respect to S

Proof

Put $x_0 = 1$, $x_j = \prod_{1 \leq i \leq j} s_i$ let $x = \prod_{1 \leq i \leq K} s_i$. because $x_{j+1} = x_j s_{j+1}$, (x_j, x_{j+1}) is an arc then (x_0, x_1)

$(x_1, x_2) \dots (x_{K-1}, x_K)$ is a path reach connects 1 to x

Inversely , let $(1, x_1) (x_1, x_2) \dots (x_{K-1}, x)$ is a path reach connects 1 to x we have by definition an arc $x_j s_{j+1} = x_{j+1}$ when $s_{j+1} \in S$

Which given $x = x_K = (x_{K-1}) s_K$ and by induction we get $x = \prod_{1 \leq i \leq K} s_i$

Lemma 3

Let G be point symmetric graph and $a \in v$, if for all $x \in G$ there exists a path connects a to x then the graph G is strong

In particular, $\text{Cay}(G, S)$ is strong if and only if for all $x \in G$, their exist a path form 1 to x

Theorem

$\text{Cay}(G, A)$ is strong if only if A is the set of generators of G

Proof

Suppose $\text{cay}(G, A)$ is strong , and let $x \in G$ by the above lemma , there exists the path connects 1 to x then there exist a factorization of x with respect to A (by lemma 2) $x = s_1 s_2 \dots s_k$ where $s_i \in A$ for $1 \leq i \leq k$ then A is the set of generators

Inversely

Suppose that A is the set of generators (we replace A by S) and let $x \in G$.

, Such that $X = \prod_{1 \leq i \leq K} t_i$ finite if it put $t_i =$ There exist a sequence $\{t_i\}_{1 \leq i \leq K}$, t_i

Because G is finite, there exist a natural P such that $\sum_{i=1}^P t_i = 1$ we have $t_i =$

By changing all $t_i \in$ by the expression of the form where $s \in S$, we obtain a factorization of x with respect to S , this prove that $\text{cay}(G, S)$ is strong (by lemma3)

Example G is the group of permutations for the set $\{a, b, c\}$ A forms the permutation (ac) : H is $\{id\}$ for the first case, H is $\{id, (ab)\}$ for the second

$[S_{\{a,b,c\}}, \{id\}, \{(ac)\}]$

Remark

We note that iff we connect xH to $a xH$ we get the? shreier graph which is the graph not in general point symmetric for example $[G, H, A]$ are take as follow

$$H = \{id\}$$

Second case

$$\begin{aligned} & \{id, (a,b)\} \\ & \{(abc), (ac)\} \\ & \{(acb), (bc)\} \end{aligned} \left. \vphantom{\begin{aligned} & \{id, (a,b)\} \\ & \{(abc), (ac)\} \\ & \{(acb), (bc)\} \end{aligned}} \right\}$$

$$[S_{\{a,b,c\}}, \{id, (ab)\}, \{(ac)\}] .$$

$$H = \{id, (ab)\}$$

Remark

There exist graphs which are point-symmetric other than Cayley graph for example, Petersen graph with 10 elements.

Conclusion

We can conclude from this study that there are many graphs which are point-symmetric other than Cayley graphs. These new graphs from this study are Petersen

Graph and Shreier graph.

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