Global existence and growth of solutions to coupled degenerately damped Klein-Gordon equations

1. Introduction

In this work, we investigate the global existence and exponential growth of solutions for the following system of nonlinear Klein-Gordon equations with viscoelastic and degenerate damping terms:

\[
\begin{align*}
    u_{tt} - \Delta u + m_1^2 u + \int_0^t \mu_1(t-s) \Delta u(s) \, ds + (|u|^k + |v|^l)|u_t|^{\eta-1} u_t &= f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\
    v_{tt} - \Delta v + m_2^2 v + \int_0^t \mu_2(t-s) \Delta v(s) \, ds + (|v|^\theta + |u|^\eta)|v_t|^{\nu-1} v_t &= f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\
    u(x, t) = v(x, t) = 0, \quad (x,t) \in \partial \Omega \times (0,T), \\
    u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\
    v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega,
\end{align*}
\]

(1)

where \( \Omega \) is a bounded open domain with smooth boundary in \( \mathbb{R}^n \) (\( n \geq 1 \)), \( m_1, m_2 > 0 \), \( \eta, \nu \geq 0 \), \( k, l, \theta, \rho > 1 \); \( \mu_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) (\( i = 1,2 \)) are positive relaxation functions.

By taking

\[
\begin{align*}
    f_1(u, v) &= a|u + v|^{2(k+1)}(u + v) + b|u|^\kappa |v|^{\kappa+2}, \\
    f_2(u, v) &= a|u + v|^{2(k+1)}(u + v) + b|v|^\kappa |u|^{\kappa+2},
\end{align*}
\]

ABSTRACT

In this work, we study a system of nonlinear Klein-Gordon equations with viscoelastic and degenerate damping in a bounded domain. By an appropriate auxiliary function (which is a small perturbation of the total energy), we prove the global existence and exponential growth of solutions for equation (1) with strong nonlinear function \( f_1 \) and \( f_2 \) satisfying appropriate conditions and the initial energy satisfying \( E(0) < 0 \).

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where $a > 0, b > 0$ and
\[
\begin{cases}
-1 < \kappa & \text{if } n = 1, 2, \\
-1 < \kappa \leq \frac{3-n}{n-2} & \text{if } n \geq 3.
\end{cases}
\]

Multiplying $f_1(u, v)$ by $u$, $f_2(u, v)$ by $v$, we get
\[
u f_1(u, v) + v f_2(u, v) = u(a|u + v|^{2(\kappa+1)}(u + v) + b|u|^\kappa u|v|^{\kappa+2})
+ v(a|u + v|^{2(\kappa+1)}(u + v) + b|v|^\kappa v|u|^{\kappa+2})
= a|u + v|^{2(\kappa+1)}(u + v)^2 + 2b|u|^{\kappa+2}|v|^{\kappa+2}
= a|u + v|^{2(\kappa+2)} + 2buv|^{\kappa+2}
= 2(\kappa + 2)F(u, v),
\]
where $\forall(u, v) \in \mathbb{R}^2$ for
\[
F(u, v) = \frac{1}{2(\kappa+2)}[a|u + v|^{2(\kappa+2)} + 2buv|^{\kappa+2}].
\]

The system (1) is a generalization of the following Klein-Gordon system:
\[
\begin{align*}
\left\{ u_{tt} - \Delta u + m_1^2 u + auv^2 = 0, \\
v_{tt} - \Delta v + m_2^2 v + auv^2 = 0,
\end{align*}
\]
where $m_1, m_2, a$ are non-negative constants, which is considered in the study of the quantum field theory. The above system defines the motion of a charged meson in an electromagnetic field and was proposed by Segal [21].

The generalized system (1) was earlier investigated by Yazid et al. [24]. The authors considered the global nonexistence of solution with positive initial energy.

Pişkin [8] investigated on coupled equations of the form
\[
\begin{align*}
\left\{ u_{tt} - \Delta u + m_1^2 u + |u|^7 u_t = f_1(u, v), \\
v_{tt} - \Delta v + m_2^2 v + |v|^7 v_t = f_2(u, v),
\end{align*}
\]
and the author considered the decay of solution by using Nakao’s inequality and the blow up of solution with negative initial energy. Then, Pişkin [9] studied same problem and proved lower bounds for the time of blow up is derived if the solutions blow up. In [10], the author studied blow up of solutions with negative initial energy in case $\eta = v = 1$. Furthermore, Ye [25] considered the problem (5) with $\eta = v$ and studied the asymptotic stability and global existence of solutions: In addition, some other authors investigated problem (5) with $\eta = v = 1$ see ([6,7,22]).

The effect of the degenerate damping terms often appear in many applications and practical problems and turns a lot of systems into different problems worth studying. Now, we state some present results in the literature: Firstly, we mention the pioneer work of Rammaha and Sakuntasathien [19] who focus on coupled equations of the form
\[
\begin{align*}
\left\{ u_{tt} - \Delta u + (|u|^k + |v|^l)|u_t|^{\eta-1} u_t = f_1(u, v), \\
v_{tt} - \Delta v + (|v|^\theta + |u|^\theta)|v_t|^{\eta-1} v_t = f_2(u, v),
\end{align*}
\]
They investigated the global well posedness of the solution under some restriction on the parameters. In [2,26], authors studied the same problem treated in [19], and they studied the growth and blow up properties. For more depth, here are some papers that focused on the study of degenerate damping [3-5, 11, 13-17, 23, 27].

It is well known fact that “Exponential Growth” phenomenon is one of the most important phenomena of asymptotic behavior but many authors omit it. It presentations us very considerable information to know the behavior of equation when time arrives at infinity, it differs from global existence and blow up in both mathematically and in applications point of view.
In this work, we investigate how we can apply the degenerate damping term for knowing the behavior of growth of solutions for a coupled nonlinear Klein-Gordon system with viscoelastic and source terms. The rest paper is organized as follows: In the next section, we present necessary assumptions that will be used later. Then, in Section 3, we establish the global existence of problem. The exponential growth of solution is presented in Section 4.

2. Preliminaries

\( W^{m,p} \) is denote the Sobolev space and

\[
\begin{align*}
W^{0,p}(\Omega) &= L^p(\Omega) \text{ if } m = 0, \\
W^{m,2}(\Omega) &= H^m(\Omega) \text{ if } p = 2.
\end{align*}
\]

Also, let denote the standard \( L^2(\Omega) \) norm by \( \| \cdot \| = \| \cdot \|_{L^2(\Omega)} \) and \( L^p(\Omega) \) norm by \( \| \cdot \|_p = \| \cdot \|_{L^p(\Omega)} \) for details see ([1,18]).

Now, we make the following assumptions:

(H1) Regarding \( \mu_i(\cdot): R^+ \to R^+ \), \( i = 1,2 \) are \( C^1 \)-nonincreasing functions satisfying

\[
\begin{align*}
\mu(\alpha) &> 0, \quad \alpha \leq 0, \quad 1 - \int_0^\infty \mu(\alpha)d\alpha = l_i > 0, \quad \alpha \geq 0. \\
1 &\leq \eta, \quad \text{if } n = 1,2, \\
1 &\leq \eta, \quad \text{if } n \geq 3.
\end{align*}
\]

(H2) Also, we use the following notation:

\[
(\mu \circ \nabla w)(t) = \int_0^t \mu_i(t-s)\| \nabla w(t) - \nabla w(s) \|^2 ds.
\]

Now, we define the energy function

\[
E(t) = \frac{1}{2}(\| u_t \|^2 + \| v_t \|^2) + \frac{1}{2}[ (\mu_1 \circ \nabla u)(t) + (\mu_2 \circ \nabla v)(t) + m_1^2 \| u \|^2 + m_2^2 \| v \|^2 ]
+ \frac{1}{2} \left[ (1 - \int_0^t \mu_1(s)ds) \| \nabla u(t) \|^2 + (1 - \int_0^t \mu_2(s)ds) \| \nabla v(t) \|^2 \right] - \int_{\Omega} F(u,v) dx. \tag{7}
\]

By computation, we have

\[
\frac{d}{dt} E(t) \leq \frac{1}{2} \left[ (\mu_1 \circ \nabla v)(t) + (\mu_2 \circ \nabla v)(t) \right]
- \frac{1}{2}(\mu_1(t)\| \Delta u \|^2 + \mu_2(t)\| \Delta v \|^2)
- \int_{\Omega} (|u|^k + |v|^k)|u_t|^\eta+1 dx - \int_{\Omega} (|v|^\theta + |u|^\theta)|v_t|^\nu+1 dx
\leq 0. \tag{8}
\]

3. Global existence

In this part, we prove the global existence of solution for problem (1). For this aim we set

\[
I(t) = m_1^2 \| u \|^2 + m_2^2 \| v \|^2 + (\mu_1 \circ \nabla u)(t) + (\mu_2 \circ \nabla v)(t) - 2(\kappa + 2) \int_{\Omega} F(u,v) dx
+ \left( 1 - \int_0^t \mu_1(s)ds \right) \| \nabla u(t) \|^2 + \left( 1 - \int_0^t \mu_2(s)ds \right) \| \nabla v(t) \|^2
\]

and

\[
J(t) = \frac{1}{2} \left[ (\mu_1 \circ \nabla u)(t) + (\mu_2 \circ \nabla v)(t) + m_1^2 \| u \|^2 + m_2^2 \| v \|^2 \right] - \int_{\Omega} F(u,v) dx
+ \frac{1}{2} \left[ (1 - \int_0^t \mu_1(s)ds) \| \nabla u(t) \|^2 + (1 - \int_0^t \mu_2(s)ds) \| \nabla v(t) \|^2 \right].
\]

Lemma 3.1. Assume that (4) holds. Then there exist \( \rho > 0 \) such that for any \( u, v \in H_0^1(\Omega) \), we get

\[
\| u + v \|_{2(\kappa + 2)}^2 + 2\| uv \|_{\kappa + 2}^2 \leq \rho (l_1 \| \nabla u \|^2 + l_2 \| \nabla v \|^2)^{\kappa + 2}
\]
is satisfied. [20]

**Lemma 3.2.** Assume that (H1) and (H2) hold. Let \( u_0, v_0 \in H^1_0(\Omega), u_1, v_1 \in H^1_0(\Omega) \). If

\[
I(0) > 0 \text{ and } \chi = \rho \left( \frac{2(\kappa+2)}{\kappa+1} E(0) \right)^{\kappa+1} < 1,
\]

then

\[
I(t) > 0, \quad \forall t > 0.
\]

**Proof.** We have \( I(0) > 0 \), then by continuity of \( I(t) \) about \( t \), there exist a maximal time \( t_m > 0 \) such that

\[
I(t) \geq 0 \text{ on } t \in (0, t_m).
\]

Let \( t_0 \) be as follows

\[
\{ I(t_0) = 0 \text{ and } I(t) > 0 \text{ for all } 0 \leq t < t_0 \}. \tag{13}
\]

By using

\[
J(t) = \frac{1}{2(\kappa+2)} I(t) + \frac{\kappa+1}{2(\kappa+2)} \left[ (\mu_1 \circ \nabla u)(t) + (\mu_2 \circ \nabla v)(t) + m^2_1 \|u\|^2 + m^2_2 \|v\|^2 \right]
\]

\[
+ \frac{\kappa+1}{2(\kappa+2)} \left[ (1 - \int_0^t \mu_1(s) ds) \|\nabla u(t)\|^2 + (1 - \int_0^t \mu_2(s) ds) \|\nabla v(t)\|^2 \right]
\]

\[
\geq \frac{\kappa+1}{2(\kappa+2)} [l_1 \|\nabla u(t)\|^2 + l_2 \|\nabla v(t)\|^2 + (\mu_1 \circ \nabla u)(t) + (\mu_2 \circ \nabla v)(t) + m^2_1 \|u\|^2 + m^2_2 \|v\|^2]. \tag{14}
\]

From (7) and (8), we have

\[
l_1 \|\nabla u(t)\|^2 + l_2 \|\nabla v(t)\|^2 \leq \frac{2(\kappa+2)}{\kappa+1} J(t)
\]

\[
\leq \frac{2(\kappa+2)}{\kappa+1} E(t)
\]

\[
\leq \frac{2(\kappa+2)}{\kappa+1} E(0), \quad \forall t \in [0, t_0]. \tag{15}
\]

By (11) and (12), we reach at

\[
2(\kappa+2) \int_\Omega F(u(t_0), v(t_0)) dx \leq \rho (l_1 \|\nabla u(t_0)\|^2 + l_2 \|\nabla v(t_0)\|^2)^{\kappa+2}
\]

\[
\leq \rho \left( \frac{2(\kappa+2)}{\kappa+1} E(0) \right)^{\kappa+1} (l_1 \|\nabla u(t_0)\|^2 + l_2 \|\nabla v(t_0)\|^2)
\]

\[
\leq l_1 \|\nabla u(t_0)\|^2 + l_2 \|\nabla v(t_0)\|^2
\]

\[
\leq \left( 1 - \int_0^t \mu_1(s) ds \right) \|\nabla u(t)\|^2 + \left( 1 - \int_0^t \mu_2(s) ds \right) \|\nabla v(t)\|^2.
\]

Then, since (9) we have

\[
I(t_0) > 0
\]

which contradicts to (13). So, \( I(t) > 0 \) on \( [0, T] \).

**Theorem 3.3.** Assume that the conditions of Lemma 3.2 hold, then the solutions (1) is bounded and global in infinite time.

**Proof.** It suffices to show that

\[
\|(u(t), v(t))\|_H := \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + m^2_1 \|u\|^2 + m^2_2 \|v\|^2
\]

is bounded independently of \( t \) (time). For this purpose, we apply (7), (8), (10) and (15) to get

\[
E(0) \geq E(t) = J(t) + \frac{1}{2} (\|u_t\|^2 + \|v_t\|^2)
\]

\[
\geq \frac{\kappa+1}{2(\kappa+2)} [l_1 \|\nabla u(t)\|^2 + l_2 \|\nabla v(t)\|^2 + (\mu_1 \circ \nabla u)(t) + (\mu_2 \circ \nabla v)(t)
\]

\[
+ \frac{\kappa+1}{2(\kappa+2)} [m^2_1 \|u\|^2 + m^2_2 \|v\|^2] + \frac{1}{2} (\|u_t\|^2 + \|v_t\|^2). \tag{16}
\]

Thus,
\[ \| (u, v) \|_H \leq CE(0), \]

where positive constant \( C \), which depends only on \( \kappa, l_1, l_2 \).

### 4. Growth

In this part, our purpose to show that the energy grow up as an exponential function as time goes to infinity.

**Theorem 4.1.** Assume that

\[ 2(\kappa + 2) > \max \{ k + \eta + 1, l + \eta + 1, \theta + v + 1, q + v + 1 \}, \]

and the initial energy \( E(0) < 0 \). Then, the solution of the system (1) grows exponentially.

**Proof.** We set

\[ H(t) = -E(t), \]

from assumption \( E(0) < 0 \) and (8) gives \( H(t) \geq H(0) > 0 \).

Then, define \( \Phi(t) \) by

\[ \Phi(t) = H(t) + \varepsilon \left( \int_{\Omega} u_tudx + \int_{\Omega} v_tvdx \right) \quad (17) \]

where \( 0 < \varepsilon \leq 1 \).

By differentiating (17) and using Eq.(1), we have

\[ \Phi'(t) = H'(t) + \varepsilon (\| u_t \|^2 + \| v_t \|^2) - \varepsilon (\| \nabla u \|^2 + \| \nabla v \|^2) + 2\varepsilon (\kappa + 2) \int_{\Omega} F(u, v)dx - \varepsilon (m_1^2 \| u \|^2 + m_2^2 \| v \|^2) + \varepsilon \int_{0}^{t} \mu_1 (t-s) \nabla u(s) \nabla u(t)dsdx + \varepsilon \int_{0}^{t} \mu_2 (t-s) \nabla v(s) \nabla v(t)dsdx - \varepsilon \left( \int_{\Omega} (\| u \|^k + \| v \|^l)u_t |u_t|^{\eta - 1}dx - \int_{\Omega} (\| u \|^\theta + \| u \|^\theta)v_t |v_t|^{\nu - 1}dx \right). \quad (18) \]

We would like to estimate the last two terms right hand side in (18) by using the following Young’s inequality

\[ AB \leq \frac{\delta^{\alpha} A^\alpha}{\alpha} + \frac{\delta^{-\beta} B^\beta}{\beta}, \]

where \( A, B \geq 0, \delta > 0, \alpha, \beta \in \mathbb{R}^+ \) such that \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \). Therefore, we have for all \( \delta_1 > 0 \)

\[ |u_tu_t|^{\eta - 1} \leq \delta_1^{\eta - 1} |u|^{\eta - 1} + \frac{\eta \delta_1}{\eta - 1} |u_t|^{\eta - 1}, \]

and therefore

\[ \int_{\Omega} (|u|^k + |v|^l)|u_tu_t|^{\eta - 1}dx \leq \delta_1^{\eta - 1} \int_{\Omega} (|u|^k + |v|^l)|u|^{\eta - 1}dx + \frac{\eta \delta_1}{\eta - 1} \int_{\Omega} (|u|^k + |v|^l)|u_t|^{\eta - 1}dx. \]

Similarly, for all \( \delta_2 > 0 \)

\[ |v_vv_t|^{\nu - 1} \leq \delta_2^{\nu - 1} |v|^{\nu - 1} + \frac{\nu \delta_2}{\nu - 1} |v_t|^{\nu - 1}, \]

which gives

\[ \int_{\Omega} (|v|^\theta + |u|^\theta)|v_vv_t|^{\nu - 1}dx \leq \delta_2^{\nu - 1} \int_{\Omega} (|v|^\theta + |u|^\theta)|v|^{\nu - 1}dx + \frac{\nu \delta_2}{\nu - 1} \int_{\Omega} (|v|^\theta + |u|^\theta)|v_t|^{\nu - 1}dx. \]

Inserting the estimates (19), (20) into (18), we have

\[ \Phi'(t) \geq H'(t) + \varepsilon (\| u_t \|^2 + \| v_t \|^2) - \varepsilon (\| \nabla u \|^2 + \| \nabla v \|^2) \]

which gives

\[ \Phi'(t) \geq H'(t) + \varepsilon (\| u_t \|^2 + \| v_t \|^2) - \varepsilon (\| \nabla u \|^2 + \| \nabla v \|^2) \]

where positive constant \( C \), which depends only on \( \kappa, l_1, l_2 \).
\[ +2\varepsilon(\kappa + 2) \int_\Omega F(u, v) dx - \varepsilon(m_1^2\|u\|^2 + m_2^2\|v\|^2) \]
\[ +\varepsilon \int_\Omega \int_0^t \mu_1(t-s)\nabla u(s)\nabla v(t) ds \ dx + \varepsilon \int_\Omega \int_0^t \mu_2(t-s)\nabla v(s)\nabla v(t) ds \ dx \]
\[ -\varepsilon \frac{\delta_{\eta+1}^\eta}{\eta+1} \int_\Omega (|u|^k + |v|^l)|u|^{\eta+1} \ dx - \varepsilon \frac{\eta\delta_{\eta+1}^\eta}{\eta+1} \int_\Omega (|u|^k + |v|^l)|u_t|^{\eta+1} \ dx \]
\[ -\varepsilon \frac{\delta_{\nu+1}^{\nu+1}}{\nu+1} \int_\Omega (|v|^\theta + |u|^\theta)|v|^{\nu+1} \ dx - \varepsilon \frac{\nu\delta_{\nu+1}^{\nu+1}}{\nu+1} \int_\Omega (|v|^\theta + |u|^\theta)|v_t|^{\nu+1} \ dx. \tag{21} \]

Now, the ninth term in the right hand side of (21) can be estimated, as follows (see [12]):
\[ \int_\Omega \nabla v(t) \int_0^t \mu_1(t-s)\nabla u(s) ds \ dx \]
\[ \leq \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2} \int_\Omega \left( \int_0^t \mu_1(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|) ds \right)^2 \ dx. \]

Thanks to Young’s inequality and assumption (H1), we have, for any \( \xi_1 > 0, \)
\[ \int_\Omega \nabla u(t) \int_0^t \mu_1(t-s)\nabla u(s) ds \ dx \leq \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2} \left( 1 + \xi_1 \right) \int_\Omega \left( \int_0^t \mu_1(t-s)\nabla u(s) ds \right)^2 \ dx \]
\[ + \frac{1}{2} \left( 1 + \xi_1 \right) \int_\Omega \left( \int_0^t \mu_1(t-s)|\nabla u(s) - \nabla u(t)| ds \right)^2 \ dx \]
\[ \leq \frac{1 + (1 + \xi_1)(1-l_1)}{2} \|\nabla u\|^2 \]
\[ + \frac{(1 + \xi_1)(1-l_1)}{2} (\mu_1 \circ \nabla u)(t). \]

Similar calculations also yield, for any \( \xi_2 > 0, \)
\[ \int_\Omega \nabla v(t) \int_0^t \mu_2(t-s)\nabla v(s) ds \ dx \leq \frac{1 + (1 + \xi_2)(1-l_2)}{2} \|\nabla v\|^2 \]
\[ + \frac{(1 + \xi_2)(1-l_2)}{2} (\mu_2 \circ \nabla v)(t). \]

Then, add \( 2H(t) \) to both side of (21), we have
\[ \Phi'(t) \geq H'(t) + 2\varepsilon(\|u_t\|^2 + \|v_t\|^2) \]
\[ +\varepsilon \left( (1 - l_1) + \frac{(1 + \xi_1)(1-l_1)^2 - 1}{2} \right) \|\nabla u\|^2 \]
\[ +\varepsilon \left( (1 - l_2) + \frac{(1 + \xi_2)(1-l_2)^2 - 1}{2} \right) \|\nabla v\|^2 \]
\[ +2\varepsilon(\kappa + 1) \int_\Omega F(u, v) dx + 2\varepsilon H(t) \]
\[ +\varepsilon \left( 1 + \frac{(1 + \xi_1)(1-l_1)}{2} \right) (\mu_1 \circ \nabla u)(t) \]
\[ +\varepsilon \left( 1 + \frac{(1 + \xi_2)(1-l_2)}{2} \right) (\mu_2 \circ \nabla v)(t) \]
\[ -\varepsilon \frac{\delta_{\eta+1}^\eta}{\eta+1} \int_\Omega (|u|^k + |v|^l)|u|^{\eta+1} \ dx - \varepsilon \frac{\eta\delta_{\eta+1}^\eta}{\eta+1} \int_\Omega (|u|^k + |v|^l)|u_t|^{\eta+1} \ dx \]
\[ -\varepsilon \frac{\delta_{\nu+1}^{\nu+1}}{\nu+1} \int_\Omega (|v|^\theta + |u|^\theta)|v|^{\nu+1} \ dx - \varepsilon \frac{\nu\delta_{\nu+1}^{\nu+1}}{\nu+1} \int_\Omega (|v|^\theta + |u|^\theta)|v_t|^{\nu+1} \ dx. \tag{22} \]

Then, by using Young’s inequality, we get
\[ \int_\Omega (|u|^k + |v|^l)|u|^{\eta+1} \ dx \leq \int_\Omega |u|^{k+\eta+1} dx + \int_\Omega |v|^l|u|^{\eta+1} dx \]
\[ \leq \int_\Omega |u|^{k+\eta+1} dx + \frac{l}{l+\eta+1} \int_\Omega |v|^{l+\eta+1} dx \]
+ \frac{\eta + l + 1}{\eta + 1} \int_{\Omega} |u|^{l + \eta + 1} dx
\]

= \|u\|^k_{\eta + 1} + \frac{l}{l + \eta + 1} \gamma_1 \int_{\Omega} |v|^{l + \eta + 1}

+ \frac{\eta + 1}{l + \eta + 1} \gamma_2 \int_{\Omega} \|v\|^{l + \eta + 1}

and

\int_{\Omega} (|v|^\theta + |u|^\theta) |v|^{l + 1} dx \leq \int_{\Omega} |v|^\theta |v|^{l + 1} dx + \int_{\Omega} |u|^\theta |v|^{l + 1} dx

\leq \int_{\Omega} |v|^\theta |v|^{l + 1} dx + \frac{\theta}{\theta + v + 1} \int_{\Omega} |u|^\theta |v|^{l + 1} dx

+ \frac{l}{l + v + 1} \gamma_1 \int_{\Omega} |v|^{l + 1}

= \|v\|^{\theta + v + 1} + \frac{\theta}{\theta + v + 1} \int_{\Omega} |u|^\theta |v|^{l + 1}

+ \frac{l}{l + v + 1} \gamma_1 \int_{\Omega} |v|^{l + 1}.

Then, (22) deduce to

\Phi'(t) \geq H'(t) + 2\varepsilon (\|u_t\|^2 + \|v_t\|^2) + 2\varepsilon H(t)

+ \varepsilon \left( 1 - l_1 \right) \left( 1 + (1 + \xi_1)(1 - l_1)^2 - 1 \right) \|\nabla u\|^2

+ \varepsilon \left( 1 - l_2 \right) \left( 1 + (1 + \xi_2)(1 - l_2)^2 - 1 \right) \|\nabla v\|^2

+ 2\varepsilon (\kappa + 1) \left[ \|u\|^{2(k+2)}_{2(k+2)} + \|v\|^{2(k+2)}_{2(k+2)} \right]

+ \varepsilon \left( 1 + \frac{1}{1 + \xi_1}(1 - l_1) \right) (\mu_1 \circ \nabla u)(t)

+ \varepsilon \left( 1 + \frac{1}{1 + \xi_2}(1 - l_2) \right) (\mu_2 \circ \nabla v)(t)

- \varepsilon \frac{\delta_1}{\eta + 1} \left( \|u\|^{k + \eta + 1} + \frac{l}{l + \eta + 1} \gamma_1 \int_{\Omega} |v|^{l + \eta + 1} + \frac{\eta + 1}{l + \eta + 1} \gamma_1 \int_{\Omega} |u|^{l + \eta + 1} \right)

- \varepsilon \frac{\delta_2}{\eta + 1} \left( \|v\|^{\theta + v + 1} + \frac{\theta}{\theta + v + 1} \gamma_2 \int_{\Omega} |u|^\theta |v|^{l + 1} + \frac{v + 1}{\theta + v + 1} \gamma_2 \int_{\Omega} \|v\|^\theta |u|^{l + 1} \right)

- \varepsilon \frac{\delta_1}{\eta + 1} \int_{\Omega} (|u|^\kappa + |v|^\kappa) |u_t|^{\eta + 1} dx - \varepsilon \frac{\delta_2}{\eta + 1} \int_{\Omega} (|v|^\kappa + |u|^\kappa) |v_t|^{\eta + 1} dx. \quad (23)

By using \(2(k + 2) > \max\{k + \eta + 1, l + \eta + 1, \theta + v + 1, \rho + v + 1\}\) assumption and the following algebraic inequality

\[ x^\sigma \leq x + 1 \leq \left( 1 + \frac{1}{\sigma} \right) (x + a), \quad \forall x \geq 0, 0 < \sigma \leq 1, \ a \geq 0, \]

we obtain for all \(t \geq 0\)

\[ \|v\|^{\theta + v + 1}_{\theta + v + 1} \leq c_1 \|v\|^{2(k+2)}_{2(k+2)} \leq d \left( \|v\|^{2(k+2)}_{2(k+2)} + H(t) \right), \]

where \(d = 1 + \frac{1}{H(0)}\). In the same way, we obtain

\[ \|u\|^{\rho + v + 1}_{\rho + v + 1} \leq c_2 \|u\|^{2(k+2)}_{2(k+2)} \leq d \left( \|u\|^{2(k+2)}_{2(k+2)} + H(t) \right), \]

\[ \|v\|^{\theta + v + 1}_{\theta + v + 1} \leq c_3 \|v\|^{2(k+2)}_{2(k+2)} \leq d \left( \|v\|^{2(k+2)}_{2(k+2)} + H(t) \right), \]

\[ \|u\|^{\rho + v + 1}_{\rho + v + 1} \leq c_4 \|u\|^{2(k+2)}_{2(k+2)} \leq d \left( \|u\|^{2(k+2)}_{2(k+2)} + H(t) \right), \]

\[ \|u\|^{\rho + v + 1}_{\rho + v + 1} \leq c_5 \|u\|^{2(k+2)}_{2(k+2)} \leq d \left( \|u\|^{2(k+2)}_{2(k+2)} + H(t) \right), \]
\[ \|u\|_{k+\eta+1} \leq c_5 \|u\|_{2(k+2)}^{(k+2)} \leq \frac{d}{4} \left( \|u\|_{2(k+2)}^{2(k+2)} + H(t) \right), \]

and
\[ \|v\|_{q+v+1} \leq c_6 \|v\|_{2(k+2)}^{(q+v+1)} \leq \frac{d}{4} \left( \|v\|_{2(k+2)}^{2(k+2)} + H(t) \right). \]

Selecting \( C_1, C_2, C_3, C_4, C_5 \) as follows
\[ C_1 = \frac{\eta \delta_1^{\eta+1}}{\eta+1}, \quad C_2 = \frac{\nu \delta_2^{\nu+1}}{\nu+1}, \]
\[ C_3 = \frac{\delta_1^{\nu+1}}{\nu+1} \left( 1 + \frac{\eta \delta_1^{\eta+1}}{\eta+1} \right) + \frac{\delta_2^{\nu+1}}{\nu+1} \frac{\epsilon}{\nu+1} \frac{1}{\eta+1}, \]
\[ C_4 = \frac{\delta_2^{\nu+1}}{\nu+1} \left( 1 + \frac{\epsilon}{\nu+1} \frac{1}{\eta+1} \right) + \frac{\delta_1^{\nu+1}}{\nu+1} \frac{\eta \delta_1^{\eta+1}}{\eta+1} \cdot \frac{\epsilon}{\nu+1}, \]

and
\[ C_5 = \frac{\delta_1^{\nu+1}}{\nu+1} \left( 1 + \frac{l}{l+\eta+1} \right) \frac{\eta \delta_1^{\eta+1}}{\eta+1} \left( 1 + \frac{\nu \delta_2^{\nu+1}}{\nu+1} \right) \frac{\eta \delta_1^{\eta+1}}{\eta+1} \cdot \frac{\epsilon}{\nu+1} \cdot \frac{1}{\eta+1}. \]

where we pick \( \delta_1, \delta_2, \gamma_1 \) and \( \gamma_2 \) to find small enough \( C_1, C_2, C_3, C_4 \) and \( C_5 \).

This implies
\[ \Phi(t) \geq H(t) + 2\epsilon (\|u_t\|^2 + \|v_t\|^2) + \epsilon (2 - dC_3) H(t) \]
\[ + \epsilon \omega_1 \|\nabla u\|^2 + \epsilon \omega_2 \|\nabla v\|^2 \]
\[ + \epsilon \theta_1 (u_\theta \circ \nabla u)(t) + \epsilon \theta_2 (u_\theta \circ \nabla v)(t) \]
\[ + (1 - \epsilon C_3) \int \Omega (|u|^k + |v|^l) |u_t|^{\eta+1} dx \]
\[ + (1 - \epsilon C_2) \int \Omega (|v|^\theta + |u|^\theta) |v_t|^{\nu+1} dx \]
\[ + \epsilon (2(\kappa + 1) - dC_3) \|u\|_{2(k+2)}^{2(\kappa+2)} + \epsilon (2(\kappa + 1) - dC_4) \|v\|_{2(k+2)}^{2(\kappa+2)} \]
(25)

where \( \omega_i = \left( 1 - l_i \right) + \frac{(1+\xi)(1-l_i)^2-1}{2} \) \( \geq 0 \) and \( \theta_i = \left( 1 + \frac{(1+\xi)(1-l_i)}{2} \right) \geq 0 \) \( i = 1, 2 \) for choosing \( \xi_i = \frac{l_i}{1-l_i} \).

We can find positive constants \( K_1, K_2, K_3 \) and \( C_6 \) such that
\[ \Phi'(t) \geq (1 - \epsilon C_6) H(t) + 2\epsilon (\|u_t\|^2 + \|v_t\|^2) + \epsilon K_1 H(t) \]
\[ + \epsilon \omega_1 \|\nabla u\|^2 + \epsilon \omega_2 \|\nabla v\|^2 + \epsilon K_2 \|u\|_{2(k+2)}^{2(k+2)} + \epsilon K_3 \|v\|_{2(k+2)}^{2(k+2)}. \]
(26)

We pick \( \epsilon \) small enough such that \( (1 - \epsilon C_6) \geq 0 \) and
\[ \Phi(0) = H(0) + \epsilon \left( \int \Omega u_t u_0 dx + \int \Omega v_t v_0 dx \right) > 0. \]

As a result, there exists \( M > 0 \) such that (26) deduce to
\[ \Phi'(t) \geq \epsilon M \left( H(t) + \|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla u\|_{2(k+2)}^{2(k+2)} + \|v\|_{2(k+2)}^{2(k+2)} \right). \]

Thus, \( \Phi(t) \) is strictly positive and increasing for all \( t \geq 0 \).

Now, by applying Holder’s and Young’s inequalities, we have
\[ \left| \int \Omega u_t u dx \right| \leq \|u_t\| \|u\| \]
\[ \leq C \|u_t\| \|u\|_{2(k+2)} \]
\[ \leq \frac{C}{2} \left( \|u_t\|^2 + \|u\|_{2(k+2)}^2 \right) \]
\[
\leq \frac{C}{2} \left( \|u_t\|^2 + \left( \|u\|_{2(k+2)}^{2(k+2)} \right)^{\frac{1}{k+2}} \right).
\]

Applying (24) for \( \|u\|_{2(k+2)}^{2(k+2)} \), we get
\[
\left| \int_\Omega u_t u dx \right| \leq \frac{C}{2} \left( \|u_t\|^2 + \left( \|u\|_{2(k+2)}^{2(k+2)} + H(t) \right) \right).
\]

Likewise, we get
\[
\left| \int_\Omega v_t v dx \right| \leq \frac{C}{2} \left( \|v_t\|^2 + \left( \|v\|_{2(k+2)}^{2(k+2)} + H(t) \right) \right).
\]

Then, we have
\[
\Phi(t) \leq C \left( H(t) + \|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{2(k+2)}^{2(k+2)} + \|v\|_{2(k+2)}^{2(k+2)} \right)
\]
and from (27) and (28), we reach
\[
\frac{d\Phi(t)}{dt} \geq \Gamma \Phi(t), \forall t \geq 0
\]
where \( \Gamma \) is a positive constant.

Integration of (29) over \((0, t)\), we obtain
\[
\Phi(t) \geq \Phi(0) \exp(\Gamma t)
\]
and this completes the proof.

**Conclusion**

In this work, we are interested in the exponential growth of solutions for coupled of Klein-Gordon equations with degenerate damping and viscoelastic term. This kind of problem is mostly found in some mathematical models in applied sciences. What interests us in this current work is the combination of Klein-Gordon system with these terms of damping (viscoelastic term, degenerate damping and source terms), which dictates the emergence of these terms in the system.

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**References**

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