Generalized permuting Tri-derivations on lattices

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ABSTRACT
In this paper, we present the notion of generalized permuting tri – derivations in lattices and looking for some related properties .

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1. Introduction
Information Theory, Information retrieval, information access controls and cryptanalysis are the various branches in which the lattice algebra play's a significant role [2,3,4,11], lately the peculiarities of lattices they were researched .

L. Ferrari and X.L. Xin in [8] and [12] respectively introduced the notions of derivations on lattices and debated some affectioned peculiarities . N. alshehri in [1] introduced the concept
of generalized derivations for a lattice and debated various properties, thereafter the symmetric biderivations on lattices notions are introduced in [5] by Y. Ceven and investigate some related properties.

In [6] Y. Ceven applied the notion of generalized symmetric biderivations on lattices and introduced some properties of it, while M.A. Ozturk in [10] introduce the notions of permuting tri derivations in lattices and some affectioned peculiarities are studied.

In the article, we apply the notion of generalized permuting tri derivations and looking for some related properties which is canvassed in [6] and [10].

2. Preliminaries

**Definition 2.1** [8]: Let \( \Gamma \) be an non-empty set endowed with operations \( \wedge \) and \( \vee \), then \( (\Gamma, \wedge, \vee) \) is said to be a lattice if it fulfilling the following requirements for everyone \( \alpha, \tau, \eta \in \Gamma \)

\[ (i) \alpha \wedge \alpha = \alpha, \alpha \vee \alpha = \alpha \]

\[ (ii) \alpha \wedge \tau = \tau \wedge \alpha, \alpha \vee \tau = \tau \vee \alpha \]

\[ (iii) (\alpha \wedge \tau) \wedge \eta = \alpha \wedge (\tau \wedge \eta), (\alpha \vee \tau) \vee \eta = \alpha \vee (\tau \vee \eta) \]

\[ (iv) (\alpha \wedge \tau) \vee \alpha = \alpha, (\alpha \vee \tau) \wedge \alpha = \alpha \]

**Definition 2.2** [3]: A lattice \( (\Gamma, \wedge, \vee) \) is namely distributive lattice if one of the following identities hold for all \( (\Gamma, \wedge, \vee) \)

\[ (v) \alpha \wedge (\tau \vee \eta) = (\alpha \wedge \tau) \vee (\alpha \wedge \eta) \]

\[ (vi) \alpha \vee (\tau \wedge \eta) = (\alpha \vee \tau) \wedge (\alpha \vee \eta) \]

**Remark 2.3** [1]: In any lattice, the properties \((v)\) and \((vi)\) are equivalent.

**Definition 2.4** [3]: let \( (\Gamma, \wedge, \vee) \) be a lattice, a binary relation \( \leq \) on \( \Gamma \) is defined by \( \alpha \leq \tau \) if and only if \( \alpha \wedge \tau = \alpha \) and \( \alpha \vee \tau = \tau \)

**Definition 2.5** [8]: A lattice \( (\Gamma, \wedge, \vee) \) is namely modular if for \( \alpha, \tau, \eta \in \Gamma \) fulfills the following requirement:

\[ (vii) \text{if } \alpha \leq \tau \text{ implies } \alpha \vee (\tau \wedge \eta) = (\alpha \vee \tau) \wedge \eta \]
Lemma 2.6 [7]: Let \((\Gamma, \wedge, \vee)\) be a lattice, let the binary relation \(\leq\) be as in definition 2.4, then \((\Gamma, \leq)\) is partially ordered set (poset) and for any \(\alpha, \tau \in \Gamma\), \(\alpha \wedge \tau\) is the g.l.b of \(\{\alpha, \tau\}\) and \(\alpha \vee \tau\) is the l.u.b. of \(\{\alpha, \tau\}\).

Definition 2.7[10]: Let \((\Gamma, \wedge, \vee)\) be a lattice, a mapping \(T(. , .): \Gamma \times \Gamma \times \Gamma \rightarrow \Gamma\) is namely permuting if

\[
T(\alpha, \tau, \eta) = T(\alpha, \eta, \tau) = T(\tau, \alpha, \eta) = T(\tau, \eta, \alpha) = T(\eta, \alpha, \tau) = T(\eta, \tau, \alpha)
\]
for all \(\alpha, \tau, z \in \Gamma\).

Definition 2.8[10]: Let \((\Gamma, \wedge, \vee)\) be a lattice, a mapping \(d: \Gamma \rightarrow \Gamma\) defined by \(d(\alpha) = T(\alpha, \alpha, \alpha)\) is called the trace of \(T(.,.,.)\). Where \(T(.,.,.): \Gamma \times \Gamma \times \Gamma \rightarrow \Gamma\) is permuting mapping.

Definition 2.9[10]: Let \((\Gamma, \wedge, \vee)\) be a lattice, a mapping \(T(.,.,.): \Gamma \times \Gamma \times \Gamma \rightarrow \Gamma\) is called permuting tri-derivation on \(\Gamma\) if

\[
T(\alpha \wedge u, y, \eta) = (T(\alpha, y, \eta) \wedge u) \vee (\alpha \wedge T(u, \tau, \eta))
\]
for all \(\alpha, \tau, \eta, u \in \Gamma\).

Obviously, a permuting tri-derivation on \(\Gamma\) satisfies the relation

\[
T(\alpha, \tau \wedge u, \eta) = (T(\alpha, \tau, \eta) \wedge u) \vee (\tau \wedge T(\alpha, u, \eta))
\]
\[
T(\alpha, \tau, \eta \wedge u) = (T(\alpha, \tau, \eta) \wedge u) \vee (\eta \wedge T(\alpha, \tau, u))
\]
for all \(\alpha, \tau, \eta, u \in \Gamma\).

Proposition 2.10[10]: Let \((\Gamma, \wedge, \vee)\) be a lattice, \(T\) be permuting tri-derivation on \(\Gamma\) with the trace \(d\). Then \(d(\alpha) \leq \alpha\) for all \(\alpha \in \Gamma\).

Proposition 2.11[10]: Let \((\Gamma, \wedge, \vee)\) be a lattice, \(T\) be permuting tri-derivation on \(\Gamma\). Then \(T(\alpha, \tau, \eta) \leq \alpha\), \(T(\alpha, \tau, \eta) \leq \tau\) and \(T(\alpha, \tau, \eta) \leq \eta\) for all \(\alpha, \tau, \eta \in \Gamma\).
3. Generalized Permuting Tri-derivations on Lattices

Definition 3.1: Let \((\Gamma, \wedge, \vee)\) be a lattice. A permuting mapping \(\kappa(\ldots, \cdot) : \Gamma \times \Gamma \times \Gamma \to \Gamma\) is namely a generalized permuting tri-derivation on \(\Gamma\), if there is a permuting tri-derivation \(T(\ldots, \cdot) : \Gamma \times \Gamma \times \Gamma \to \Gamma\) and fulfills the following requirement

\[
\kappa(\alpha \wedge u, \tau, \eta) = (\kappa(\alpha, y, \eta) \wedge u) \vee (x \wedge T(u, \tau, \eta))
\]

for all \(\alpha, \tau, \eta, u \in \Gamma\).

The mapping \(\vartheta : \Gamma \to \Gamma\) defined by \(\vartheta(\alpha) = \kappa(\alpha, \alpha, \alpha)\) is namely the trace of the generalized permuting tri-derivation \(\kappa\).

Obviously that a generalized permuting tri-derivation \(\kappa\) on \(\Gamma\) satisfies the relations

\[
\kappa(\alpha, \tau \wedge u, \eta) = (\kappa(\alpha, y, \eta) \wedge u) \vee (y \wedge T(p, u, \eta))
\]

\[
\kappa(\alpha, \tau, \eta \wedge u) = (\kappa(\alpha, y, \eta) \wedge u) \vee (z \wedge T(\alpha, \tau, u))
\]

for all \(\alpha, \tau, \eta, u \in \Gamma\).

Example 3.2: Let \((L, \wedge, \vee)\) be a lattice with least element \(0\) and the mapping \(D(\ldots, \cdot) : L \times L \times L \to L\) defined by \(D(x, y, z) = 0\) for all \(x, y, z \in L\) is permuting tri-derivation on \(L\). Then the mapping \(\kappa(x, y, z) : L \times L \times L \to L\) defined by \(\kappa(x, y, z) = (x \wedge y) \wedge z\) for all \(x, y, z \in L\) is a generalized permuting tri-derivation on \(L\).

Proposition 3.3: Let \(\kappa\) be a generalized permuting tri-derivation related to a permuting tri-derivation \(T\) on a lattice \(\Gamma\), then the mappings \(p_1 : \Gamma \to \Gamma\), \(p_2 : \Gamma \to \Gamma\) and \(p_3 : \Gamma \to \Gamma\) defined by \(p_1(\alpha) = \kappa(\alpha, \tau, \eta)\), \(p_2(\tau) = \kappa(\alpha, \tau, \eta)\) and \(p_3(\eta) = \kappa(\alpha, \tau, \eta)\) respectively are generalized derivations on \(\Gamma\).

Proof:

\[
p_1(\alpha \wedge u) = \kappa(\alpha \wedge u, y, z)
\]

\[
= (\kappa(x, y, z) \wedge u) \vee (x \wedge T(u, \tau, \eta))
\]

\[
= (p_1(\alpha) \wedge u) \vee (\alpha \wedge d_1(u))
\]

Where \(d_1 : \Gamma \to \Gamma\) is a derivation on \(\Gamma\) defined by \(d_1(u) = T(u, \tau, \eta)\).

Hence \(p_1\) is a generalized derivation on \(\Gamma\).

Similarly for the mappings \(p_2\) and \(p_3\).
Theorem 3.4: let \((\Gamma, \wedge, \vee)\) be a lattice, \(\kappa\) be a generalized permuting tri-derivation related to a permuting tri-derivation \(T\) on \(\Gamma\), \(\vartheta\) be the trace of \(\kappa\) and \(d\) be the trace of \(T\). then

i) \(T(\alpha, \tau, \eta) \leq \kappa(\alpha, \tau, \eta) \quad \forall \alpha, \tau, \eta \in \Gamma\)

If \(\Gamma\) is distributive lattice, then

ii) \(\kappa(\alpha, \tau, \eta) \leq \alpha, \kappa(\alpha, \tau, \eta) \leq \beta\) and \(\kappa(\alpha, \tau, \eta) \leq \eta\)

iii) \(\kappa(\alpha, \tau, \eta) \leq \alpha \wedge \tau, \kappa(\alpha, \tau, \eta) \leq \tau \wedge \eta\) and \(\kappa(\alpha, \tau, \eta) \leq \alpha \wedge \eta\) and \(\kappa(\alpha, \tau, \eta) \leq \alpha \wedge \tau \wedge \eta\)

iv) \(d(\alpha) \leq \vartheta(\alpha) \leq \alpha\)

v) \(d(\alpha) = \alpha\) then \(\vartheta(\alpha) = \alpha\)

Proof:

i) \(\kappa(\alpha, \tau, \eta) = \kappa(\alpha \wedge \alpha, \tau, \eta)\)

\[= (\kappa(\alpha, \tau, \eta) \wedge \alpha) \lor (\alpha \land T(\alpha, \tau, \eta))\]

\[= (\kappa(\alpha, \tau, \eta) \wedge \alpha) \lor T(\alpha, \tau, \eta)\]

By proposition 2.11, then \(T(\alpha, \tau, \eta) \leq \kappa(\alpha, \tau, \eta)\)

ii) If \(\Gamma\) is distributive, then

\(\kappa(\alpha, \tau, \eta) = \kappa(\alpha \wedge \alpha, \tau, \eta)\)

\[= (\kappa(\alpha, \tau, \eta) \wedge \alpha) \lor (\alpha \land T(\alpha, \tau, \eta))\]

\[= \kappa(\alpha, \tau, \eta) \land \alpha\] by (i) and proposition 2.11

Hence \(\kappa(\alpha, \tau, \eta) \leq \alpha\)

Since \(\kappa\) is permuting and by the same way, we get

\(\kappa(\alpha, \tau, \eta) \leq \tau\) and \(\kappa(\alpha, \tau, \eta) \leq \eta\)

iii) Since \(\kappa(\alpha, \tau, \eta) \leq \alpha, \kappa(\alpha, \tau, \eta) \leq \tau\) and \(\kappa(\alpha, \tau, \eta) \leq \eta\)

Then \(\kappa(\alpha, \tau, \eta) \land \kappa(\alpha, \tau, \eta) \leq \alpha \land \tau\)

So that \(\kappa(\alpha, \tau, \eta) \leq \alpha \land \tau\)

By the same way we can prove

\(\kappa(\alpha, \tau, \eta) \leq \tau \land \eta, \kappa(\alpha, \tau, \eta) \leq \alpha \land \eta\) and \(\kappa(\alpha, \tau, \eta) \leq \alpha \land \tau \land \eta\)

iv) By (i) and (ii) we can conclude \(d(\alpha) \leq \vartheta(\alpha) \leq \alpha\)

v) It is clear by (iv)
**Corollary 3.5:** Let $(\Gamma, \wedge, \vee)$ be a lattice, $\kappa$ be a generalized permuting tri-derivation related to a permuting tri-derivation $T$ on $\Gamma$, if $0$ is the least element and $1$ is the greatest element of $\Gamma$, then

$$\kappa(\alpha, \tau, 0) = 0 \text{ and } \kappa(\alpha, \tau, 1) \leq \alpha \text{ for all } \alpha, \tau \in \Gamma.$$

**Proof:** trivially by (ii) of theorem 3.4.

**Theorem 3.6:** Let $(\Gamma, \wedge, \vee)$ be a distributive lattice, $\kappa$ be a generalized permuting tri-derivation related to a permuting tri-derivation $T$ on $\Gamma$, $\vartheta$ be the trace of $\kappa$ and $d$ be the trace of $T$ then

$$\vartheta(\alpha \wedge \tau) = (\vartheta(\alpha) \wedge \tau) \vee (\alpha \wedge d(\tau)) \vee T(\alpha, \alpha, \tau) \vee T(\alpha, \tau, \tau)$$

for all $\alpha, \tau \in \Gamma$.

**Proof:**

$$\vartheta(\alpha \wedge \tau) = \kappa(\alpha \wedge \tau, \alpha \wedge \tau, \alpha \wedge \tau)$$

$$= (\kappa(\alpha, \alpha, \alpha \wedge \tau) \wedge \tau) \vee (\alpha \wedge T(\tau, \alpha \wedge \tau, \alpha \wedge \tau))$$

$$= \{[(\kappa(\alpha, \alpha, \alpha \wedge \tau) \wedge \tau) \vee (\alpha \wedge T(\alpha, \tau, \alpha \wedge \tau)) \wedge \tau\}$$

$$\vee \{\alpha \wedge [D(\tau, \alpha, \alpha \wedge \tau) \wedge \tau) \vee (\alpha \wedge T(\alpha, \tau, \alpha \wedge \tau))\}$$

$$= \{[(\vartheta(\alpha) \wedge \tau) \vee (\alpha \wedge \tau \wedge T(\alpha, \alpha, \tau)) \vee [(\alpha \wedge \tau \wedge T(\alpha, \alpha, \tau) \vee (\alpha \wedge T(\alpha, \tau, \tau))] \wedge \tau\}$$

$$\vee [\alpha \wedge \{[(T(\alpha, \alpha, \tau) \wedge \tau) \vee (\alpha \wedge \tau \wedge T(\alpha, \tau, \tau)) \vee [(\alpha \wedge \tau \wedge T(\alpha, \tau, \tau)) \vee (\alpha \wedge d(\tau)))]\}$$

$$= \{[(\vartheta(\alpha) \wedge \tau) \vee (\alpha \wedge \beta \wedge T(\alpha, \alpha, \tau)) \wedge \tau\}$$

$$\vee \{[(\alpha \wedge \tau \wedge T(\alpha, \alpha, \tau) \vee (\alpha \wedge T(\alpha, \tau, \tau))] \wedge \tau\}$$

$$\vee \{\alpha \wedge [(T(\alpha, \alpha, \tau) \wedge \tau) \vee (\alpha \wedge \tau \wedge T(\alpha, \tau, \tau))\}$$

$$\vee \{\alpha \wedge [(\alpha \wedge \tau \wedge T(\alpha, \tau, \tau)) \vee (\alpha \wedge d(\tau))]\}$$

$$= (\vartheta(\alpha) \wedge \tau) \vee (\alpha \wedge \tau \wedge T(\alpha, \alpha, \tau) \vee (\alpha \wedge \tau \wedge T(\alpha, \alpha, \tau))$$

$$\vee (\alpha \wedge \tau \wedge T(\alpha, \tau, \tau)) \vee (\alpha \wedge \tau \wedge T(\alpha, \alpha, \tau))$$

$$\vee (\alpha \wedge \tau \wedge T(\alpha, \tau, \tau)) \vee (\alpha \wedge d(\tau))$$

$$= (\vartheta(\alpha) \wedge \tau) \vee T(\alpha, \alpha, \tau) \vee T(\alpha, \alpha, \tau) \vee T(\alpha, \tau, \tau) \vee T(\alpha, \alpha, \tau)$$

$$\vee T(\alpha, \tau, \tau) \vee T(\alpha, \tau, \tau) \vee (\alpha \wedge d(\tau))$$
\[
= (\vartheta(\alpha) \wedge \tau) \lor (\alpha \wedge d(\tau)) \lor T(\alpha, \alpha, \tau) \lor T(\alpha, \tau, \tau)
\]

Since \(T(\alpha, \alpha, \tau) \leq \alpha\), \(T(\alpha, \alpha, \tau) \leq \tau\), \(T(\alpha, \tau, \tau) \leq \alpha\) and \(T(\alpha, \tau, \tau) \leq \tau\).

**Corollary 3.7:** Let \((\Gamma, \wedge, \lor)\) be a distributive lattice, \(\kappa\) be a generalized permuting tri-derivation related to a permuting tri derivation \(T\) on \(\Gamma\), \(\vartheta\) be the trace of \(\kappa\) and \(d\) be the trace of \(T\). Then

\begin{align*}
\text{i)} & \quad T(\alpha, \alpha, \tau) \leq \vartheta(\alpha \wedge \tau) \text{ and } T(\alpha, \tau, \tau) \leq \vartheta(\alpha \wedge \tau) \\
\text{ii)} & \quad \vartheta(\alpha) \wedge \tau \leq \vartheta(\alpha \wedge \tau) \\
\text{iii)} & \quad \alpha \wedge d(\tau) \leq \vartheta(\alpha \wedge \tau)
\end{align*}

for all \(\alpha, \tau \in \Gamma\).

**Proof:** (i), (ii) and (iii) are directly from theorem 3.6.

**Theorem 3.8:** Let \((\Gamma, \wedge, \lor)\) be a distributive lattice, \(\kappa\) be a generalized permuting tri-derivation related to a permuting tri derivation \(T\) on \(\Gamma\), \(\vartheta\) be the trace of \(\kappa\) and \(d\) be the trace of \(T\). Then

\[\vartheta(\alpha) \wedge \vartheta(\tau) \leq \vartheta(\alpha \wedge \tau)\]

for all \(\alpha, \tau \in \Gamma\).

**Proof:**

Since \(\vartheta(\alpha) \wedge \tau \leq \vartheta(\alpha \wedge \tau)\) and \(\vartheta(\tau) \leq \tau\), then \(\vartheta(\alpha) \wedge \tau \wedge \vartheta(\tau) \leq \vartheta(\alpha \wedge \tau) \wedge \tau\).

Also since \(\vartheta(\alpha) \leq \alpha\), we have

\[\vartheta(\alpha) \wedge \tau \wedge \vartheta(\tau) \wedge \vartheta(\alpha) \leq \vartheta(\alpha \wedge \tau) \wedge \tau \wedge \alpha\]

Hence \(\vartheta(\alpha) \wedge \vartheta(\tau) \wedge \vartheta(\alpha) \leq \vartheta(\alpha \wedge \tau) \wedge \alpha \wedge \tau\).

So that \(\vartheta(\alpha) \wedge \tau \wedge \vartheta(\tau) \leq \vartheta(\alpha \wedge \tau)\) since \((\alpha \wedge \tau) \wedge \alpha \wedge \tau = \vartheta(\alpha \wedge \tau)\).

**Corollary 3.9:** Let \((\Gamma, \wedge, \lor)\) be a distributive lattice, \(\kappa\) be a generalized permuting tri-derivation related to a permuting tri derivation \(T\) on \(\Gamma\), \(\vartheta\) be the trace of \(\kappa\) and \(d\) be the trace of \(T\). Then

\[d(\alpha) \wedge d(\tau) \leq \vartheta(\alpha \wedge \tau)\]

for all \(\alpha, \tau \in \Gamma\).

**Proof:**

Since \(d(\alpha) \leq \vartheta(\alpha)\) and \(d(\tau) \leq \vartheta(\tau)\), then \(d(\alpha) \wedge d(\tau) \leq \vartheta(\alpha) \wedge \vartheta(\tau)\).

By theorem 3.8, we have \(d(\alpha) \wedge d(\tau) \leq \vartheta(\alpha \wedge \tau)\) for all \(\alpha, \beta \in \Gamma\).

**Theorem 3.10:** Let \((\Gamma, \wedge, \lor)\) be a lattice, \(\kappa\) be a generalized permuting tri-derivation related to a permuting tri derivation \(T\) on \(\Gamma\), \(\vartheta\) be the trace of \(\kappa\). Then

\[\vartheta^2(\alpha) = \vartheta(\alpha)\]

for all \(\alpha \in \Gamma\).

**Proof:**
It is clear that $\vartheta^2(\alpha) = \vartheta(\vartheta(\alpha)) \leq \vartheta(\alpha) \leq \alpha$ for all $\alpha \in \Gamma$ ...(1)

Nou

\[
\vartheta^2(\alpha) = \vartheta(\vartheta(\alpha)) = \vartheta(\vartheta(\alpha) \wedge \alpha)
\]

\[
= (\vartheta(\vartheta(\alpha)) \wedge \alpha) \lor (\vartheta(\alpha) \wedge \delta(\alpha)) \lor D(\vartheta(\alpha), \vartheta(\alpha), \alpha) \lor D(\delta(x), \alpha, \alpha)
\]

\[
= (\vartheta^2(\alpha) \wedge \alpha) \lor (\vartheta(\alpha) \wedge \delta(\alpha)) \lor D(\vartheta(\alpha), \vartheta(\alpha), \alpha) \lor D(\delta(x), \alpha, \alpha)
\]

\[
\leq \vartheta^2(\alpha) \lor \vartheta(\alpha) \lor \delta(x) \lor \delta(x)
\]

\[
= \vartheta^2(\alpha) \lor \vartheta(\alpha)
\]

Hence

\[
\vartheta^2(\alpha) \leq \vartheta(\alpha) \text{ for all } \alpha \in \Gamma \quad \text{...(2)}
\]

From (1) and (2), we have $\vartheta^2(\alpha) = \vartheta(\alpha)$.

**Theorem 3.10**: Let $(\Gamma, \wedge, \lor)$ be a distributive lattice, $\kappa$ be a generalized permuting tri-derivation related to a permuting tri derivation $T$ on $\Gamma$, $\vartheta$ be the trace of $\kappa$ and $d$ be the trace of $T$.

If 0 and 1 are the least and greatest elements of $\Gamma$ respectively, then

i) when $\alpha \leq \vartheta(1)$ then $\vartheta(\alpha) = \alpha$

ii) when $\vartheta(1) \leq \alpha$ then $\vartheta(\alpha) \geq \vartheta(1)$

iii) when $\alpha \leq \tau$ and $(\tau) = \tau$, we have $\vartheta(\alpha) = \alpha$.

**Proof**:

i) By corollary 3.7 (2) we have

\[
\vartheta(1) \wedge x \leq \delta(1 \wedge x) = \delta(x)
\]

Now if $\alpha \leq \vartheta(1) \Rightarrow \alpha \wedge \vartheta(1) = \alpha$

Hence $\alpha \leq \vartheta(\alpha)$ but $\vartheta(\alpha) \leq \alpha$ by theorem 3.3 (iv)

So that $\vartheta(\alpha) = \alpha$

ii) when $\vartheta(1) \leq \alpha$ this implies $\vartheta(1) \wedge \alpha = \vartheta(1)$

and since $\vartheta(1) \wedge \alpha \leq \vartheta(1 \wedge \alpha) = \vartheta(\alpha)$

but $\vartheta(1) \wedge \alpha = \vartheta(1)$

hence $\vartheta(1) \leq \vartheta(\alpha)$
iii) when \( \alpha \leq \tau \) then \( \alpha \wedge \tau = \alpha \) and since \( d(\tau) = \tau \), \( \vartheta(\alpha) \leq \alpha \), \( \alpha \leq \tau \) and \( T(\alpha, \tau, \eta) \leq \alpha \), we have

\[
\vartheta(\alpha) = \vartheta(\alpha \wedge \tau) \\
= (\vartheta(\alpha) \wedge \beta) \vee (x \wedge d(\tau)) \vee T(\alpha, \alpha, \tau) \vee T(\alpha, \tau, \tau) \\
= \vartheta(\alpha) \vee \alpha \vee \alpha \vee \alpha \\
= \vartheta(\alpha) \vee \alpha = \alpha
\]

**Definition 3.11**: let \((\Gamma, \wedge, \vee)\) be a lattice, the mapping \(\kappa : \Gamma \times \Gamma \times \Gamma \to \Gamma\) in which satisfying

\[
\kappa(\alpha \lor \tau, \eta, u) = \kappa(\alpha, \eta, u) \lor \kappa(\tau, \eta, u)
\]

for all \(\alpha, \tau, \eta, u \in \Gamma\) is called joinitive mapping.

**Theorem 3.12**: let \((\Gamma, \wedge, \vee)\) be a lattice, the mapping \(\kappa : \Gamma \times \Gamma \times \Gamma \to \Gamma\) be a joinitive mapping with the trace \(\vartheta\), then

i) \(\vartheta(\alpha \lor \tau) = \vartheta(\alpha) \lor \vartheta(\tau) \lor \kappa(\alpha, \alpha, \alpha) \lor \kappa(\alpha, \alpha, \tau)\)

ii) \(\vartheta(\alpha) \lor \vartheta(\tau) \leq \vartheta(\alpha \lor \tau)\)

For all \(\alpha, \tau \in \Gamma\)

**Proof**:

i) For all \(\alpha, \tau \in \Gamma\) we have

\[
\vartheta(\alpha \lor \tau) = \kappa(\alpha \lor \tau, \alpha \lor \tau, \alpha \lor \tau) \\
= \kappa(\alpha, \alpha \lor \tau, \alpha \lor \tau) \lor \kappa(\tau, \alpha \lor \tau, \alpha \lor \tau) \\
= \kappa(\alpha, \alpha, \alpha \lor \tau) \lor \kappa(\tau, \alpha, \alpha \lor \tau) \lor \kappa(\alpha, \tau, \alpha \lor \tau) \lor \kappa(\tau, \tau, \alpha \lor \tau) \\
= \vartheta(\alpha) \lor \vartheta(\tau) \lor \kappa(\alpha, \alpha, \tau) \lor \kappa(\alpha, \tau, \tau) \lor \kappa(\tau, \alpha, \alpha) \lor \kappa(\tau, \tau, \tau)
\]

ii) Directly from (i)

**Theorem 3.12**: let \(\kappa_1\) and \(\kappa_2\) are two generalized permuting tri-derivations related to the same permuting tri-derivation \(T\) on the distributive lattice. Then the mapping \(\kappa_1 \wedge \kappa_2\) defined by

\[
(\kappa_1 \wedge \kappa_2)(\alpha, \tau, \eta) = \kappa_1(\alpha, \tau, \eta) \wedge \kappa_2(\alpha, \tau, \eta)
\]

for all \(\alpha, \tau, \eta \in \Gamma\)
Is generalized permuting tri–derivation related to a permuting tri–derivation $T$.

**Proof**: let $\alpha, \tau, \eta, u \in \Gamma$

\[
(\kappa_1 \land \kappa_2)(\alpha \land u, \tau, \eta) = \kappa_1(\alpha \land u, \tau, \eta) \land \kappa_2(\alpha \land u, \tau, \eta)
\]

\[
= [(\kappa_1(\alpha, \tau, \eta) \land u) \lor (\alpha \land T(u, \tau, \eta)]
\land [(\kappa_2(\alpha, \tau, \eta) \land u) \lor (\alpha \land T(u, \tau, \eta))]
\]

\[
= [(\kappa_1(\alpha, \tau, \eta) \land u) \land (\kappa_2(\alpha, \tau, \eta) \land u)] \lor (\alpha \land T(u, \tau, \eta))
\]

\[
= [(\kappa_1(\alpha, \tau, \eta) \land \kappa_2(\alpha, \tau, \eta)) \land u] \lor (\alpha \land T(u, \tau, \eta))
\]

So that $\kappa_1 \land \kappa_2$ is generalized permuting tri–derivation related to a permuting tri–derivation $T$ on a lattice.

**Theorem 3.13**: let $\kappa_1$ and $\kappa_2$ are two generalized permuting tri–derivations related to the same permuting tri–derivation $T$ on the distributive lattice. Then the mapping $\kappa_1 \lor \kappa_2$ defined by

\[
(\kappa_1 \lor \kappa_2)(\alpha, \tau, \eta) = \kappa_1(\alpha, \tau, \eta) \lor \kappa_2(\alpha, \tau, \eta)
\]

is generalized permuting tri–derivation related to a permuting tri–derivation $T$.

**Proof**: let $\alpha, \tau, \eta, u \in \Gamma$

\[
(\kappa_1 \lor \kappa_2)(\alpha \land u, \tau, \eta) = \kappa_1(\alpha \land u, \tau, \eta) \lor \kappa_2(\alpha \land u, \tau, \eta)
\]

\[
= [(\kappa_1(\alpha, \tau, \eta) \land u) \lor (\alpha \land T(u, \tau, \eta)]
\]

\[
\lor [(\kappa_2(\alpha, \tau, \eta) \land u) \lor (\alpha \land T(u, \tau, \eta))]
\]

\[
= [(\kappa_1(\alpha, \tau, \eta) \land u) \lor (\kappa_2(\alpha, \tau, \eta) \land u)] \lor (\alpha \land T(u, \tau, \eta))
\]

\[
= [(\kappa_1(\alpha, \tau, \eta) \lor \kappa_2(\alpha, \tau, \eta)) \land u] \lor (\alpha \land T(u, \tau, \eta))
\]

So that $\kappa_1 \lor \kappa_2$ is generalized permuting tri–derivation related to a permuting tri–derivation $T$ on a lattice.
**Definition 3.13**: let $(\Gamma, \Lambda, \nu)$ be a lattice , $\kappa$ be a generalized permuting tri–derivation related to a permuting tri– derivation $T$, $\vartheta$ is the trace of $\kappa$, $\vartheta$ is called isotone mapping if when $\alpha \leq \tau$ implies $\vartheta(\alpha) \leq \vartheta(\tau)$.

**Theorem 3.14**: let $\kappa_1$ and $\kappa_2$ are tuo generalized permuting tri– derivations related to the same permuting tri– derivation $T$ on the distributive lattice $\Gamma$, and $\vartheta_1$, $\vartheta_2$ are the traces of $\kappa_1$ and $\kappa_2$ respectively . If $\vartheta_1$ and $\vartheta_2$ are isotone mapping , then $\vartheta_1 = \vartheta_2$ if and only if $F_i\alpha_{\vartheta_1}(\Gamma) = F_i\alpha_{\vartheta_2}(\Gamma)$.

**Proof**: suppose that $\vartheta_1 = \vartheta_2$ then

\[
F_i\alpha_{\vartheta_1}(\Gamma) = \{\alpha \in \Gamma | \vartheta_1(\alpha) = \alpha\}
\]

\[= \{\alpha \in \Gamma | \vartheta_2(\alpha) = \alpha\} \quad \text{since} \ \vartheta_1 = \vartheta_2
\]

\[= F_i\alpha_{\vartheta_2}(\Gamma)
\]

Conversely if $F_i\alpha_{\vartheta_1}(\Gamma) = F_i\alpha_{\vartheta_2}(\Gamma)$

Suppose that $\alpha \in F_i\alpha_{\vartheta_1}(\Gamma) \Rightarrow \vartheta_1(\alpha) = \alpha$

and $\vartheta_1(\vartheta_1(\alpha)) = \vartheta_1(\alpha)$

$\Rightarrow \vartheta_1(\alpha) \in F_i\alpha_{\vartheta_1}(\Gamma)$

but $F_i\alpha_{\vartheta_1}(\Gamma) = F_i\alpha_{\vartheta_2}(\Gamma)$

so that $\vartheta_1(\alpha) \in F_i\alpha_{\vartheta_2}(\Gamma)$

hence $\vartheta_2(\vartheta_1(\alpha)) = \vartheta_1(\alpha)$

by the same way we can prove that $\vartheta_1(\vartheta_2(\alpha)) = \vartheta_2(\alpha)$

since $\vartheta_1(\alpha) \leq \alpha$ and $\vartheta_2(\alpha) \leq \alpha$ and since $\vartheta_1$ and $\vartheta_2$ are isotone ,we can conclude

$\vartheta_2(\vartheta_1(\alpha)) \leq \vartheta_1(\alpha)$ and $\vartheta_1(\vartheta_2(\alpha)) \leq \vartheta_2(\alpha)$

but $\vartheta_2(\vartheta_1(\alpha)) = \vartheta_1(\alpha)$ and $\vartheta_1(\vartheta_2(\alpha)) = \vartheta_2(\alpha)$

hence $\vartheta_2(\vartheta_1(\alpha)) \leq \vartheta_1(\vartheta_2(\alpha))$ and $\vartheta_1(\vartheta_2(\alpha)) \leq \vartheta_2(\vartheta_1(\alpha))$

so $\vartheta_1(\alpha) = \vartheta_2(\alpha)$ for all $\alpha \in \Gamma$ \Rightarrow $\vartheta_1 = \vartheta_2$.
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