Anti Fuzzy k-Ideal of Ternary Semiring

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Abstract
In this paper, we introduce the notion of anti fuzzy k-ideal of ternary semi ring and study some properties of it.

Introduction
The notion of fuzzy subset of a set was introduced by Zadeh in 1965 [4]. The notion of fuzzy subgroup was made by Rosenfeld in 1971 [1]. Fuzzy ideal in a ring were introduced by W. Liu in 1982 [9]. In 1996, Kim and Park studied fuzzy ideal in semirings [2]. The notion of ternary semi rings introduced by Dutta and Kar in 2003 [8]. In 2007, J. kavikumar and Azme Bin khamis introduced The notion of fuzzy ideals in ternary semirings [3]. R. Biswas given the notion of anti fuzzy in 1999 [5]. The notion of fuzzy k-ideals in ternary semirings was introduced by Sathinee Malee and Ronnason Chinram in 2010 [7].

The main purpose of the paper is to introduce the notion of anti fuzzy k-ideal of ternary semiring and study some properties of it.

1- Preliminaries
In this section we review some basic definition which will be used in this paper.

Definition (1.1) [6] A non empty set R together with a binary operation, called addition and ternary multiplication, is said to be a ternary semi ring if R is an additive commutative semigroup satisfying the following conditions
(i) (abc)de = a(bcd)e = ab(cde),
(ii) (a+b)cd = a(cd) + bcd,
(iii) a(b+c)d = abd + acd,
(iv) ab(c+d) = abc + abd, for all a, b, c, d, e ∈ R

Definition (1.2) [6] Let R be a ternary semiring. If there exists an element 0 ∈ R such that
0 + x = x and 0xy = x0y = xy0, for all x, y ∈ R then “0” is called the zero element or simply the zero of ternary semirings. In this case we say that R is a ternary semiring with zero

Definition (1.3) [6] An additive subsemigroup I of ternary semiring R is called a left (resp., right and lateral) ideal of R if
s, s₂i(resp., i₁s₁, s₂, i₂s₂) ∈ I ∀ s, s₂ ∈ R and i ∈ I
If I is both left and right ideal of R, then I is called a two sided ideal of R.
If I is a left, a right and a lateral ideal of R, then I is called an ideal of R.

**Definition (1.4) [4]** A function \( \mu \) from a non empty set \( X \) to the interval \([0,1]\) is called a fuzzy subset of \( X \).

**Definition (1.5) [4]** The complement of a fuzzy subset \( \mu \) of a set \( X \) is denoted by \( \mu^c \) and defined as 
\[
\mu^c(x) = 1 - \mu(x), \forall x \in X
\]

**Definition (1.6)[3]** Let \( \nu \) and \( \mu \) be any two fuzzy subsets of \( X \) then 
\[
\nu \cap \mu \quad \text{and} \quad \nu \cup \mu
\]
are fuzzy subset of \( X \) and defined by 
\[
(\nu \cap \mu)(x) = \min\{\nu(x), \mu(x)\}
\]
\[
(\nu \cup \mu)(x) = \max\{\nu(x), \mu(x)\}, \forall x \in X
\]

**Definition (1.7) [3]** A fuzzy subsemigroup \( \mu \) of a ternary semiring \( R \) is called a fuzzy ideal of \( R \) if the function \( \mu : R \rightarrow [0,1] \) satisfying the following conditions:
(i) \( \mu(x + y) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in R \)
(ii) \( \mu(xyz) \geq \mu(z) \)
(iii) \( \mu(xyz) \geq \mu(x) \)
(iv) \( \mu(xyz) \geq \mu(y), \forall x, y, z \in R \)

A fuzzy set \( \mu \) with conditions (i)and (ii) is called an fuzzy left ideal of \( R \). If a fuzzy set \( \mu \) satisfies (i) and (iii) then it is called a fuzzy right ideal of \( R \). Also if \( \mu \) satisfy (i) and (iv) then it is called a fuzzy lateral ideal of \( R \). If \( \mu \) is fuzzy left ideal, fuzzy right ideal and fuzzy lateral ideal then it is called fuzzy ideal of a ternary semiring \( R \).

**Definition (1.8)[7]** A fuzzy ideal \( \mu \) of a ternary semiring \( R \) is said to be a fuzzy k-ideal of \( R \) if 
\[
\mu(x) \geq \min\{\mu(x + y), \mu(y)\}, \forall x, y \in R
\]

**Definition(1.9)[7]** Let \( S \) and \( R \) be two ternary semirings. a mapping \( f : S \rightarrow R \) is said to be a homomorphism if 
\[
f(x+y)=f(x)+f(y) \quad \text{and} \quad f(xyz)=f(x)f(y)f(z)
\]
, \( \forall x, y, z \in S \)
If \( S \) and \( R \) are ternary semirings with zero 0, then \( f(0)=0 \).

**Definition (1.10) [7]** Let \( f : S \rightarrow R \) be a homomorphism of ternary semirings and \( \mu \) be a fuzzy subset of \( S \), we define a fuzzy subset \( f(\mu) \) of \( R \) by 
\[
f(\mu)(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} & \text{if} & f^{-1}(y) \neq \emptyset \\
0 & \text{otherwise} 
\end{cases}
\]
We call \( f(\mu) \) the image of \( \mu \) under \( f \).

**Definition (1.11)[7]** Let \( R_1 \) and \( R_2 \) be two ternary semirings and \( f \) be a function of \( R_1 \) into \( R_2 \). If \( \mu \) is a fuzzy subset of \( R_2 \), then the preimage of \( \mu \) under \( f \) is a fuzzy subset of \( R_1 \) defined by 
\[
f^{-1}(\mu)(x) = \mu(f(x)) \quad \forall x \in R_1
\]

**Definition (1.12)[9]** Let \( \phi : R_1 \rightarrow R_2 \) be any function. An anti fuzzy ideal \( \mu \) of \( R_1 \) is called \( \phi \)-invariant if \( \phi(x) = \phi(y) \) implies \( \mu(x) = \mu(y) \) where \( x, y \in R_1 \).
2-The Main Results

**Definition (2.1)** A fuzzy subset $\mu$ of a ternary semiring $R$ is said to be an anti fuzzy left (right, lateral) ideal of $R$ if
1- $1 - \mu(x + y) \leq \max \{\mu(x), \mu(y)\}$,
2- $\mu(xy)\leq \mu(z)$, \[ \mu(xyz)\leq \mu(x), \mu(xyz)\leq \mu(y)\] for all $x, y, z \in R$.

$\mu$ is an anti fuzzy ideal of $R$ if it is anti fuzzy left ideal, anti fuzzy right ideal and anti fuzzy lateral ideal of $R$.

**Definition (2.2)** An anti fuzzy ideal $\nu$ of a ternary semiring $R$ is said to be a fuzzy $k$-ideal of $R$ if $\mu(x)\leq \max \{\mu(x+y), \mu(y)\}$, for all $x, y \in R$.

**Example (2.3)** Let $R$ be the set of nonpositive integer with zero. $R$ is ternary semiring with the usual addition and ternary multiplication.

Let $\mu$ a fuzzy subset of $R$ defined by
$\mu(x) = \begin{cases} 0 & \text{if } x \text{ is an even or 0} \\ 1 & \text{if } x \text{ is an odd} \end{cases}$

Then $\mu$ is an anti fuzzy $k$-ideal of $R$.

**Proposition (2.4)** Let $R$ be a ternary semiring and $\mu$ be a fuzzy subset of $R$. Then $\mu$ is an anti fuzzy $k$-ideal of $R$ if and only if $\mu^c$ is a fuzzy $k$-ideal of $R$.

**Proof:**
Suppose $\mu$ be an anti fuzzy $k$-ideal of $R$. Let $x, y, z \in R,$
$\mu^c(x + y) = 1\mu^c(x + y), \text{ since } \mu$ is an anti fuzzy $k$-ideal of $R$
$\geq 1 - \max \{\mu(x), \mu(y)\}$
$= \min \{1 - \mu(x), 1 - \mu(y)\}$

$= \min \{\mu^c(x), \mu^c(y)\}$,

$\mu^c(xyz) = 1 - \mu(xyz)$ since $\mu$ is an anti fuzzy left $k$-ideal of $R$
$\geq 1 - \mu(z)$.

Hence $\mu^c$ is fuzzy left ideal of $R$

$\mu^c(xyz) = 1 - \mu(xyz)$
$\geq 1 - \mu(x)$

$= \mu^c(x).$ Hence $\mu^c$ is fuzzy right ideal of $R$

$\mu^c(xyz) = 1 - \mu(xyz)$
$\geq 1 - \mu(y)$

$= \mu^c(y).$ Hence $\mu^c$ is fuzzy lateral ideal of $R$

Then $\mu^c$ is a fuzzy ideal of $R$.

Let $x, y \in R$ Then
$\mu^c(x) = 1 - \mu(x)$
$\geq 1 - \max \{\mu(x+y), \mu(y)\}$
$= \min \{1 - \mu(x+y), 1 - \mu(y)\}$
$= \min \{\mu^c(x+y), \mu^c(y)\}.

Hence $\mu^c$ is a fuzzy $k$-ideal of $R$.

Conversely, let $\mu^c$ be a fuzzy $k$-ideal of $R$.

For $x, y, z \in R$, we have
$\mu(x + y) = 1 - \mu^c(x + y)$
$\leq 1 - \min \{\mu^c(x), \mu^c(y)\}$
$= \max \{\mu(x), \mu(y)\}$
$\mu(xyz) = 1 - \mu^c(xyz)$
$\leq 1 - \mu^c(z)$

$= \mu(z).$ Hence $\mu$ is an anti fuzzy left ideal of $R$.

Similarly we can prove that $\mu$ is an anti fuzzy right and lateral ideal of $R$.

Let $x, y \in R$ Then
$\mu(x) = 1 - \mu^c(x)$
\[ \mu \leq I - \min\{ \mu^c(x+y), \mu^c(y)\} \]
\[ = \max\{\mu(x+y), \mu(y)\} \]
Hence \( \mu \) is an anti fuzzy k-ideal of \( R \).

**Proposition (2.5)** Let \( \mu \) and \( \nu \) are anti fuzzy k-ideal of ternary semiring \( R \). Then \( \mu \cup \nu \) is also an anti fuzzy k-ideal of ternary semiring \( R \).

**Proof**

Let \( \mu \) and \( \nu \) be two anti fuzzy k-ideals of a ternary semiring \( R \) and \( x, y, z \in R \). Then we have

\[ (\mu \cup \nu)(x+y) = \max\{\mu(x+y), \nu(x+y)\} \]
\[ \leq \max\{\max\{\mu(x), \nu(x)\}, \max\{\mu(y), \nu(y)\}\} \]
\[ = \max\{\max\{\mu(x), \nu(x)\}, \max\{\mu(y), \nu(y)\}\} \]
\[ = \max\{\mu(x), \nu(x)\} \]
\[ \leq \max\{\mu(z), \nu(z)\} \]
\[ = (\mu \cup \nu)(z), \]

Hence \( \mu \cup \nu \) is an anti fuzzy left ideal of \( R \) similarly we can prove that \( \mu \cup \nu \) is an anti fuzzy right and lateral ideal of \( R \) then \( \mu \cup \nu \) is an anti fuzzy ideal of \( R \).

Let \( x, y \in R \), since \( \mu \) and \( \nu \) are anti fuzzy k-ideal Then

\( \mu(x) \leq \max\{\mu(x+y), \mu(y)\} \) and \( \nu(x) \leq \max\{\nu(x+y), \nu(y)\} \),

\[ (\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\} \]
\[ \leq \max\{\max\{\mu(x), \nu(x)\}, \max\{\nu(x+y), \nu(y)\}\} \]
\[ = \max\{\max\{\mu(x+y), \nu(x+y)\}, \max\{\mu(y), \nu(y)\}\} \]
\[ = \max\{\mu(x+y), (\mu \cup \nu)(y)\} \]

then \( \mu \cup \nu \) is an anti fuzzy k-ideal of \( R \).

**Theorem (2.6)** Let \( f : R_1 \to R_2 \) be an onto homomorphism of ternary semirings \( R_1 \) and \( R_2 \). If \( \mu \) is an anti fuzzy k-ideal of \( R_2 \), then \( f^{-1}(\mu) \) is an anti fuzzy k-ideal of \( R_1 \).

**Proof**

Let \( \mu \) be an anti fuzzy k-ideal of \( R_2 \) and let \( x, y, z \in R_1 \).

Then we have

\[ f^{-1}(\mu)(x+y) = \mu(f(x+y)), \text{ since } f \text{ is a homomorphism} \]
\[ \leq \max\{\mu(f(x)), \mu(f(y))\} \]
\[ = \max\{ f^{-1}(\mu)(x), f^{-1}(\mu)(y)\} \]
\[ = f^{-1}(\mu)(z). \]

Hence \( f^{-1}(\mu) \) is an anti fuzzy left ideal of \( R_1 \).

Similarly we \( f^{-1}(\mu) \) is an anti fuzzy right and lateral ideal of \( R_1 \) then

\( f^{-1}(\mu) \) is an anti fuzzy ideal of \( R_1 \).

Let \( x, y \in R_1 \). Then

\[ f^{-1}(\mu)(x) = \mu(f(x)) \]
\[ \leq \max\{\mu(f(x)+f(y)), \mu(f(y))\}, \text{ since } f \text{ is a homomorphism} \]
\[ = \max\{\mu(f(x+y)), \mu(f(y))\} \]
\[ = f^{-1}(\mu)(x+y), f^{-1}(\mu)(y). \]

Hence \( f^{-1}(\mu) \) is an anti fuzzy k-ideal of \( R_1 \).
Lemma (2.7)[7] Let \( R_1 \) and \( R_2 \) be two ternary semirings and \( \phi : R_1 \rightarrow R_2 \) be a homomorphism. Let \( \mu \) be a \( \phi \)-invariant anti fuzzy ideal of \( R_1 \) if \( x = x \in R_1 \), such that \( \mu x = \phi(a), a \in R_1 \), then \( \phi(\mu)(x) = \mu(a) \).

**Proof:** If \( r \in \phi^{-1}(x) \), then \( \phi(r) = x = \phi(a) \).

Since \( \mu \) is a \( \phi \)-invariant \( \mu(r) = \mu(a) \), then by definition (1.10), we have \( \phi(\mu)(x) = \sup_{r \in \phi^{-1}(x)} \mu(r) = \mu(a) \).

Hence \( \phi(\mu)(x) = \mu(a) \).

**Theorem (2.8)** Let \( \phi : R_1 \rightarrow R_2 \) be an onto homomorphism of ternary semirings \( R_1 \) and \( R_2 \). If \( \mu \) is a \( \phi \)-invariant anti fuzzy k-ideal of \( R_1 \), then \( \phi(\mu) \) is an anti fuzzy k-ideal of \( R_2 \).

**Proof:** Let \( \phi : R_1 \rightarrow R_2 \) be an onto homomorphism and \( \mu \) is \( \phi \)-invariant anti fuzzy k-ideal of \( R_1 \).

Let \( x, y, z \in R_2 \). Since \( \phi \) is surjective then there exist \( a, b, c \in R_1 \) such that \( \phi(a) = x, \phi(b) = y \) and \( \phi(c) = z \).

Since \( \phi \) is a homomorphism then \( x + y = \phi(a) + \phi(b) = \phi(a + b) \) and \( xyz = \phi(a)\phi(b)\phi(c) = \phi(ab) \). Then we have \( \phi(\mu)(x + y) = \mu(a + b) \leq \max\{\mu(a), \mu(b)\} \).

Since \( \mu \) is \( \phi \)-invariant by lemma (2.9), \( \phi(\mu)(xyz) = \mu(abc) \leq \mu(c) \).

Hence \( \phi(\mu)(x) = \mu(a) \).

Hence \( \phi(\mu) \) is an anti fuzzy left ideal of \( R_2 \).

Similarly \( \phi(\mu) \) is an anti fuzzy right and lateral ideal of \( R_2 \).

Let \( x, y \in R_2 \) since \( \phi \) is onto then there exists \( a, b \in R_1 \) such that \( \phi(a) = x \) and \( \phi(b) = y \).

Then \( \phi(\mu)(x) = \mu(a) \leq \max\{\mu(a + b), \mu(b)\} \).

Hence \( \phi(\mu) \) is an anti fuzzy k-ideal of \( R_2 \).

**Definition (2.9)** An anti fuzzy k-ideal \( \mu \) of a ternary semiring \( R \) is said to be normal if \( \mu(0) = 1 \).

**Theorem (2.10)** Let \( \mu \) be an anti fuzzy k-ideal of a ternary semiring \( R \) and \( \mu^{*} \) be a fuzzy subset of \( R \) defined by \( \mu^{*}(x) = \mu(x) + 1 - \mu(0) \) for all \( x \in R \).

Then \( \mu^{*} \) is a normal anti fuzzy k-ideal of \( R \).

**Proof:** Let \( x, y, z \in R \). Then \( \mu^{*}(x + y) = \mu(x + y) + 1 - \mu(0) \), since \( \mu \) is an anti fuzzy k-ideal \( \leq \max\{\mu(x), \mu(y)\} + 1 - \mu(0) \).

Then \( \mu^{*} \) is a normal anti fuzzy k-ideal of \( R \).

Hence \( \phi(\mu) \) is an anti fuzzy left k-ideal then \( \leq \mu(z) + 1 - \mu(0) \).
Hence $\mu^*$ is an anti fuzzy left ideal of R similarly we can prove that $\mu^*$ is an anti fuzzy right and lateral ideal of R then $\mu^*$ is an anti fuzzy ideal of R.

$$\mu^*(x) = \mu(x) + 1 - \mu(0) \leq \max\{\mu(x+y), \mu(y)\} + 1 - \mu(0) = \max\{\mu(x+y) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\}$$

Hence $\mu^*$ is a normal anti fuzzy k-ideal of R.

References